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# Pointwise multipliers for functions of bounded mean oscillation

By Eiichi NAKAI and Kôzô YABUTA*

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## 1. Introduction.

The purpose of this paper is to characterize the set of pointwise multipliers on $bmo_{\phi}(\boldsymbol{R}^n)$, which is the function space defined using the mean oscillation and a growth function $\phi$.

Janson [2] has characterized pointwise multipliers on $bmo_{\phi}(\boldsymbol{T}^n)$ on the $n$-dimensional torus $\boldsymbol{T}^n$. We extend his result to the case of the $n$-dimensional Euclidean space $\boldsymbol{R}^n$.

To define $bmo_{\phi}(\boldsymbol{R}^n)$, let $I(a, r)$ be the cube $\{x\in\boldsymbol{R}^n ; |x_i-a_i|\leqq r/2, i=1, 2, \cdots, n\}$ whose edges have length $r$ and are parallel to the coordinate axes. For a cube $I$, we denote by $|I|$ the Lebesgue measure of $I$, by $M(f, I)$ or $f_I$ the mean value of a function $f$ on $I$, i.e. $|I|^{-1}\int_I f(x)dx$, and by $MO(f, I)$ the mean oscillation of $f$ on $I$, i.e. $|I|^{-1}\int_I |f(x)-f_I|dx$.

We now define

$$bmo_{\phi}(\boldsymbol{R}^n)=\left\{f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n) \ ; \ \sup_{I(a,r)} \frac{MO(f, I(a, r))}{\phi(r)}<+\infty\right\},$$

where $\phi$ is assumed to be a positive non-decreasing function on $\boldsymbol{R}_+=(0, \infty)$. Such a function is called a growth function. If two growth functions $\phi_1$ and $\phi_2$ are equivalent ($\phi_1\sim\phi_2$) i.e. there is a constant $C>0$ such that $C^{-1}\phi_1(r)\leqq\phi_2(r)\leqq C\phi_1(r)$, then $bmo_{\phi_1}(\boldsymbol{R}^n)=bmo_{\phi_2}(\boldsymbol{R}^n)$.

A function $g$ on $\boldsymbol{R}^n$ is called a pointwise multiplier on $bmo_{\phi}(\boldsymbol{R}^n)$, if the pointwise multiplication $fg$ belongs to $bmo_{\phi}(\boldsymbol{R}^n)$ for all $f$ belonging to $bmo_{\phi}(\boldsymbol{R}^n)$.

Janson's characterization is the following. If $\phi$ is a growth function and $\phi(r)/r$ is almost decreasing, then a function $g$ is a pointwise multiplier on $bmo_{\phi}(\boldsymbol{T}^n)$ if and only if $g$ belongs to $bmo_{\psi}(\boldsymbol{T}^n)\cap L^{\infty}(\boldsymbol{T}^n)$ where $\psi(r)=\phi(r)\Big/\int_r^1\phi(t)t^{-1}dt$. (A positive function $h(t)$ is said to be almost decreasing if there

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is a constant $A$ such that $h(t)\leqq Ah(t')$ if $t\geqq t'$.)

However the case of $\boldsymbol{R}^n$ is more complicated, and we must introduce a new function space similar to $bmo_\phi$ as follows. Let $w(x, r)$ be a positive function on $\boldsymbol{R}^n\times\boldsymbol{R}_+$. We define

$$bmo_w(\boldsymbol{R}^n)=\left\{f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)\ ;\ \|f\|_{BMO_w}=\sup_{I(a,r)}\frac{MO(f, I(a, r))}{w(a, r)}<+\infty\right\}.$$

With a growth function $\phi(r)$, we always associate the function $w_\phi(x, r)$, defined by

$$w_\phi(x, r)=\phi(r)\Big/\left(\left|\int_r^1\phi(t)\frac{dt}{t}\right|+\int_1^{2+|x|}\phi(t)\frac{dt}{t}\right).$$

Then our main result is the following.

THEOREM 1. *Suppose $\phi(r)/r$ is almost decreasing. Then a function $g$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$ if and only if $g$ belongs to $bmo_{w_\phi}(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$.*

We consider $bmo_\phi(\boldsymbol{R}^n)$ with the norm

$$\|f\|_{bmo_\phi}=|M(f, I(0, 1))|+\sup_{I(a,r)}\frac{MO(f, I(a, r))}{\phi(r)}.$$

Usually (see Janson [2]), $bmo_\phi(\boldsymbol{R}^n)$ is denoted by $BMO_\phi(\boldsymbol{R}^n)$ equipped with the seminorm

$$\|f\|_{BMO_\phi}=\sup_{I(a,r)}\frac{MO(f, I(a, r))}{\phi(r)}.$$

Then $BMO_\phi(\boldsymbol{R}^n)$ modulo constants is a Banach space, but $bmo_\phi(\boldsymbol{R}^n)$ is itself a Banach space modulo null-functions. To consider pointwise multipliers, the space $bmo_\phi(\boldsymbol{R}^n)$ is a more suitable one than $BMO_\phi(\boldsymbol{R}^n)$.

If we consider subspaces $bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$, we obtain a similar result as follows.

THEOREM 2. *Suppose $\phi(r)/r$ is almost decreasing.*

(i) *Let $1\leqq p<\infty$. Then a function $g$ is a pointwise multiplier from $bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$ to $bmo_\phi(\boldsymbol{R}^n)$ if and only if $g\in bmo_\psi(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$, where $\psi(r)=\phi(r)\Big/\int_{\min(1,r)}^2\phi(t)t^{-1}dt$.*

(ii) *A function $g$ is a pointwise multiplier from $bmo_\phi(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$ to $bmo_\phi(\boldsymbol{R}^n)$ if and only if $g\in bmo_\phi(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$.*

In these cases, $g$ is a pointwise multiplier from $bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$ into itself ($1\leqq p\leqq\infty$).

If we define the Banach space $UBM\text{-}BMO_\phi(\boldsymbol{R}^n)$ by

$$\{f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)\ ;\ \|f\|_{UBM\text{-}BMO_\phi}=\|f\|_{BMO_\phi}+\sup_{a\in\boldsymbol{R}^n}M(f, I(a, 1))<+\infty\},$$

then we have the following theorem similar to the torus case.

THEOREM 3. *Suppose $\phi(r)/r$ is almost decreasing. Then a function $g$ is a pointwise multiplier from $UBM\text{-}BMO_{\phi}(\boldsymbol{R}^n)$ to $bmo_{\phi}(\boldsymbol{R}^n)$ if and only if $g\in bmo_{\psi}(\boldsymbol{R}^n)\cap L^{\infty}(\boldsymbol{R}^n)$, where $\psi$ is as in Theorem 2. In this case, $g$ is a pointwise multiplier on $UBM\text{-}BMO_{\phi}(\boldsymbol{R}^n)$.*

It is known that $UBM\text{-}BMO_1(\boldsymbol{R}^n)$ is the dual space of the local Hardy space $h^1(\boldsymbol{R}^n)$, introduced by D. Goldberg [**1**]. Hence by duality we have, as in the torus case [**2**], the following:

COROLLARY 4. *A function $g$ is a pointwise multiplier from $h^1(\boldsymbol{R}^n)$ to itself, if and only if $g\in bmo_{\psi}(\boldsymbol{R}^n)\cap L^{\infty}(\boldsymbol{R}^n)$, where $\psi(r)=1\Big/\int_{\min(1,r)}^{2}t^{-1}dt$.*

Our Theorem 1 answers a problem, which is implicitly stated in Johnson [**3**]. Stegenga [**5**] also treated the one dimensional torus case with $\phi\equiv 1$, and applied it to the boundedness problem of Toeplitz operators on the Hardy space $H^1(\boldsymbol{T})$. Applications of this paper will be treated in future.

Sections 2 and 3 are for the preliminaries and lemmas. In section 4 we give the proofs of Theorems 1, 2 and 3, and in section 5 we give some sufficient conditions for pointwise multipliers, and examples. The letter $C$ will always denote a constant and does not necessarily denote the same one.

We note that the almost-decreasingness of $\phi(t)/t$ combined with the non-decreasingness of $\phi(t)$ implies that $\phi(t)$ is equivalent to a nondecreasing concave function. We have learned this from J. Peetre.

We would like to express our thanks to the referee. He gave us a proof of Lemma 3.4 simpler than ours, valid for $1\leqq p\leqq\infty$, by which we could improve the case $p=\infty$ in Theorem 2.

## 2. Preliminaries.

First, we state some simple lemmas without proofs. (See for example Spanne [**4**].) We write

$$\rho(f, r)=\sup_{a\in R^n,\, t\leq r} MO(f, I(a, t)).$$

LEMMA 2.1. $$MO(f, I)\leqq 2\inf_{c}|I|^{-1}\int_{I}|f(x)-c|dx.$$

LEMMA 2.2. *If $|F(x)-F(y)|\leqq C|x-y|$, then $MO(F(f), I)\leqq 2CMO(f, I)$.*

LEMMA 2.3. *Suppose that $I(a', r')\subset I(a, r)$. Then*

$$|M(f, I(a', r'))-M(f, I(a, r))|\leqq C\int_{r'}^{2r}\frac{\rho(f, t)}{t}dt.$$

In the sequel, we always assume that $\phi(t)$ denotes a positive non-decreasing function and that $\phi(t)/t$ is almost decreasing. For each $\phi$, we define strictly positive functions $\Phi^*(r)$ and $\Phi_*(r)$:

$$\Phi^*(r)=\begin{cases}\int_1^r \phi(t)/t\,dt & (2\leqq r)\\ \int_1^2 \phi(t)/t\,dt & (0<r<2)\,,\end{cases}$$

$$\Phi_*(r)=\begin{cases}\int_r^2 \phi(t)/t\,dt & (0<r\leqq 1)\\ \int_1^2 \phi(t)/t\,dt & (1<r)\,.\end{cases}$$

Then, by a slight modification of the proof of the theorem 2 (a) in Spanne [**4**, p. 601], we see that $\Phi^*(r)$ and $\Phi_*(r)$ belong to $bmo_\phi(\boldsymbol{R}_+)$. One can easily see that $f(|x|)\in bmo_\phi(\boldsymbol{R}^n)$ if $f(r)\in bmo_\phi(\boldsymbol{R}_+)$. Hence we have:

LEMMA 2.4. $\Phi^*(|x|),\quad \Phi_*(|x|)\in bmo_\phi(\boldsymbol{R}^n)$.

Next we state some other properties of the functions $\Phi^*(r)$ and $\Phi_*(r)$.

LEMMA 2.5. (i) *For any $k>0$, there exists a constant $C_k>0$ such that*

$$C_k^{-1}\Phi^*(kr)\leqq\Phi^*(r)\leqq C_k\Phi^*(r/k)\qquad \textit{for all } r>0\,.$$

(ii) *For any $k>0$, there is a constant $C_k>0$ such that*

$$C_k^{-1}\Phi_*(kr)\geqq\Phi_*(r)\geqq C_k\Phi_*(r/k)\qquad \textit{for all } r>0\,.$$

(iii) *There is a constant $C>0$, depending only on the dimension $n$, such that*

$$r^{-n}\int_0^r \Phi^*(t)t^{n-1}dt\geqq C\Phi^*(r/2)\qquad \textit{for all } r>0\,.$$

(iv) *There is a constant $C>0$ such that*

$$r^{-1}\int_0^r \frac{dt}{\Phi^*(t)}\leqq C\frac{\phi(r)}{\Phi^*(r)}\qquad \textit{for all } r\geqq 2\,.$$

PROOF. (i) Since $\Phi^*(r)$ is non decreasing, it is clear for $0<k\leqq 1$. So we assume $k>1$. If $r\leqq kr\leqq 2$, then $\Phi^*(kr)=\Phi^*(r)$. If $r\leqq 2\leqq kr$, then $\Phi^*(kr)\leqq \Phi^*(2k)\leqq C_k\Phi^*(2)=C_k\Phi^*(r)$. And if $2\leqq r\leqq kr$, then, since $\phi(t)/t$ is almost decreasing, we get

$$\Phi^*(kr)=\int_1^{kr}\phi(t)\frac{dt}{t}=\int_{1/k}^r \phi(kt)\frac{dt}{t}\leqq\int_{1/k}^r Ak\phi(t)\frac{dt}{t}$$
$$\leqq C_k\Phi^*(r)\,.$$

Therefore we get $\Phi^*(kr)\leqq C_k\Phi^*(r)$ for all $r>0$. And hence $\Phi^*(r)\leqq C_k\Phi^*(r/k)$ for all $r>0$.

(ii) In a way similar to the case (i) we get (ii).

(iii) Since $\Phi^*(t)$ is non-decreasing, we have

$$r^{-n}\int_0^r \Phi^*(t)t^{n-1}dt \geqq r^{-n}\int_{r/2}^r \Phi^*(t)t^{n-1}dt$$
$$\geqq n^{-1}(1-2^{-n})\Phi^*(r/2)\,.$$

(iv) Since

$$\Phi^*(t) \geqq \max\{\phi(1)\log t,\ \phi(1)\log 2\} > \frac{1}{4}\phi(1)\log(e^2+t)\,,$$

and

$$\frac{1}{\log(e^2+t)} \leqq \frac{2}{\log(e^2+t)}\left(1-\frac{1}{\log(e^2+t)}\right) = 2\frac{d}{dt}\left(\frac{e^2+t}{\log(e^2+t)}\right),$$

we have

$$\frac{1}{r}\int_0^r \frac{dt}{\Phi^*(t)} \leqq \frac{8}{\phi(1)r}\left[\frac{e^2+t}{\log(e^2+t)}\right]_0^r < \frac{8}{\phi(1)r}\frac{e^2+r}{\log(e^2+r)}$$
$$< C/\log r, \qquad \text{as } r \geqq 2\,.$$

Hence we have the desired inequality, since

$$\phi(r)\log r = \int_1^r \phi(r)\frac{dt}{t} \geqq \int_1^r \phi(t)\frac{dt}{t} = \Phi^*(r) \qquad \text{as } r \geqq 2\,.$$

q.e.d.

REMARK 2.1. By this lemma there is a constant $C>0$ such that

$$C^{-1}(\Phi_*(r)+\Phi^*(r)+\Phi^*(|x|)) \leqq \left|\int_r^1 \phi(t)\frac{dt}{t}\right| + \int_1^{2+|x|} \phi(t)\frac{dt}{t} \tag{2.1}$$
$$\leqq C(\Phi_*(r)+\Phi^*(r)+\Phi^*(|x|))\,.$$

Finally in this section, we note one more fact (Spanne [**4**, p. 601]).

LEMMA 2.6. *If* $\int_0^1 \phi(t)t^{-1}dt < +\infty$, *then*

$$\omega(f, r) = \operatorname*{ess\,sup}_{|x-y|\leqq r} |f(x)-f(y)| \leqq C\int_0^r \phi(t)\frac{dt}{t}\|f\|_{BMO_\phi}\,,$$

*for any* $f \in bmo_\phi(\boldsymbol{R}^n)$.

## 3. Lemmas.

To prove the theorems, we show a few lemmas in this section.

LEMMA 3.1. *There is a constant* $C>0$ *such that*

$$|M(f, I(a, r))| \leqq C\|f\|_{bmo_\phi}(\Phi_*(r)+\Phi^*(r)+\Phi^*(|a|))$$

*for any* $f \in bmo_{\phi}(\boldsymbol{R}^n)$ *and for any cube* $I(a, r)$.

PROOF. Case 1: $r \geqq 1$, $|a| \geqq r$. Since $I(a, r)$, $I(0, 1) \subset I(0, r+2|a|)$, by Lemma 2.3 and Lemma 2.5 (i), we have

$$\begin{aligned} &|M(f, I(a, r))-M(f, I(0, 1))| \\ &\leqq |M(f, I(a, r))-M(f, I(0, r+2|a|))|+|M(f, I(0, 1))-M(f, I(0, r+2|a|))| \\ &\leqq C\int_r^{2(r+2|a|)} \rho(f, t)\frac{dt}{t}+C\int_1^{2(r+2|a|)} \rho(f, t)\frac{dt}{t} \\ &\leqq 2C\int_1^{6|a|} \rho(f, t)\frac{dt}{t} \leqq 2C\|f\|_{BMO_{\phi}}\int_1^{6|a|} \phi(t)\frac{dt}{t} \\ &=2C\|f\|_{BMO_{\phi}}\Phi^*(6|a|) \leqq C'\|f\|_{BMO_{\phi}}\Phi^*(|a|). \end{aligned}$$

Case 2: $1 \leqq r$, $|a| \leqq r$. Since $I(a, r)$, $I(0, 1) \subset I(0, r+2|a|)$, in a way similar to the case 1, we have

$$|M(f, I(a, r))-M(f, I(0, 1))| \leqq C\|f\|_{BMO_{\phi}}\Phi^*(r).$$

Case 3: $r \leqq 1$, $1 \leqq |a|$. Since $I(a, r)$, $I(0, 1) \subset I(0, r+2|a|)$, by Lemma 2.3 and Lemma 2.5 (i), we have

$$\begin{aligned} &|M(f, I(a, r))-M(f, I(0, 1))| \\ &\quad \leqq C\int_r^{2(r+2|a|)} \rho(f, t)\frac{dt}{t}+C\int_1^{2(r+2|a|)} \rho(f, t)\frac{dt}{t} \\ &\quad \leqq C\int_r^1 \rho(f, t)\frac{dt}{t}+2C\int_1^{6|a|} \rho(f, t)\frac{dt}{t} \\ &\quad \leqq C\|f\|_{BMO_{\phi}}\int_r^1 \phi(t)\frac{dt}{t}+2C\|f\|_{BMO_{\phi}}\int_1^{6|a|} \phi(t)\frac{dt}{t} \\ &\quad \leqq C'\|f\|_{BMO_{\phi}}(\Phi_*(r)+\Phi^*(|a|)). \end{aligned}$$

Case 4: $r \leqq 1$, $|a| \leqq 1$. Since $I(a, r)$, $I(0, 1) \subset I(0, 3)$, by Lemma 2.3 we get

$$\begin{aligned} &|M(f, I(a, r))-M(f, I(0, 1))| \\ &\leqq |M(f, I(a, r))-M(f, I(0, 3))|+|M(f, I(0, 1))-M(f, I(0, 3))| \\ &\leqq C\int_r^6 \rho(f, t)\frac{dt}{t}+C\int_1^6 \rho(f, t)\frac{dt}{t} \leqq 2C\|f\|_{BMO_{\phi}}\int_r^6 \phi(t)\frac{dt}{t} \\ &\leqq C'\|f\|_{BMO_{\phi}}\Phi_*(r). \end{aligned}$$

Summing up the above cases, we obtain

$$|M(f, I(a, r))| \leqq |M(f, I(0, 1))|+C\|f\|_{BMO_{\phi}}(\Phi_*(r)+\Phi^*(r)+\Phi^*(|a|))$$

$$\leqq C'\|f\|_{bmo_\phi}(\Phi_*(r)+\Phi^*(r)+\Phi^*(|a|)).$$

q. e. d.

REMARK 3.1. The estimate in Lemma 3.1 is sharp. In fact, consider the functions $\Phi^*(|x|)$ and $\Phi_*(|x-a|)$. Then $\|\Phi^*(|x|)\|_{bmo_\phi}$, $\|\Phi_*(|x-a|)\|_{bmo_\phi}<C$, independently of $a\in \boldsymbol{R}^n$, since $\Phi_*(r)\leqq\phi(2)\max(\log(2/r), \log 2)$. If $4|a|\leqq r$, using $\{x ; |x|\leqq r/4\}\subset I(a, r)$ we get

$$\begin{aligned} M(\Phi^*(|x|), I(a, r)) &\geqq |I(a, r)|^{-1}\int_{|x|\leqq r/4}\Phi^*(|x|)dx \\ &= C_1 r^{-n}\int_0^{r/4}\Phi^*(t)t^{n-1}dt \\ &\geqq C_2\Phi^*(r/8)\geqq C_3\Phi^*(r)\geqq C_3\Phi^*(|a|), \end{aligned}$$

by using Lemma 2.5 (i) and (iii). If $r<4|a|$, by Lemma 2.5 (i)

$$\begin{aligned} M(\Phi^*(|x|), I(a, r)) &\geqq C_1 M\left(\Phi^*(|x|), I\left(a, \frac{r}{4n}\right)\right)\geqq C_2\Phi^*(|a|) \\ &\geqq C_2\Phi^*\left(\frac{r}{4}\right)\geqq C_3\Phi^*(r), \end{aligned}$$

since $\Phi^*(|x|)\geqq C_4\Phi^*(|a|)$ on $I(a, r/(4n))$ by Lemma 2.5 (i). Next we consider $\Phi_*(|x-a|)$. Since $\Phi_*(r)$ is non-increasing and $\{x ; |x-a|<r/2\}\subset I(a, r)$, we have

$$\begin{aligned} M(\Phi_*(|x-a|), I(a, r)) &\geqq |I(a, r)|^{-1}\int_{|x-a|<r/2}\Phi_*(|x-a|)dx \\ &\geqq C\Phi_*\left(\frac{r}{2}\right)\geqq C'\Phi_*(r) \end{aligned}$$

by using Lemma 2.5 (ii).

LEMMA 3.2. *Suppose* $1\leqq p\leqq\infty$. *There is a constant* $C>0$ *such that*

$$|M(f, I(a, r))|\leqq C(\|f\|_{BMO_\phi}+\|f\|_{L^p})\Phi_*(r)$$

*for any* $f\in bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$, *and for any cube* $I(a, r)$.

PROOF. If $1\leqq r$, we have by Hölder's inequality

$$|M(f, I(a, r))|\leqq\left(|I|^{-1}\int_I|f(x)|^p dx\right)^{1/p}\leqq\|f\|_{L^p}\leqq C\Phi_*(r)\|f\|_{L^p}.$$

If $0<r<1$, since $I(a, r)\subset I(a, 1)$, by Lemma 2.3 we have

$$\begin{aligned} |M(f, I(a, r))| &\leqq |M(f, I(a, r))-M(f, I(a, 1))|+|M(f, I(a, 1))| \\ &\leqq C\int_r^2\rho(f, t)\frac{dt}{t}+|M(f, I(a, 1))| \\ &\leqq C\|f\|_{BMO_\phi}\Phi_*(r)+\|f\|_{L^p} \end{aligned}$$

$$\leqq C'(\|f\|_{BMO_\phi}+\|f\|_{L^p})\Phi_*(r).$$

q. e. d.

REMARK 3.2. Let $f(x)=\Phi_*(|x-a|)-\Phi_*(1)$. Then, since $\Phi_*(r)\leqq\phi(2)\times\max(\log 2/r, \log 2)$, $\|f\|_{L^p}\leqq C_p$ $(1\leqq p<\infty)$, $\|f\|_{BMO_\phi}\leqq C$, independently of $a$. As in Remark 3.1, we have $M(f, I(a, r))\geqq C\Phi_*(r)$ for $r\leqq 1$.

LEMMA 3.3. *Suppose* $f\in bmo_\phi(\boldsymbol{R}^n)$ *and* $g\in L^\infty(\boldsymbol{R}^n)$. *Then*, $fg$ *belongs to* $bmo_\phi(\boldsymbol{R}^n)$ *if and only if*

$$F(f, g)=\sup_{I(a,r)}|M(f, I(a, r))|MO(g, I(a, r))/\phi(r)<+\infty.$$

*In this case,*

$$F(f, g)\leqq\|fg\|_{BMO_\phi}+2\|g\|_\infty\|f\|_{BMO_\phi}.$$

PROOF. For any cube $I=I(a, r)$, by elementary calculation (see for example Stegenga [**5**, p. 582]), we have

$$|MO(fg, I)-|f_I|MO(g, I)|\leqq 2\|g\|_\infty MO(f, I),$$

and therefore

$$\left|\frac{MO(fg, I)}{\phi(r)}-\frac{|f_I|MO(g, I)}{\phi(r)}\right|\leqq 2\|g\|_\infty\|f\|_{BMO_\phi}.$$

This implies the assertion by the definition of $bmo_\phi(\boldsymbol{R}^n)$. q. e. d.

LEMMA 3.4. *Suppose* $1\leqq p\leqq\infty$. *If* $g$ *is a pointwise multiplier from* $bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$ *to* $bmo_\phi(\boldsymbol{R}^n)$, *then it follows that* $g\in L^\infty(\boldsymbol{R}^n)$.

PROOF. First of all, since $bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$ is a Banach space, equipped with the norm $\|f\|_{BMO_\phi}+\|f\|_{L^p}$, and $bmo_\phi(\boldsymbol{R}^n)$ is also a Banach space, we have by the closed graph theorem that

$$\|gf\|_{bmo_\phi}\leqq C(\|f\|_{BMO_\phi}+\|f\|_{L^p})$$

for all $f\in bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$.

For any cube $I=I(a, r)$ with $r<1$, we define a function $h\in bmo_\phi(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$ as follows:

$$h(x)=\begin{cases}0 & r\leqq|x-a|\\ \exp(i\Phi_*(|x-a|))-\exp(i\Phi_*(r)) & |x-a|<r.\end{cases}$$

Then, by Lemma 2.2, we get $\|h\|_{BMO_\phi}\leqq C_0\|\Phi_*(|x|)\|_{BMO_\phi}$. And, since supp $h\subset I(a, 2)$ and $|h(x)|\leqq 2$, we have $\|h\|_{L^p}\leqq C_p$. Hence $\|gh\|_{bmo_\phi}\leqq C(\|h\|_{BMO_\phi}+\|h\|_{L^p})\leqq C_1$, independently of $I$. This gives

$$MO(gh, I(a, 4r))\leqq C_1\phi(4r). \tag{3.1}$$

Let $C_2$ and $C_3$ be constants such that $\log C_2=\pi/\phi(1)$, $1<C_2C_3<C_2$, and let $L_r=$

$\{x\,;\,r/C_2 \leqq |x-a| \leqq r/(C_2C_3)\}$. If $x \in L_r$, then, since $\phi(r)/r$ is almost decreasing, we have

$$(\phi(r)/(AC_2C_3)) \log C_2C_3 \leqq \phi(r/(C_2C_3)) \log C_2C_3 \leqq \int_{r/(C_2C_3)}^{r} \phi(t)\frac{dt}{t}$$

$$\leqq \Phi_*(|x-a|) - \Phi_*(r) \leqq \int_{r/C_2}^{r} \phi(t)\frac{dt}{t} \leqq \phi(1) \log C_2 = \pi\,.$$

So, the inequality $|e^{i\theta}-1| \geqq 2\theta/\pi$ $(0 \leqq \theta \leqq \pi)$ implies that $|h(x)| \geqq C_4\phi(r)$ for $x \in L_r$. Let $\sigma = M(gh, I(a, 4r))$. Then we have, by considering the support of $h$,

$$MO(gh, I(a, 4r))|I(a, 4r)| = \int_{I(a,4r)} |gh(x)-\sigma|\,dx$$

$$\geqq \int_{L_r} |gh(x)-\sigma|\,dx + \int_{I(a,4r)\setminus I(a,2r)} |\sigma|\,dx$$

$$\geqq \int_{L_r} (|gh(x)-\sigma| + |\sigma|)dx \geqq \int_{L_r} |gh(x)|\,dx$$

$$\geqq C_4\phi(r) \int_{L_r} |g(x)|\,dx\,,$$

and so

$$|L_r|^{-1}\int_{L_r} |g(x)|\,dx \leqq C_5 MO(gh, I(a, 4r))/\phi(r) \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$|L_r|^{-1}\int_{L_r} |g(x)|\,dx \leqq C_6\,.$$

Letting $r$ tend to zero, we have

$$|g(a)| \leqq C_6 \qquad \text{a.e.}$$

q. e. d.

## 4. Proofs of the theorems.

PROOF OF THEOREM 1. Suppose that $g$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$. Then $g \in L^\infty$ by Lemma 3.4. Since $gf \in bmo_\phi(\boldsymbol{R}^n)$ for all $f \in bmo_\phi(\boldsymbol{R}^n)$, by Lemma 3.3 and the closed graph theorem we have

$$\sup_{I(a,r)} \frac{|f_I|\,MO(g, I)}{\phi(r)} < C\|f\|_{bmo_\phi}\,. \tag{4.1}$$

Hence, taking $f(x) = \Phi^*(|x|)$ or $\Phi_*(|x-a|)$, we have by Remark 3.1

$$\sup_{I(a,r)} (\Phi_*(r) + \Phi^*(r) + \Phi^*(|a|))MO(g, I)/\phi(r) < +\infty\,, \tag{4.2}$$

and hence

$$\sup_{I(a,r)} \frac{MO(g, I(a, r))}{w_\phi(a, r)} < +\infty ,$$

by using Remark 2.1. Consequently $g \in bmo_{w_\phi}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$.

Conversely, suppose $g \in bmo_{w_\phi}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$. For any $I = I(a, r)$ and any $f \in bmo_\phi(\boldsymbol{R}^n)$, by Lemma 3.1 we get

$$\frac{|f_I| MO(g, I)}{\phi(r)} \leqq C \|f\|_{bmo_\phi} (\Phi_*(r) + \Phi^*(r) + \Phi^*(|a|)) MO(g, I)/\phi(r)$$

$$\leqq C' \|f\|_{bmo_\phi} \frac{MO(g, I)}{w_\phi(a, r)} \leqq C' \|f\|_{bmo_\phi} \|g\|_{BMO_{w_\phi}} .$$

Therefore $fg \in bmo_\phi(\boldsymbol{R}^n)$ by Lemma 3.3, which shows that $g$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$. This proves Theorem 1.

PROOF OF THEOREM 2. (i) Case $1 \leqq p < \infty$. Suppose that $g$ is a pointwise multiplier from $bmo_\phi(\boldsymbol{R}^n) \cap L^p(\boldsymbol{R}^n)$ to $bmo_\phi(\boldsymbol{R}^n)$. Then $g$ is bounded by Lemma 3.4. Hence by Lemma 3.3

$$\sup_{I(a,r)} \frac{|f_I| MO(g, I)}{\phi(r)} \leqq C(\|f\|_{BMO_\phi} + \|f\|_{L^p}) .$$

Taking $f(x) = \Phi_*(|x-a|) - \Phi_*(1)$, we have by Remark 3.2

$$\sup_{r \leqq 1,\, a \in \boldsymbol{R}^n} \Phi_*(r) MO(g, I)/\phi(r) < +\infty .$$

According to $g \in L^\infty(\boldsymbol{R}^n)$, $MO(g, I) \leqq 2\|g\|_\infty$. Since $\Phi_*(r)$ is constant and $\phi(r) \geqq \phi(1)$ for $r \geqq 1$, we have

$$\sup_{r > 1,\, a \in \boldsymbol{R}^n} \Phi_*(r) MO(g, I)/\phi(r) < +\infty .$$

Thus $g \in bmo_\psi(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$. Sufficiency can be proved in the same way as in Theorem 1, using Lemma 3.2 in place of Lemma 3.1. (ii) Case $p = \infty$. (Necessity) Since $1 \in bmo_\phi(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$, $g$ must belong to $bmo_\phi(\boldsymbol{R}^n)$. By Lemma 3.4, $g$ is bounded. (Sufficiency) We have, for any cube $I$,

$$|f_I| MO(g, I)/\phi(r) \leqq \|f\|_\infty MO(g, I)/\phi(r) \leqq \|f\|_\infty \|g\|_{BMO_\phi} .$$

So, since $g$ is bounded, by Lemma 3.3 we have $gf \in bmo_\phi(\boldsymbol{R}^n)$. This completes the proof.

PROOF OF THEOREM 3. (Necessity) Clearly we have $bmo_\phi(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n) \subset UBM\text{-}BMO_\phi(\boldsymbol{R}^n)$. Hence by Theorem 2 we have the desired conclusion. (Sufficiency) For all $r \geqq 1$ and all $a \in \boldsymbol{R}^n$, we get

$$(4.3) \qquad |M(|f|, I(a, r))| \leqq 2^n [\sup_{b \in \boldsymbol{R}^n} |M(f, I(b, 1))| + \sup_{b \in \boldsymbol{R}^n} MO(f, I(b, 1))] .$$

(To show this, let $j$ be the smallest integer satisfying $r \leqq 2^j$ and take the cube

$I(a, 2^j)$, and then divide it into non-overlapping $2^{jn}$ cubes with side length 1. Then by the definition we get the above inequality.) Hence we get $\sup_{r\geq 1, a\in R^n}|M(f, I(a, r))|/\phi(r)\leqq C\|f\|_{UBM\text{-}BMO_\phi}$. As in Case 4 in Lemma 3.1 we get $|M(f, I(a, r))|\leqq C\|f\|_{UBM\text{-}BMO_\phi}\Phi_*(r)$. Therefore, since $g$ is bounded, we have $gf\in bmo_\phi(\boldsymbol{R}^n)$ by Lemma 3.3. q.e.d.

REMARK 4.1. By (4.3), one can easily show that $\|f\|_{UBM\text{-}BMO_\phi}$ is equivalent to

$$\sup_{0<r\leq 1, a\in R^n}|MO(f, I(a, r))|/\phi(r)+\sup_{r\geq 1, a\in R^n}|M(|f|, I(a, r))|/\phi(r). \tag{4.4}$$

For the case $\phi(t)\equiv 1$, Goldberg [**1**, Corollary 1] introduced $UBM\text{-}BMO_1(\boldsymbol{R}^n)$, using (4.4), by the symbol *bmo*, and showed that it is the dual of the local Hardy space $h^1(\boldsymbol{R}^n)$.

## 5. Some sufficient conditions and examples.

As consequences of our theorems, we give some sufficient conditions for pointwise multipliers, corresponding to those in the torus case, Stegenga [**5**, Corollary 2.8] and Janson [**2**, p. 196].

PROPOSITION 5.1. *Suppose $g$ satisfies the following conditions:*

(5.1) *There is a constant $M_1>0$ such that*

$$|g(x+y)-g(x)|\leqq M_1\phi(|y|)/[\Phi_*(|y|)+(1-\operatorname{sgn}\phi(0+))\Phi^*(|x|)]$$

*for all $x, y\in\boldsymbol{R}^n$ with $|y|\leqq 1$, where $\phi(0+)=\lim_{r\downarrow 0}\phi(r)$.*

(5.2) *There are constants $M_2>0$ and $B\in\boldsymbol{C}$ such that*

$$|g(x)-B|\leqq M_2/\Phi^*(|x|) \qquad \text{for all } x\in\boldsymbol{R}^n.$$

*Then, $g$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$.*

PROOF. We omit the detailed proof. One has only to treat the four cases; $\{r\leqq 1/\sqrt{n}, \phi(0+)=0\}$, $\{r\leqq 1/\sqrt{n}, \phi(0+)>0\}$, $\{1/\sqrt{n}<r\leqq |a|/\sqrt{n}\}$, and $\{r\geqq \max(1, |a|)/\sqrt{n}\}$.

As a consequence we have the following corollary, whose proof we omit.

COROLLARY 5.2. *If $g=g_1/g_2$ satisfies the following conditions:*

(5.3) *$g_1$ is bounded and there is a $C_1>0$ such that $|g_1(x)-g_1(y)|\leqq C_1|x-y|$, $x, y\in\boldsymbol{R}^n$;*

(5.4) *There are $C_2, C_3>0$ such that $|g_2(x)|\geqq C_2\Phi^*(|x|)$ and $|g_2(x)-g_2(y)|\leqq C_3|x-y|$, $x, y\in\boldsymbol{R}^n$.*

*Then, $g$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$.*

For pointwise multipliers from $bmo_\phi \cap L^p$ to $bmo_\phi$, we have:

PROPOSITION 5.3. *If g is bounded and satisfies*

$$|g(x+y)-g(x)| \leqq C\phi(|y|)/\Phi_*(|y|), \qquad x, y \in \boldsymbol{R}^n, \ |y|<1, \tag{5.5}$$

*then g is a pointwise multiplier from* $bmo_\phi(\boldsymbol{R}^n) \cap L^p(\boldsymbol{R}^n)$ *to* $bmo_\phi(\boldsymbol{R}^n)$, $(1 \leqq p \leqq \infty)$.

EXAMPLES. By Corollary 5.2

$$\frac{1}{\Phi^*(|x|)}, \quad \frac{\sin|x|}{\Phi^*(|x|)}, \quad \frac{1}{1+|x|}, \quad \frac{\sin \Phi^*(|x|)}{1+|x|}$$

are pointwise multipliers on $bmo_\phi(\boldsymbol{R}^n)$. And, for any $\phi$, for which $\phi(t)/(t\Phi_*(t))$ is almost decreasing, put $\Psi_*(r)=\int_r^2 \phi(t)/(t\Phi_*(t))\,dt$ for $0<r\leqq 1$ and $=\int_1^2 \phi(t)/(t\Phi_*(t))\,dt$ for $1<r$. Then, $\sin\Psi_*(|x|)/\Phi^*(|x|)$ is a pointwise multiplier on $bmo_\phi(\boldsymbol{R}^n)$. This gives a pointwise multiplier, which is not continuous, as in [**2**, p. 196].

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Eiichi NAKAI
Yuki Daiichi Senior High School
1076 Yuki, Yuki-shi
Ibaraki-ken 307
Japan

Kôzô YABUTA
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310
Japan