Pointwise Green's Function Approach to Stability for Scalar Conservation Laws

PETER HOWARD
Indiana University

Abstract

We study the pointwise behavior of perturbations from a viscous shock solution to a scalar conservation law, obtaining an estimate independent of shock strength. We find that for a perturbation with initial data decaying algebraically or slower, the perturbation decays in time at the rate of decay of the integrated initial data in any L^p norm, $p \geq 1$. Stability in any L^p norm is a direct consequence. The approach taken is that of obtaining pointwise estimates on the perturbation through a Duhamel's principle argument that employs recently developed pointwise estimates on the Green's function for the linearized equation. © 1999 John Wiley & Sons, Inc.

1 Introduction

We consider the scalar viscous conservation law

$$u_t + f(u)_x = u_{xx}, \quad f, u, x \in \mathbb{R},\ t \in \mathbb{R}_+,\ u(0,x) = u_0(x), \tag{1.1}$$

where $f \in C^2(\mathbb{R})$ and $u_0(x) \to u_\pm$ as $x \to \pm\infty$. Physical contexts in which equations of form (1.1) arise are discussed, for example, in [9]. We will be concerned with *traveling wave solutions* to (1.1), that is, solutions of the form $\bar{u}(x - st)$ that satisfy $\bar{u}(\pm\infty) = u_\pm$ and the Rankine-Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-).$$

We note that by a translation of coordinates we may take $s = 0$ without loss of generality. In particular, we will consider Lax shocks, that is, u satisfying $f'(u_+) < s < f'(u_-)$. In the scalar case with diffusion only, this just rules out *sonic* shocks for which $f'(u_-)$ or $f'(u_+)$ is equal to s.

The result obtained is a pointwise estimate on perturbations from viscous profile solutions to (1.1), an estimate that agrees with the exact analysis of Burgers' equation carried out by Nishihara [18]. From our estimate, nonlinear orbital stability follows in any L^p norm, $p \geq 1$. Our key observation is a precise formulation of how spatial decay of initial data leads directly to temporal decay of the perturbation. Nishihara observed this fact for Burgers' equation and general algebraically decaying data, and Liu [11] observed it for n-dimensional systems of conservation laws for weak shocks and data decaying as $(1+|x|)^{-3/2}$. A method by which the constant diffusion term in our analysis may be replaced by the more general $(b(u)u_x)_x$ (where $b(\bar{u}) > b_0 > 0$ and $b \in C^{1+\bar{\alpha}}, \bar{\alpha} > 0$) is given in [23] but not employed here.

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The study of stability for scalar viscous conservation laws was initiated by Hopf in his study of the large-time behavior of Burgers' equation [4]. Il'in and Oleĭnik [8] proved through a maximum principle argument that viscous profiles of (1.1) are stable in the L^∞ norm under the assumptions of a convex nonlinear term and exponentially decaying initial data. Peletier then employed energy estimates—a more generalizable method—to gain an alternate proof of Il'in and Oleĭnik's result [20].

In 1976 Sattinger [21] extended these results to a nonconvex nonlinear term (still exponentially decaying initial data), using a weighted norm approach that took advantage of the semigroup structure of the solution operator. Under a mild assumption on the initial data, Osher and Ralston [19] used the semigroup framework of Sattinger to prove the stability of viscous profile solutions in the L^1 norm. Jones, Gardner, and Kapitula [9] obtained the first stability result for algebraically decaying initial data (general f), employing a new technique for getting estimates on the resolvent and explicitly trading spatial decay for temporal decay during the analysis. The results of [9] were extended by Matsumura and Nishihara [16], who obtained a better rate of decay while also proving the stability of sonic shocks, extending the work of Mei [17]. Freistühler and Serre achieved L^1 stability for data in $L^1 + L^\infty$, extending the result of Osher and Ralston by eliminating their assumption on the initial data [2].

Here, we employ the pointwise approach developed in (among others) [5, 6, 10, 11, 12, 14, 15, 22, 23] to get a stability result for algebraically and slower decaying initial data. In particular, we employ the pointwise estimates of [5, 6] made on the Green's function of the convection-diffusion equation resulting from the linearization of (1.1) about a viscous shock profile to get estimates on the perturbation.

1.1 Definitions

Before stating our main theorem, we make the following definitions:

DEFINITION 1.1 (Class of Initial Data) Denote by Δ the space of functions $d \geq 0$ such that $d \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $d(\cdot)$ nonincreasing on $x \geq 0$, nondecreasing on $x \leq 0$, and $d(\gamma t) \leq C(\gamma) d(t)\ \forall \gamma > 0$. (The paradigmatic element of Δ is $(1+|x|)^{-r}$, $r > 1$.) We will denote by D the asymptotically decaying antiderivative of d,

$$D(x) := \begin{cases} \int_{-\infty}^{x} d(y)dy\,, & x < 0\,, \\ \int_{x}^{+\infty} d(y)dy\,, & x \geq 0\,. \end{cases}$$

We remark that the class Δ contains all functions steadily decaying at an algebraic or slower rate. The analysis for exponentially decaying data has already been carried out by Sattinger [21]. Our interest here, following [9, 11], is to study slower decaying initial data. The analysis could be altered to accommodate faster decaying data, but the proof, along with the precise statement, of Theorem 1.4 would have to change appreciably.

DEFINITION 1.2 (Asymptotic Stability) We say that a traveling wave solution $\bar{u}$ to (1.1) is *asymptotically stable* in norm $\|\cdot\|$ if there exists an $\varepsilon > 0$ such that if another solution, u, to (1.1) satisfies $\|u(0,x) - \bar{u}(x)\| < \varepsilon$, then $\|u(t,x) - \bar{u}(t,x)\|$ decays to zero in time.

In general, we will not be able to get asymptotic stability for viscous profile solutions to (1.1), because u and $\bar{u}$ are both solutions to the conservation law (1.1), so they must be conserved quantities, satisfying (assuming $u - \bar{u} \in L^1(\mathbb{R})$)

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(u-\bar{u})dx = 0\,.$$

In order to have hope of convergence, we must then have $\int_{\mathbb{R}}(u-\bar{u})dx = 0$. This may not be true for generic $\bar{u}$, but it will certainly be true for some translate of $\bar{u}$, say $\bar{u}_l := \bar{u}(x-l)$, as can readily be seen through

$$\frac{d}{dl}\int_{-\infty}^{+\infty}(u(x)-\bar{u}(x-l))dx = u_+ - u_-$$

so that

$$\int_{-\infty}^{+\infty}(u(x)-\bar{u}(x-l))dx = (u_+-u_-)l + \int_{-\infty}^{+\infty}(u(x)-\bar{u}(x))dx\,.$$

Clearly, we can choose l to make the right-hand side zero. With this in mind, we make the following more apposite stability definition:

DEFINITION 1.3 (Orbital Stability) We say that a traveling wave solution $\bar{u}$ to (1.1) is *orbitally stable* in norm $\|\cdot\|$ if there exists an $\varepsilon > 0$ and a translate of $\bar{u}$, say $\bar{u}_l := \bar{u}(x-l)$, such that if another solution, u, to (1.1) satisfies $\|u(0,x)-\bar{u}_l(x)\| < \varepsilon$, then $\|u(t,x) - \bar{u}_l(t,x)\|$ decays to zero in time.

1.2 Main Result

We now state the main result of the paper, from which orbital stability follows in any L^p norm:

THEOREM 1.4 *Suppose $\bar{u}$ is a traveling wave solution to* (1.1) *and $f \in C^2(\mathbb{R})$ satisfies $f'(u_+) < s < f'(u_-)$. Then for initial data*

$$u_0(x) - \bar{u}(x) \in \mathcal{A}_\zeta\,,$$

$$\mathcal{A}_\zeta := \{v_0(x) : |v_0(x)| \le \zeta d(x),\ \textit{some } d \in \Delta, \int_{\mathbb{R}} v_0(x)dx = 0\}\,,$$

and ζ sufficiently small, we get the estimate

$$|u(t,x) - \bar{u}(x-st)| \le C\zeta[e^{-\delta_r|x-st|}D(t) + d(|x-st|+t)]$$

for $0 < \delta_r < \delta$, where δ_r depends on the rate of decay of $D(\cdot)$ and $\delta > 0$ is a positive constant as in (2.6) *below.*

Before giving the proof of Theorem 1.4, we make two remarks and mention two applications.

Remarks. Using an explicit solution, Nishihara has obtained a precise estimate on solutions of Burgers' equation [18]. His estimate shows that if the integral of initial disturbance is $O(|x|^{1-r})(r > 1)$, then the solution converges in the L^∞ norm to the traveling wave solution at the same algebraic rate in t, $O(t^{1-r})$. He also notes that, in general, this time decay rate is optimal for convergence to the asymptotically selected translate.

The significance of our result is that we get an explicit decay rate, $D(t)$, from the analysis for general data, and can consequently directly observe that x-decay of the initial data leads in general to t-decay of the solution.

1.3 Applications

In the case that $|u_0(x) - \bar{u}(x)| \leq \zeta d(x)$, where $d(x) = (1+|x|)^{-r}$, we get that (letting $v = u - \bar{u}$)

$$\begin{aligned} |v(t,x)| &\leq C[e^{-\delta_r|x|}(1+t)^{1-r} + (1+|x|+t)^{-r}] \\ &\leq C(1+|x|+t)^{1-r}(1+|x|)^{-1}. \end{aligned}$$

In theorem 1.1 of [9], Jones, Gardner, and Kapitula obtain an estimate on v (under the same assumptions made here) of the form

$$||(1+|x|)v||_{L^\infty} \leq C_k(1+t)^{-\frac{k}{2}}||(1+|x|)^{1+k}v_0||_{L^\infty}$$

for all $k \geq 2$ ($||(1+|x|)^{1+k}v_0||_{L^\infty}$ sufficiently small).

In order to draw a comparison with our result, we write

$$|(1+|x|)^{1+k}v_0| \leq \varepsilon,$$

some ε sufficiently small, so that $v_0 \leq \varepsilon(1+|x|)^{-1-k}$, $k \geq 2$. That is, for $r = 1+k \geq 3$, they obtain the estimate

$$|v(t,x)| \leq C_r(1+t)^{\frac{1-r}{2}}(1+|x|)^{-1}.$$

In particular, we extend this result to $r > 1$, with improved decay in both time and space.

We can also get an L^1 stability result from Theorem 1.4 in the spirit of Freistühler and Serre [2], though under much more stringent assumptions on initial data (we assume initial data in $\mathcal{A}_\zeta$, whereas they assume only initial data in $L^1 + L^\infty$). We compute

$$\begin{aligned} \int_{-\infty}^{+\infty}|v(t,x)|dx &\leq C\int_{-\infty}^{+\infty}(e^{-\delta_r|x|}D(t) + d(|x|+t))dx \\ &= CD(t)\int_{-\infty}^{+\infty}e^{-\delta_r|x|}\,dx + C\int_{-\infty}^{+\infty}d(|x|+t)dx \leq CD(t), \end{aligned}$$

where the constant C has changed from step to step.

Further, we note that for any $1 < p < \infty$, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} |v|^p\, dx &\leq C \int_{-\infty}^{+\infty} (e^{-\delta_r|x|}D(t) + d(|x|+t))^p\, dx \\ &\leq C(D(t)+d(t))^{p-1} \int_{-\infty}^{+\infty} (e^{-\delta_r|x|}D(t) + d(|x|+t))dx \\ &\leq C(D(t)+d(t))^{p-1}D(t) \leq CD(t)^p\,.\end{aligned}$$

Hence we arrive at the estimate

$$\Big(\int_{-\infty}^{+\infty} |v|^p dx\Big)^{1/p} \leq CD(t)\,.$$

Thus, for $v \in \mathcal{A}_\zeta$, the decay rate of v in L^p for *any* $1 \leq p \leq \infty$ is again $D(t)$ (cf. corollary 1.2 of [11]).

1.4 Plan of the Paper

In Section 2 we give an outline of the general procedure based on [15, 23], reducing the proof of Theorem 1.4 to gaining tight estimates on certain Duhamel integrals. In Section 3 we carry out the analysis by obtaining the necessary estimates.

2 General Procedure

We begin by letting $\bar{u}(x-st)$ denote a traveling wave solution to (1.1). Letting $u = \bar{u} + v$ be another solution, our goal will be to obtain pointwise estimates on the perturbation v.

First, we choose a translate of $\bar{u}$ (renamed $\bar{u}$ for convenience) so that (see discussion after Definition 1.2)

$$\int_{-\infty}^{+\infty} (u - \bar{u})dx = 0\,.$$

Substituting $u = \bar{u} + v$ into (1.1), we arrive at the linearized equation for v

$$v_t + (f'(\bar{u})v)_x = v_{xx} + O(v^2)_x\,. \tag{2.1}$$

We now make the change $w(\cdot,x) = \int_{-\infty}^{x} v(\cdot,\xi)d\xi$. Note that we have chosen our profile so that $\int_{-\infty}^{+\infty} v(\cdot,\xi)d\xi = 0$, giving

$$\int_{-\infty}^{x} v(\cdot,\xi)d\xi = -\int_{x}^{+\infty} v(\cdot,\xi)d\xi\,,$$

and so we also have that $w(0,x) = -\int_{x}^{+\infty} v(0,\xi)d\xi$. In the following analysis, we will mean by $\int^{x} v(0,\xi)d\xi$ whichever representation is most useful.

Integrating (2.1) from $-\infty$ to x, we get

$$\int_{-\infty}^{x} v_t(t,\xi)d\xi + f'(\bar{u})v(t,x) = v_x(t,x) + O(v^2)$$

or

$$w_t + f'(\bar{u})w_x = w_{xx} + O(v^2)\,,$$

the integrated form of (2.1). We note that the big O term is left as a function of v^2 for later convenience.

The need to work with the integrated equation is a consequence of the pointwise Green's function estimates in [5, 6] holding only in situations for which there are no eigenvalues (point spectrum) at the origin. In the Lax case for the nonintegrated equation, there is an eigenvalue at the origin. In the integrated case, however, there is never an eigenvalue at the origin, as is easy to see through a direct computation.

We remark that this eigenvalue at the origin is a permanent fixture of viscous conservation laws, and that in general (in the case of higher-order scalar equations and systems of any order) this eigenvalue cannot be projected out (integrated out). In these cases, a similar analysis maintains, in which the eigenvalue is taken into account through a residue analysis (see [7] for the case of higher-order scalar equations and [23] for the case of systems).

Define now the linear operator

$$Lw := w_{xx} - (f'(\bar{u})w)\,.$$

With $a(x) := f'(\bar{u})$, theorem 1.1 of [6] (see Proposition 2.2 below) gives bounds on the Green's function, $G(t,x;y)$, for

$$w_t = Lw\,. \tag{2.2}$$

What we are interested in here, however, is the forced equation

$$w_t - Lw = O(v^2)\,. \tag{2.3}$$

Applying Duhamel's principle to (2.3), we get

$$\begin{aligned} w(t,x) &= e^{Lt}w(0,x) + \int_0^t e^{L(t-s)}O(v^2)(s,x)ds \\ &= \int_{-\infty}^{+\infty} G(t,x;y)w(0,y)dy + \int_0^t \int_{-\infty}^{+\infty} G(t-s,x;y)O(v^2)(s,y)dy\,ds\,. \end{aligned}$$

We now take an x-derivative of this integral equation to arrive at

$$\begin{aligned} w_x(t,x) = &\int_{-\infty}^{+\infty} G_x(t,x;y)w(0,y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y)O(v^2)(s,y)dy\,ds\,. \end{aligned}$$

Writing the above integral equation again in terms of v yields

$$\begin{aligned} v(t,x) = &\int_{-\infty}^{+\infty} G_x(t,x;y)\int^y v(0,\xi)d\xi\,dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y)O(v^2)(s,y)dy\,ds\,. \end{aligned}$$

The following lemma, an integrated form of lemma 1.5 from [23], will provide a direct means for using this representation to obtain a pointwise estimate on v:

LEMMA 2.1 *Let C_1 and C_2 be constants, and let $h_0(x), h(t,x) \geq 0$ satisfy the relations*

$$\int_{-\infty}^{+\infty} |G_x(t,x;y)| \int^y h_0(\xi)\,d\xi\,dy \leq C_1 h(t,x) \quad \text{or}$$
$$\int_{-\infty}^{+\infty} \left| \int^y G_x(t,x;\xi)\,d\xi \right| h_0(y)\,dy \leq C_1 h(t,x)$$

and

$$\int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| h(s,y)^2\,dy\,ds \leq C_1 h(t,x) \tag{2.4}$$

for all $t > 0$ and $x \in \mathbb{R}$, and where $\int^x$ can be chosen either as $\int_{-\infty}^x$ or $\int_x^{+\infty}$ for each x. If then $|v(0,x)| \leq \zeta_0 h_0(x)$ for ζ_0 sufficiently small, then $|v(t,x)| \leq C_2 \zeta_0 h(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$, where v is the solution to (2.1).

PROOF: We define

$$\zeta(t) := \sup_{y, s \leq t} |v/h|(s,y)$$

and $\zeta(0) := \zeta_0$. Then,

$$|v(t,x)| \leq \left| \int_{-\infty}^{+\infty} G_x(t,x;y) \int^y v(0,\xi)\,d\xi\,dy \right| + \left| \int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y) O(v^2)\,dy\,ds \right|.$$

Employing the assumption $|v(0,y)| \leq \zeta_0 h_0(y)$, the definition of $\zeta(t)$ and the inequality $|O(v^2)| \leq M v^2$ for some M, we get

$$\begin{aligned} |v(t,x)| &\leq \int_{-\infty}^{+\infty} |G_x(t,x;y)| \zeta_0 \int^y h_0(\xi)\,d\xi\,dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| M \zeta(t)^2 h(s,y)^2\,dy\,ds \\ &= \zeta_0 \int_{-\infty}^{+\infty} |G_x(t,x;y)| \int^y h_0(\xi)\,d\xi\,dy \\ &\quad + M\zeta(t)^2 \int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| h(s,y)^2\,dy\,ds, \end{aligned}$$

where in order to take advantage of the second condition, we integrate by parts before pulling the absolute values inside. The bounds (2.4) then give

$$|v(t,x)| \leq \zeta_0 C_1 h(t,x) + \zeta(t)^2 M C_1 h(t,x).$$

Dividing by $h(t,x)$ and taking the supremum on both sides, we get

$$\zeta(t) \leq \zeta_0 C_1 + \zeta(t)^2 M C_1 \leq C(\zeta_0 + \zeta(t)^2),$$

where $C := \max\{C_1, C_1M\}$. Taking ζ_0 small enough so that $4C^2\zeta_0 < 1$, we apply *continuous induction* to show that $\zeta(t) \le 2C\zeta_0$. That is, by the continuity of $\zeta(t)$, we can take C large enough so that we have *strict* inequality ($\zeta(t) < 2C\zeta_0$) on some sufficiently small interval $t \in [0,T]$. Let T be the first time for which equality occurs ($\zeta(T) = 2C\zeta_0$). If no such T exists, we are done. If such a T does exist, we compute

$$\zeta(T) \le C(\zeta_0 + \zeta(T)^2) = C(\zeta_0 + 4C^2\zeta_0^2) < C\zeta_0 + C\zeta_0 = 2C\zeta_0\,.$$

This contradiction completes the proof. □

We will achieve the estimates assumed in Lemma 2.1 through the pointwise Green's function estimates of theorem 1.1 in [6], given below for the Lax case as Proposition 2.2.

PROPOSITION 2.2 *Under the hypotheses of Theorem* 1.4*, the Green's function for*

$$w_t + f'(\bar{u})w_x = w_{xx}\,, \tag{2.5}$$

where $a(x) := f'(\bar{u})$ *and* $a_\pm := \lim_{x\to\pm\infty} a(x)$ *in the Lax case* $(a_+ < 0 < a_-)$ *satisfies the estimate* (*for* $x \ge 0$)

$$|G_x(t,x;y)| \le \begin{cases} \frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-y-a_+t)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-y-a_+t)^2}{Mt}}\,, & y \ge 0\,, \\ \frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-y-a_-t)^2}{Mt}} + \frac{C_1 e^{-\delta|x|}}{t} e^{-\frac{(x-y-a_-t)^2}{Mt}}\,, & y \le 0\,, \end{cases} \tag{2.6}$$

where we have subsumed certain constants into M *and* C_1 (*the subscript follows the notation of* [6] *and is indicative of the first derivative*), *and where* M, C_1, *and* δ *each depend upon the asymptotic behavior of* $f'(\bar{u}(x))$ *and on the spectrum of the associated linearized operator* L (*defined above*). *Symmetric estimates hold for* $x \le 0$.

PROOF: As discussed in [6], the elliptic eigenvalue equation associated with (2.5) has no eigenvalues on or to the left of the imaginary axis. Further, $f \in C^2(\mathbb{R})$ gives $a \in C^1(\mathbb{R})$, satisfying the assumptions needed in theorem 1.1 of [6] for first-order derivative estimates. □

We note before proceeding that results similar to that of Proposition 2.2 can be obtained for a broad class of linear equations, including those obtained through the linearization of higher-order scalar conservation laws [7], those obtained through the linearization of systems of conservation laws [23], and (through a somewhat similar analysis in a different setting) a very general class of second-order elliptic operators with complex, bounded, measurable coefficients in $\mathbb{R}^n$ [1].

In each of the cases arising from the analysis of conservation laws, the spectral approach of [13] is employed and extended to the nonconstant coefficient case through the semigroup framework of [9, 21]. In particular, the analysis breaks into two parts: a small-time/large-eigenvalue portion that follows the outline of a Fourier transform analysis of the constant-coefficient evolution equation with only the highest-order spatial derivative considered, and a large-time/small-eigenvalue portion that follows the outline of a Fourier transform analysis of the constant-coefficient evolution equation with only the lowest two-order spatial derivatives considered. The analysis of [1] is similar to the small-time analysis mentioned above.

In the next section we find an appropriate *template function*, $h(t,x)$ (see [11, 14, 15, 23]), and employ it to prove Theorem 1.4.

3 Proof of Theorem 1.4

In the scalar case with diffusion only, we only have a viscous profile when the Lax entropy condition holds ($f'(\bar{u}_+) < s < f'(\bar{u}_-)$ or with $s = 0$, $a_+ < 0 < a_-$, that is $n+1 = 2$ incoming characteristics). Without loss of generality (by symmetry of the estimates of Proposition 2.2), we may take $x \geq 0$. The following lemma will greatly simplify the forthcoming analysis:

LEMMA 3.1 *Let $f(y) \geq 0$ be a nonincreasing function on $\mathbb{R}_+$, with $f(0) < C_1$. Assume further that there exist constants $\gamma > 0$ and $\omega > 1$ so that $f(y) \geq \gamma e^{-\frac{a}{2}(1-\frac{1}{\omega})^2 y^2}$ on $\mathbb{R}_+$. Then, for $a, z > 0$*

$$\int_0^{+\infty} e^{-a(z-y)^2} f(y)dy \leq \frac{C(\omega)}{\sqrt{a}} f(z/\omega)\,.$$

Remark 3.2. By symmetry we have for $z < 0$

$$\int_{-\infty}^{0} e^{-a(z-y)^2} f(y)dy \leq \frac{C(\omega)}{\sqrt{a}} f(z/\omega)$$

for $\omega < -1$ and f nonincreasing on $\mathbb{R}_-$ and satisfying the same lower bound as above. This *peak estimate* characterizes how the kernel $\sqrt{a}e^{-a(z-y)^2}$ behaves like a delta-function.

Remark 3.3. Also convenient for the forthcoming analysis is the observation that for $z \geq 0$, $y \leq 0$, and $a > 0$, we have the *tail estimate*

$$\int_{-\infty}^{y} e^{-a(z-\xi)^2} d\xi \leq \frac{1}{2}\sqrt{\frac{2\pi}{a}} e^{-\frac{a}{2}(z-y)^2}\,,$$

with a symmetric result true for $z \leq 0$ and $y \geq 0$. This observation is easily recognized through the computation

$$\begin{aligned}\int_{-\infty}^{y} e^{-a(z-\xi)^2} d\xi &= \int_{-\infty}^{y} e^{-\frac{a}{2}(z-\xi)^2-\frac{a}{2}(z-\xi)^2} d\xi = \int_{-\infty}^{y} e^{-\frac{a}{2}(z-\xi)^2} e^{-\frac{a}{2}(z-\xi)^2} d\xi \\ &\leq e^{-\frac{a}{2}(z-y)^2} \int_{-\infty}^{y} e^{-\frac{a}{2}(z-\xi)^2} d\xi \leq e^{-\frac{a}{2}(z-y)^2} \frac{1}{2}\sqrt{\frac{2\pi}{a}}\,.\end{aligned}$$

PROOF OF LEMMA 3.1: We break the integration into two regions as follows:

$$\int_0^{+\infty} e^{-a(z-y)^2} f(y)dy = \int_0^{z/\omega} e^{-a(z-y)^2} f(y)dy + \int_{z/\omega}^{+\infty} e^{-a(z-y)^2} f(y)dy. \tag{3.1}$$

We note that the result follows immediately for the second integral on the right-hand side of (3.1), since we can write

$$\int_{z/\omega}^{+\infty} e^{-a(z-y)^2} f(y)dy \le f(z/\omega)\int_{x/\omega}^{+\infty} e^{-a(z-y)^2}\,dy \le \frac{C}{\sqrt{a}} f(z/\omega).$$

For the first integral on the right-hand side of (3.1), we compute

$$\begin{aligned}\int_0^{z/\omega} e^{-a(z-y)^2} f(y)dy &\le f(0)(z/\omega)e^{-a(1-\frac{1}{\omega})^2 z^2}\\ &\le f(0)\frac{\omega}{\sqrt{a}}e^{-\frac{a}{2}(1-\frac{1}{\omega})^2 z^2} \le \frac{C(\omega)}{\sqrt{a}} f(z/\omega),\end{aligned}$$

where we have used above that $ze^{-\frac{a}{2}(1-\frac{1}{\omega})^2 z^2} \le C(\omega)/\sqrt{a}$ for some constant $C(\omega)$. This completes the proof. □

We are now prepared to prove two lemmas regarding the behavior of elements of Δ integrated against $G_x(t,x;y)$.

LEMMA 3.4 *Under the assumptions of Theorem* 1.4 *and with* $G(t,x;y)$ *being the Green's function for* (2.2), *we have for* $d(x) \in \Delta$,

$$\int_{-\infty}^{+\infty} |G_x(t,x;y)| \int^y d(\xi)d\xi\,dy \le C\big[e^{-\delta_r|x|}D(t) + d(|x|+t)\big]$$

or

$$\int_{-\infty}^{+\infty} \left|\int^y G_x(t,x;\xi)d\xi\right| d(y)dy \le C\big[e^{-\delta_r|x|}D(t) + d(|x|+t)\big]$$

for $0 < \delta_r < \delta$, *where* δ_r *depends on the rate of decay of* $D(\cdot)$, δ *is as in* (2.6), *and* $\int^y = \int_{-\infty}^y$ *or* $\int_y^{+\infty}$.

Remark 3.5. We will take advantage in the proof of Lemma 3.4 of the identity

$$\int_{-\infty}^{+\infty} G_x(t,x;y)dy = 0$$

by using

$$\int_{-\infty}^{y} G_x(t,x;\xi)d\xi = -\int_y^{+\infty} G_x(t,x;\xi)d\xi.$$

This identity is clear from the relationship between G and $\tilde{G}$, the Green's function for the unintegrated problem, namely, $G_x(t,x;y) \overset{\mathrm{d}}{=} \tilde{G}_y(t,x;y)$. We can see this as follows: Let w solve the integrated equation and v the unintegrated equation so that $w_t = Lw$ and $v_t = \tilde{L}v$, where $\tilde{L}$ is the spatial operator for the unintegrated equation

(2.1) (minus the error term). We then have that $w = G * w_0$ and $v = \tilde{G} * v_0$, where w_0 and v_0 are initial data. Since $v = w_x$, we get $v = G_x * w_0$, but since $w_{0y} = v_0$, we also have $v = \tilde{G} * v_0 = \tilde{G} * w_{0y} = \tilde{G}_y * w_0$. Comparing gives the claim under distribution, which is what we mean by $\stackrel{\text{d}}{=}$.

PROOF OF LEMMA 3.4: Without loss of generality (by symmetry), we need only concern ourselves with the case $x \geq 0$.

3.1 Large-Time Estimates

Assume $t \geq 1$. In this case we can use $t^{-1/2} \leq C(1+t)^{-1/2}$. The estimate will follow immediately in the case $y \leq 0$ from the additional exponential decay in x.

We compute

$$\int_{-\infty}^{+\infty} |G_x(t,x;y)| \int^{y} d(\xi)d\xi\, dy = \int_{-\infty}^{0} |G_x(t,x;y)| D(y) dy + \int_{0}^{+\infty} |G_x(t,x;y)| D(y) dy\,.$$

For the integral over $y \leq 0$ we have a bound by

$$\int_{-\infty}^{0} \frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-y-a_-t)^2}{Mt}} D(y) dy + \int_{-\infty}^{0} \frac{C_1 e^{-\delta|x|}}{t} e^{-\frac{(x-y-a_-t)^2}{Mt}} D(y) dy\,.$$

We consider two cases: $x \geq \varepsilon t$ and $x \leq \varepsilon t$ for some $0 < \varepsilon \ll a_-$. In the case that $x \geq \varepsilon t$, the exponential x-decay also gives exponential t-decay, better than our claimed estimate. In the case $x \leq \varepsilon t$, we have that $x - a_- t < 0$, so that Remark 3.2 applies, giving a bound by

$$C_1 e^{-\delta|x|} D(-x + a_- t) + C_1 e^{-\delta|x|} t^{-1/2} D(-x + a_- t) \leq C_1 e^{-\delta|x|} D((a_- - \varepsilon)t) + C_1 t^{-1/2} e^{-\delta|x|} D((a_- - \varepsilon)t)\,.$$

Note that here and in the following computations, we will use our assumption that d is radial to keep the argument of d and D positive.

We now consider the integral over $y \geq 0$, which, as above, is bounded by

$$\int_{0}^{+\infty} \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-y-a_+t)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-y-a_+t)^2}{Mt}} \right] D(y) dy \leq C_1 e^{-\delta|x|} D(x + |a_+|t) + C_1 t^{-1/2} D(x + |a_+|t)\,,$$

where the inequality is an application of Lemma 3.1.

We note that for the second term in the last expression we can do better. In fact, as we need an L^1 bound on v, we *must* do better. Motivated by the observation that we would like to achieve $d(|x|+t)$ decay, we now estimate the *parts* integral of Lemma 2.1. Recalling that by Remark 3.5 we can take $\int^y G_x d\xi = \int_{-\infty}^{y} G_x d\xi$ or $-\int_{y}^{+\infty} G_x d\xi$ (but keeping in mind we must then be careful which estimate we use on G_x), we now arrange terms so as never to integrate over the peak at $y = x - a_+ t$.

For $y \le x - a_+ t$, we will integrate on $(-\infty, y]$, and for $y \ge x - a_+ t$ we will integrate on $[y, +\infty)$. That is, we write

$$\begin{aligned}\int_0^{+\infty} \left| \int^y G_x(t,x;\xi) d\xi \right| d(y) dy \\ = \int_0^{+\infty} \left| \int_{-\infty}^y G_x(t,x;\xi) d\xi \right| d(y) I_{\{y \le x - a_+ t\}} dy \\ + \int_0^{+\infty} \left| \int_y^{+\infty} G_x(t,x;\xi) d\xi \right| d(y) I_{\{y \ge x - a_+ t\}} dy,\end{aligned}$$

where I_A represents an indicator function on the set A. Recalling that $x \ge 0$, $a_+ \le 0$ so that $x - a_+ t \ge 0$, we then obtain a bound by

$$\begin{aligned}&\int_0^{+\infty} \int_{-\infty}^0 \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} + \frac{C_1 e^{-\delta|x|}}{t} \right] e^{-\frac{(x-\xi-a_- t)^2}{Mt}} d\xi\, d(y) I_{\{y \le x - a_+ t\}} dy \\ &+ \int_0^{+\infty} \int_0^y \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} + \frac{C_1}{t} \right] e^{-\frac{(x-\xi-a_+ t)^2}{Mt}} d\xi\, d(y) I_{\{y \le x - a_+ t\}} dy \\ &+ \int_0^{+\infty} \int_y^{+\infty} \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} + \frac{C_1}{t} \right] e^{-\frac{(x-\xi-a_+ t)^2}{Mt}} d\xi\, d(y) I_{\{y \ge x - a_+ t\}} dy.\end{aligned} \tag{3.2}$$

We now estimate each of the integrals of (3.2). First, for $\xi \in (-\infty, 0]$, we note that, extending the ξ-integration over all space and using that $d(\cdot)$ is integrable, we get a bound by $Ce^{-\delta|x|}$ with no t-decay. For the second integral ($\xi \in [0,y]$), we note that $y \le x - a_+ t$ so that the integration does not run across the peak. Hence, Remark 3.3 applies, giving a bound by

$$\begin{aligned}\int_0^{+\infty} \left[C_1 e^{-\delta|x|} e^{-\frac{(x-\xi-a_+ t)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-\xi-a_+ t)^2}{Mt}} \right] d(y) I_{\{y \le x - a_+ t\}} dy \le \\ C_1 e^{-\delta|x|} \sqrt{t} d(x + |a_+|t) + C_1 d(x + |a_+|t),\end{aligned} \tag{3.3}$$

the second term of which is as claimed and the first term of which is bounded by $C_1 e^{-\delta|x|} D(t)$ by the integrability of $d(\cdot)$. For the third integral in (3.2), we similarly arrive at exactly the same estimates. Combining these estimates with (3.3), we have

$$\int_0^{+\infty} |G_x(t,x;y)| D(y) dy \le C_1 \min \left\{ e^{-\delta|x|}, t^{-1/2} D(x+t) \right\},$$

where on the left-hand side we mean either the integral listed or the parts integral.

We now achieve the desired estimates by breaking the tx-plane into regions and observing that one of the above bounds always yields the claim. Consider first the region in which $t^{1/2} \ge e^{(\delta/r)|x|}$, so that $\ln t^{1/2} \ge (\delta/r)|x|$ and thus $|x| \le (r/2\delta) \ln t$. In this region, our bound of $t^{-1/2} D(x+t)$ gives rise to a bound of $e^{-(\delta/r)|x|} D(x+t)$, or $e^{-\delta_r |x|} D(x+t)$, with $\delta_r := (\delta/r)$. We remark here that this argument is the only place in our proof in which it arises that the exponential space decay depends on the rate r.

Next, we consider the estimate we can obtain when $|x| \geq (r/2\delta)\ln t$. In this case we use our bound by $e^{-\delta|x|}$ to compute a bound by

$$e^{-\delta|x|} = e^{-\frac{\delta}{2}|x|}e^{-\frac{\delta}{2}|x|} \leq e^{-\frac{\delta}{2}|x|}e^{-\frac{\delta}{2}\frac{r}{2\delta}\ln t} = e^{-\frac{\delta}{2}|x|}t^{-r/4}, \tag{3.4}$$

where the constant has been omitted. Thus, if $D(t)$ decays slower than $t^{-r/4}$ and $\delta_r \leq \delta/2$, this is better than the claimed result. We conclude that for each rate of decay $r/4$ there exists a δ_r, namely, $\delta_r := \delta/2r$ (redefined as smaller than in the previous paragraph) such that we can achieve the claim for that algebraic rate. Hence, we have the claim for all algebraic rates of decay. Note that for decay slower than algebraic we can take $r = 1$.

3.2 Small-Time Estimates

Assume now $t \leq 1$. We note that small-time behavior of $\tilde{G}$ ($= \int G_x\,dy$ by Remark 3.5) is known and can be found, for example, in Friedman [3]. An analysis is included here to keep the work self-contained. In this case our goal is to get x-decay only. We will take advantage here of the fact that there exist constants C_1 and C_2 so that

$$\frac{(x-y)^2}{t} \leq C_1\frac{(x-y-a_+t)^2}{t} + C_2, \tag{3.5}$$

and similarly for $(x-y-a_-t)/t$. This is clear since

$$\frac{(x-y-a_+t)^2}{t} = \frac{(x-y)^2}{t} - 2a_+(x-y) + a_+^2 t \geq \tilde{C}_1\frac{(x-y)^2}{t} - \tilde{C}_2$$

for some $\tilde{C}_1$ and $\tilde{C}_2$, since for $|x-y| \leq 1$, $2a_+|x-y| \leq \tilde{C}_2$, and for $|x-y| \geq 1$, $|x-y| \leq (x-y)^2$.

In this case, we again want to estimate the parts integral of Lemma 2.1. For $y \leq 0$ and by using (3.5), we compute

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left|\int^{y} G_x(t,x;\xi)d\xi\right| d(y)dy \\
&\quad = \int_{-\infty}^{0}\int^{y}\left[\frac{C_1e^{-\delta|x|}}{\sqrt{t}}e^{-\frac{(x-\xi-a_-t)^2}{Mt}} + \frac{C_1e^{-\delta|x|}}{t}e^{-\frac{(x-\xi-a_-t)^2}{Mt}}\right]d\xi\, d(y)dy \\
&\quad \leq \int_{-\infty}^{0}\int^{y}\left[\frac{C_1e^{-\delta|x|}}{\sqrt{t}}e^{-\frac{(x-\xi)^2}{Mt}} + \frac{C_1e^{-\delta|x|}}{t}e^{-\frac{(x-\xi)^2}{Mt}}\right]d\xi\, d(y)dy,
\end{aligned}$$

where our constants C_1 and M have changed. According to Remark 3.3 and our assumption that $d(y) \in L^\infty(\mathbb{R})$, this is bounded by

$$\int_{-\infty}^{0} C_1\left[e^{-\delta|x|}e^{-\frac{(x-y)^2}{Mt}} + \frac{e^{-\delta|x|}}{\sqrt{t}}e^{-\frac{(x-y)^2}{Mt}}\right]dy \leq C_1e^{-\delta|x|}(\sqrt{t}+1).$$

For $y \geq 0$, we have a bound by

$$\int_0^{+\infty}\int^y \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-\xi)^2}{Mt}}\right] d\xi\, d(y) dy$$
$$= \int_0^{+\infty}\int_y^{+\infty} \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-\xi)^2}{Mt}}\right] d\xi\, d(y) I_{\{y>x\}}(y) dy$$
$$+ \int_0^{+\infty}\int_{-\infty}^y \left[\frac{C_1 e^{-\delta|x|}}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{Mt}} + \frac{C_1}{t} e^{-\frac{(x-\xi)^2}{Mt}}\right] d\xi\, d(y) I_{\{y<x\}}(y) dy$$
$$\leq \int_0^{+\infty} C_1 \left[e^{-\delta|x|} e^{-\frac{(x-y)^2}{Mt}} + \frac{C_1}{\sqrt{t}} e^{-\frac{(x-y)^2}{2Mt}}\right] d(y) dy\,,$$

where this last inequality is a result of Remark 3.3. We can now apply Lemma 3.1 to get a bound by

$$C_1 e^{-\delta|x|}\sqrt{t} d(x) + C_1 d(x)\,,$$

completing the proof. □

LEMMA 3.6 *Under the assumptions of Theorem* 1.4 *and with* $G(t,x;y)$ *being the Green's function for* (2.2), *we have for* $d(x) \in \Delta$,

$$\int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| d^2(|y|+s) dy\, ds \leq C\left[e^{-\delta|x|} D(t) + d(|x|+t)\right],$$

δ *as in* (2.6).

PROOF: As before, we will make the computation through the G_x bounds of (2.6) and by breaking the integral into a region of $y \leq 0$ and a region of $y \geq 0$. In particular, we write

$$\begin{aligned}(3.6)\qquad &\int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y) d^2(|y|+s) dy\, ds \\ &= \int_0^t \int_{-\infty}^{0} G_x(t-s,x;y) d^2(|y|+s) dy\, ds \\ &+ \int_0^t \int_{0}^{+\infty} G_x(t-s,x;y) d^2(|y|+s) dy\, ds\,.\end{aligned}$$

We first analyze the integral over $y \leq 0$. In this case, we get a bound by

$$\begin{aligned}&C_1 \int_0^t \int_{-\infty}^{0} \frac{e^{-\delta|x|}}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d^2(|y|+s) dy\, ds \\ (3.7)\qquad &+ C_1 \int_0^t \int_{-\infty}^{0} \frac{e^{-\delta|x|}}{t-s} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d^2(|y|+s) dy\, ds\,.\end{aligned}$$

As in the previous proof, we now consider two regions: $x \geq \varepsilon t$ and $x \leq \varepsilon t$ for $\varepsilon \ll a_-$ (in particular, $\varepsilon < a_-/2$). In the case $x \geq \varepsilon t$, the exponential x-decay implies exponential t-decay, and we get decay faster than that claimed. In the case $x \leq \varepsilon t$, we break the s-integration into two intervals, $[0,\frac{t}{2}]$ and $[\frac{t}{2},t]$, obtaining a bound for the first integral on the right-hand side of (3.6) of

$$
\begin{aligned}
&C_1\int_0^{t/2}\int_{-\infty}^0 \frac{e^{-\delta|x|}}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d^2(|y|+s)\,dy\,ds\\
&+C_1\int_{t/2}^{t}\int_{-\infty}^0 \frac{e^{-\delta|x|}}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d^2(|y|+s)\,dy\,ds\,.
\end{aligned}
\tag{3.8}
$$

For the first integral in (3.8), $t-s\geq\frac{t}{2}$, so that $x-a_-(t-s)<\varepsilon t-a_-(t/2)<0$. Remark 3.2 applies, then, giving a bound by

$$
\begin{aligned}
&C_1e^{-\delta|x|}\int_0^{t/2} d^2(-x+|a_-|(t-s)+s)\,ds\\
&\quad\leq C_1e^{-\delta|x|}\,d(t)\int_0^{t/2} d(-x+|a_-|(t-s)+s)\,ds\\
&\quad\leq C_1e^{-\delta|x|}\,d(t)D(t)\,,
\end{aligned}
$$

better than the claimed decay.

For the second integral in (3.8), we have a bound by

$$
\begin{aligned}
&C_1e^{-\delta|x|}d(t)\int_{t/2}^{t}\int_{-\infty}^0 \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d(|y|+s)\,dy\,ds\\
&\quad= C_1e^{-\delta|x|}\,d(t)\int_{t/2}^{t^*}\int_{-\infty}^0 \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d(|y|+s)\,dy\,ds\\
&\qquad+C_1e^{-\delta|x|}d(t)\int_{t^*}^{t}\int_{-\infty}^0 \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} d(|y|+s)\,dy\,ds\,,
\end{aligned}
$$

where t^* is defined as before by the relation $x=a_-(t-t^*)$. Hence, on the first integral ($[\frac{t}{2},t^*]$) $x-a_-(t-s)\leq x-a_-(t-t^*)=0$, so that Remark 3.2 applies. On the second integral, $x-a_-(t-s)\geq x-a_-(t-t^*)=0$, so that $x-y-a_-(t-s)\geq 0$, and we can apply Remark 3.3. Combining these two observations, we get a bound by

$$
C_1e^{-\delta|x|}d(t)\int_{t/2}^{t^*} d(-x+|a_-|(t-s)+s)\,ds + C_1e^{-\delta|x|}d^2(t)\int_{t^*}^{t} e^{-\frac{(x-a_-(t-s))^2}{M(t-s)}}\,ds\,.
\tag{3.9}
$$

Both integrals in (3.9) give decay better than that claimed, by the integrability of $d(\cdot)$.

For the second integral in (3.7), we get better decay by $t^{-1/2}$ on the interval $[0,t/2]$. For the integral over $[t/2,t]$ we arrive by an analysis precisely as above at a bound by

$$C_1 e^{-\delta|x|} d(t) \int_{\frac{t}{2}}^{t^*} \frac{1}{\sqrt{t-s}} d(-x+|a_-|(t-s)+s)ds + C_1 e^{-\delta|x|} d^2(t) \int_{t^*}^{t} \frac{1}{\sqrt{t-s}} e^{-\frac{(x-a_-(t-s))^2}{M(t-s)}} ds, \tag{3.10}$$

where t^* is defined as above. In the first integral of (3.10), we make the change of variable $\xi = \sqrt{t-s}$, which leads to the representation

$$\begin{aligned} & C_1 e^{-\delta|x|} d(t) \int_{\sqrt{t-t^*}}^{\sqrt{t/2}} 2d(-x+|a_-|\xi^2 + t - \xi^2) d\xi \\ & \leq C_1 e^{-\delta|x|} d(t) \int_{\sqrt{t-t^*}}^{\sqrt{t/2}} 2d(-x+|a_-|\xi^2 + t/2) d\xi \\ & \leq C_1 e^{-\delta|x|} d(t) \int_{\sqrt{t-t^*}}^{\sqrt{t/2}} 2d(-x+|a_-|\xi^2 + \xi) d\xi \\ & \leq C_1 e^{-\delta|x|} d(t), \end{aligned} \tag{3.11}$$

by the integrability of $d(\cdot)$, and where the last inequality is valid for t large (for t small the estimate is obvious). A similar argument leads to precisely the same bound on the second integral of (3.9).

This completes the analysis of the case with $y \leq 0$. We now consider the case $y \geq 0$. For the second integral on the right-hand side of (3.6), we have

$$\begin{aligned} & \int_0^t \int_0^{+\infty} |G_x(t,x;y)| d^2(|y|+s) dy\, ds \\ & \leq C_1 \int_0^t \int_0^{+\infty} \frac{e^{-\delta|x|}}{\sqrt{t-s}} e^{-\frac{(x-y-a_+(t-s))^2}{M(t-s)}} d^2(y+s) dy\, ds \\ & + C_1 \int_0^t \int_0^{+\infty} \frac{1}{t-s} e^{-\frac{(x-y-a_+(t-s))^2}{M(t-s)}} d^2(y+s) dy\, ds. \end{aligned}$$

An application of Lemma 3.1 ($a_+ < 0$, so $x - a_+(t-s) > 0$) yields a bound by

$$\begin{aligned} & C_1 e^{-\delta|x|} \int_0^t d^2(x+|a_+|(t-s)+s) ds \\ & + C_1 \int_0^t \frac{1}{\sqrt{t-s}} d^2(x+|a_+|(t-s)+s) ds \\ & \leq C_1 e^{-\delta|x|} d(x+t) \int_0^t d(x+|a_+|(t-s)+s) ds \\ & + C_1 d(x+t) \int_0^t \frac{1}{\sqrt{t-s}} d(x+|a_+|(t-s)+s) ds \\ & \leq C_1 e^{-\delta|x|} d(x+t) + C_1 d(x+t), \end{aligned}$$

where these last estimates are the result of an argument similar to that in (3.11).

This completes the proof of Lemma 3.6. □

LEMMA 3.7 *Under the assumptions of Theorem* 1.4 *and with* $G(t,x;y)$ *being the Green's function for* (2.2)*, we have for* $d(x) \in \Delta$*,*

$$\int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| e^{-2\delta_r |y|} D^2(s)\,dy\,ds \le C e^{-\delta_r |x|} D^2(t)\,,$$

δ_r *as in the statement of Lemma* 3.6*.*

PROOF: The proof of Lemma 3.7 follows similarly (more directly, in fact) to the proof of Lemma 3.6 and is here omitted. The interested reader is referred to [5] for details. □

PROOF OF THEOREM 1.4: Letting

$$h(t,x) = e^{-\delta_r |x|} D(t) + d(|x|+t)\,,$$

Lemma 3.4 combined with Lemmas 3.6 and 3.7 will yield the result through Lemma 2.1. In order to apply Lemma 2.1 to solutions for (1.1), we need two estimates. First, we need

$$\int_{-\infty}^{+\infty} |G_x(t,x;y)| \int^y h_0(\xi)\,d\xi \le C_1 h(t,x)\,.$$

With $h_0(\xi) \le d(\xi)$ we can achieve this directly from Lemma 3.4. A similar argument works for the parts assumption. We also need the estimate

$$\int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| h(s,y)^2\,dy\,ds \le C_1 h(t,x)\,.$$

We can achieve this through Lemmas 3.6 and 3.7 and Young's inequality as follows:

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| h(s,y)^2\,dy\,ds \\
&\quad \le C \int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| [e^{-\delta_r |x|} D(t) + d(x+t)]^2\,dy\,ds \\
&\quad \le C \int_0^t \int_{-\infty}^{+\infty} |G_x(t-s,x;y)| e^{-2\delta_r |x|} D^2(t) + d^2(x+t)\,dy\,ds\,.
\end{aligned}$$

Applying then Lemma 2.1 with $v_0 \in \mathcal{A}_{\zeta_0}$ and ζ_0 sufficiently small, we get

$$|v(t,x)| \le C\zeta_0 \left[e^{-\delta_r |x|} D(t) + d(x+t)\right].$$

By the above-mentioned symmetry for the case $x \le 0$, this completes the proof of Theorem 1.4. □

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PETER HOWARD
Brown University
Division of Applied Mathematics
182 George Street
Providence, RI 02912
E-mail: phoward@cfm.brown.edu

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