POINTWISE PROPERTIES OF FUNCTIONS OF BOUNDED VARIATION IN METRIC SPACES

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Abstract. We study pointwise properties of functions of bounded variation on a metric space equipped with a doubling measure and a Poincaré inequality. In particular, we obtain a Lebesgue type result for BV functions. We also study approximations by Lipschitz continuous functions and a version of the Leibniz rule. We give examples which show that our main result is optimal for BV functions in this generality.

1. Introduction

This paper studies Lebesgue points for functions of bounded variation on a metric measure space (X, d, μ) equipped with a doubling measure and supporting a Poincaré inequality. Here the metric d and the measure μ will be fixed, and we denote the triple (X, d, μ) simply by X. We say that $x \in X$ is a Lebesgue point of a locally integrable function u, if

$$\lim_{r\to 0} \fint_{B(x,r)} |u - u(x)|\, d\mu = 0.$$

Observe that if $x \in X$ is a Lebesgue point of u, then

$$\lim_{r\to 0} \fint_{B(x,r)} u\, d\mu = u(x).$$

By the Lebesgue differentiation theorem for doubling measures, almost every point with respect to the underlying measure is a Lebesgue point of a locally integrable function. In this work, we focus on pointwise properties of functions outside exceptional sets of codimension one. The set of non-Lebesgue points of a classical Sobolev function is a set of measure zero with respect the Hausdorff measure of codimension one and this holds true also in metric spaces supporting a doubling measure and a Poincaré inequality, see [11] and [12]. More precisely, the set of non-Lebesgue points is of zero capacity, but we do not need this refinement here.

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The situation is more delicate for BV functions. Indeed, easy examples show that already in the Euclidean case with Lebesgue measure the characteristic function of a set of finite perimeter may fail to be continuous on a set of positive Hausdorff measure of codimension one. However, it is known that in the Euclidean case with Lebesgue measure, a $BV(\mathbb{R}^n)$ function has the property

$$\lim_{r\to 0} \fint_{B(x,r)} u\,dy = \frac{1}{2}\big(u^\wedge(x) + u^\vee(x)\big) \tag{1.1}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \mathbb{R}^n$, see [2], [6, Corollary 1 of page 216], [7, Theorem 4.5.9] and [20, Theorem 5.14.4]. Here $u^\wedge(x)$ and $u^\vee(x)$ are the lower and upper approximate limits of u at x, see Definition 2.5. The proof of this result lies rather deep in the theory of BV functions and it seems to be very sensitive to the measure. Indeed, we give a simple example which shows that the corresponding result is not true even in the Euclidean case with a weighted measure. However, we are able to show the following metric space analogue of the result. The Hausdorff measure of codimension one is denoted by $\mathcal{H}$. The precise definitions will be given in Section 2.

Theorem 1.1. *Assume that μ is doubling and that X supports a weak $(1,1)$-Poincaré inequality. If $u \in BV(X)$, then for $\mathcal{H}$-almost every $x \in X$, we have*

$$\begin{aligned}(1-\gamma)u^\wedge(x) + \gamma u^\vee(x) &\le \liminf_{r\to 0} \fint_{B(x,r)} u\,d\mu \\ &\le \limsup_{r\to 0} \fint_{B(x,r)} u\,d\mu \le \gamma u^\wedge(x) + (1-\gamma)u^\vee(x),\end{aligned}$$

where $0 < \gamma \le \frac{1}{2}$ and γ depends only on the doubling constant and the constants in the in the weak $(1,1)$-Poincaré inequality.

We also give examples which show that, unlike in the classical Euclidean setting with the Lebesgue measure, in this generality we cannot hope to get $\gamma = \frac{1}{2}$ or the existence of the limit of the integral averages. In this sense, the above result seems to be the best we can have in the metric setting.

As an application of Theorem 1.1, we study approximations of BV functions. By definition, a BV function can be approximated by locally Lipschitz continuous functions in $L^1(X)$ so that the integral of upper gradients converges to the total variation measure. In some applications, a better control on pointwise convergence would be desirable. We construct two approximation procedures and apply one of the approximations in proving a version of the Leibniz rule for bounded BV functions. In the Euclidean case, the corresponding result has been studied in [5], [18] and [19]. An unexpected feature is that a multiplicative constant appears, which is not present for Sobolev functions.

2. Preliminaries

In this paper, (X, d, μ) is a complete metric measure space with a Borel regular outer measure μ. The measure is assumed to be doubling. This means that there exists a constant $c_D > 0$ such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. This implies that

$$\frac{\mu(B(y, R))}{\mu(B(x, r))} \leq c \left(\frac{R}{r}\right)^Q$$

for every $r \leq R$ and $y \in B(x, r)$ for some $Q > 1$ and $c \geq 1$ that only depend on c_D. We recall that complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact.

A nonnegative Borel function g on X is an upper gradient of an extended real valued function u on X if for all paths γ in X we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds, \tag{2.1}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. Here x and y are the end points of γ. If g is a nonnegative measurable function on X and if (2.1) holds for almost every path with respect to the 1-modulus, then g is a 1-weak upper gradient of u. By saying that (2.1) holds for 1-almost every path we mean that it fails only for a path family with zero 1-modulus. A family Γ of curves is of zero 1-modulus if there is a non-negative Borel measurable function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the path integral $\int_\gamma \rho\, ds$ is infinite, see [10].

The collection of all upper gradients, together, play the role of the modulus of the weak gradient of a Sobolev function in the metric setting. We consider the following norm

$$\|u\|_{N^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)}$$

with the infimum taken over all upper gradients g of u. The Newton-Sobolev space considered in this note is the space

$$N^{1,1}(X) = \{u : \|u\|_{N^{1,1}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,1}(X)} = 0.$$

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [14].

Definition 2.1. For $u \in L^1_{\text{loc}}(X)$, we define

$$\|Du\|(X) = \inf \left\{ \liminf_{k\to\infty} \int_X g_{u_k}\, d\mu : u_k \in \text{Lip}_{\text{loc}}(X), u_k \to u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where g_{u_k} is a 1-weak upper gradient of u_k. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in BV(X)$, if $\|Du\|(X) < \infty$. Moreover, a measurable set $E \subset X$ is said to have finite perimeter if $\|D\chi_E\|(X) < \infty$. By replacing X with an open set $U \subset X$, we may define $\|Du\|(U)$ and we denote the perimeter of E in U as

$$P(E,U) = \|D\chi_E\|(U).$$

By [14, Theorem 3.4], we have the following result.

Theorem 2.2. *Let $u \in BV(X)$. For an arbitrary set $A \subset X$, we define*

$$\|Du\|(A) = \inf\big\{\|Du\|(U) : U \supset A,\, U \subset X \text{ is open}\big\}.$$

Then $\|Du\|(\cdot)$ is a finite Borel outer measure.

Assume that E be a set of finite perimeter in X and let $A \subset X$ be an arbitrary set. As above, we denote

$$P(E,A) = \|D\chi_E\|(A).$$

We say that X supports a weak $(1,1)$-Poincaré inequality if there exist constants $c_P > 0$ and $\tau > 1$ such that for all balls $B = B(x,r)$, all locally integrable functions u and for all 1-weak upper gradients g of u, we have

$$\fint_B |u - u_B|\, d\mu \le c_P r \fint_{\tau B} g\, d\mu, \tag{2.2}$$

where

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu.$$

If the space supports a weak $(1,1)$-Poincaré inequality, then for every $u \in BV(X)$, we have

$$\fint_B |u - u_B|\, d\mu \le c_P r \frac{\|Du\|(\tau B)}{\mu(B)}, \tag{2.3}$$

where the constant c_P and the dilation factor τ are the same constants as in (2.2).

The $(1,1)$-Poincaré inequality implies the Sobolev-Poincaré inequality

$$\Big(\fint_B |u - u_B|^{Q/(Q-1)}\, d\mu\Big)^{(Q-1)/Q} \le cr \frac{\|Du\|(\tau B)}{\mu(B)}, \tag{2.4}$$

see [9, Theorem 9.7]. Here the constant $c > 0$ depends only on the doubling constant and the constants in the Poincaré inequality. We assume, without further notice, that the measure μ is doubling and that the space supports a weak $(1,1)$-Poincaré inequality. For brevity, the weak $(1,1)$-Poincaré inequality is called the Poincaré inequality later on.

The following coarea formula holds for BV functions. For a proof, see [14, Proposition 4.2]. If $u \in L^1_{\mathrm{loc}}(X)$ and $A \subset X$ is open, then

$$\|Du\|(A) = \int_{-\infty}^{\infty} P(\{u > t\}, A)\, dt. \tag{2.5}$$

In particular, if $u \in BV(X)$, then the set $\{u > t\}$ has finite perimeter for almost every $t \in \mathbb{R}$ and formula (2.5) holds for all Borel sets $A \subset X$.

The restricted spherical Hausdorff content of codimension one of a set $A \subset X$ is

$$\mathcal{H}_R(A) = \inf\Big\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R \Big\},$$

where $0 < R \le \infty$. When $R = \infty$, the infimum is taken over coverings with finite radii. The number $\mathcal{H}_\infty(A)$ is the Hausdorff content of A. The Hausdorff measure of codimension one is

$$\mathcal{H}(A) = \lim_{R \to 0} \mathcal{H}_R(A).$$

A combination of [3, Theorems 4.4 and 4.6] gives the equivalence of the perimeter measure, and the Hausdorff measure of codimension one, of measure theoretic boundaries of sets of finite perimeter. The measure theoretic boundary of a set $A \subset X$, denoted by $\partial^* A$, is the set of points $x \in X$, where both A and its complement have positive density, that is,

$$\limsup_{r \to 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} > 0.$$

The following result is extremely useful for us. For the proof, we refer to [1] and [3].

Theorem 2.3. *Assume that E is a set of finite perimeter and A is an arbitrary subset of X. Then*

$$\frac{1}{c} P(E, A) \le \mathcal{H}(\partial^* E \cap A) \le c P(E, A), \tag{2.6}$$

where $c \ge 1$ depends only on the doubling constant and the constants in the Poincaré inequality.

Moreover, the following theorem shows that we can consider even a smaller part of the measure theoretic boundary. For a proof, see [1, Theorem 5.3].

Theorem 2.4. *Let E be a set of finite perimeter. For $\gamma > 0$, we define $\Sigma_\gamma(E)$ to be the set consisting of all points $x \in X$ for which*

$$\liminf_{r \to 0} \min\Big\{ \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))}, \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \Big\} \ge \gamma.$$

Then there exists $\gamma > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that

$$\mathcal{H}(\partial^* E \setminus \Sigma_\gamma(E)) = 0.$$

Let $A \subset X$ be a Borel set. The upper density of A at a point $x \in X$ is

$$\overline{D}(A, x) = \limsup_{r \to 0} \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))}$$

and the lower density

$$\underline{D}(A, x) = \liminf_{r \to 0} \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))}.$$

If $\overline{D}(A, x) = \underline{D}(A, x)$ then the limit exists and we denote it by $D(A, x)$. By the differentiation theory of doubling measures, we have

$$D(A, x) = 1 \quad \text{for } \mu\text{-almost every } x \in A$$

and

$$D(A, x) = 0 \quad \text{for } \mu\text{-almost every } x \in X \setminus A.$$

Following the notation of [3], we define upper and lower approximate limits.

Definition 2.5. Let $u \colon X \to [-\infty, \infty]$ be a measurable function. The upper and lower approximate limit of u at $x \in X$ are

$$u^\vee(x) = \inf\big\{t : D(\{u > t\}, x) = 0\big\}$$

and

$$u^\wedge(x) = \sup\big\{t : D(\{u < t\}, x) = 0\big\}.$$

If $u^\vee(x) = u^\wedge(x)$, then the common value is denoted by $\widetilde{u}(x)$ and called the approximate limit of u at x. The function u is approximately continuous at x if $\widetilde{u}(x)$ exists and $\widetilde{u}(x) = u(x)$. The approximate jump set of u is

$$S_u = \{u^\wedge < u^\vee\}.$$

By the Lebesgue differentiation theorem, a locally integrable function u is approximately continuous μ-almost everywhere and hence $\mu(S_u) = 0$. Therefore we can define $\widetilde{u}$ as we like on the approximate jump set S_u. A similar argument as in the classical case of [20, Remark 5.9.2] shows that a function u is approximately continuous at x if and only if there exists a measurable set E such that $x \in E$, $D(E, x) = 1$ and the restriction of u to E is continuous.

We need the following standard measure theoretic lemma. We recall the proof here to emphasize the fact that it also applies in the metric context.

Lemma 2.6. *Assume that $u \in BV(X)$ and $\lambda > 0$. Let $A \subset X$ be a Borel set such that*

$$\limsup_{r\to 0} r\frac{\|Du\|(B(x,r))}{\mu(B(x,r))} > \lambda$$

for all $x \in A$. Then there is a constant $c > 0$, depending only on the doubling constant, such that

$$\|Du\|(A) \geq c\lambda\mathcal{H}(A).$$

Proof. Let $\varepsilon > 0$. Let U be an open set such that $A \subset U$. For each $x \in A$ there exists r_x with

$$0 < r_x \leq \frac{1}{5}\min\{\varepsilon, \operatorname{dist}(x, X\setminus U)\}$$

such that

$$\frac{\mu(B(x,r_x))}{r_x} < \frac{\|Du\|(B(x,r_x))}{\lambda}. \tag{2.7}$$

By a covering argument, there is a subfamily of disjoint balls $B_i = B(x_i, r_i) \subset U$, $i = 1, 2, \dots$, such that (2.7) holds for each B_i and $A \subset \cup_{i=1}^{\infty} 5B_i$. Using the doubling condition, (2.7) and the pairwise disjointedness of the balls B_i, we obtain

$$\begin{aligned}\mathcal{H}_\varepsilon(A) &\leq \sum_{i=1}^{\infty}\frac{\mu(5B_i)}{5r_i} \leq c\sum_{i=1}^{\infty}\frac{\mu(B_i)}{r_i}\\ &\leq \frac{c}{\lambda}\sum_{i=1}^{\infty}\|Du\|(B_i) \leq \frac{c}{\lambda}\|Du\|(U).\end{aligned}$$

Since

$$\|Du\|(A) = \inf\{\|Du\|(U) : U \text{ is open, } A \subset U\},$$

the claim follows by letting $\varepsilon \to 0$ and then taking the infimum over open sets U. □

3. Lebesgue points

In this section, we use the approximately continuous representative $\widetilde{u}$ of u, and denote it by u. Our first lemma shows that the Sobolev-Poincaré inequality holds infinitesimally without the integral average on the left-hand side if the approximate limit of the function is zero in the center of the ball.

Lemma 3.1. *Let $u \in BV(X)$ and let $x_0 \in X$. If the approximate limit $\widetilde{u}(x_0)$ exists and $\widetilde{u}(x_0) = 0$, then*

$$\limsup_{r\to 0}\left(\fint_{B(x_0,r)}|u|^{Q/(Q-1)}\,d\mu\right)^{(Q-1)/Q} \leq c\limsup_{r\to 0} r\frac{\|Du\|(B(x_0,r))}{\mu(B(x_0,r))}.$$

Here the constant $c > 0$ depends only on the doubling constant and the constants in the Poincaré inequality.

Proof. Let $0 < \varepsilon < 1$. Since 0 is the approximate limit of u at x_0, we have

$$D(\{|u| > t\}, x_0) = 0$$

for all $t > 0$. For a fixed $t > 0$, there exists $r_\varepsilon > 0$ such that

$$\mu(\{|u| > t\} \cap B(x_0, r)) < \varepsilon\mu(B(x_0, r))$$

whenever $0 < r \le r_\varepsilon$. This implies that for the sets $B = B(x_0, r)$ and $E = \{|u| \le t\} \cap B$, we have

$$\mu(E) \ge (1 - \varepsilon)\mu(B).$$

From this, together with the Sobolev-Poincaré inequality (2.4), we conclude that

$$\begin{aligned}
&\Big(\fint_B |u - u_E|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\quad\le 2\frac{\mu(B)}{\mu(E)}\Big(\fint_B |u - u_B|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\quad\le \frac{2cr}{1-\varepsilon}\frac{\|Du\|(\tau B)}{\mu(B)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Big(\fint_B |u|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} &\le \Big(\fint_B |u - u_E|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} + |u|_E \\
&\le \frac{2cr}{1-\varepsilon}\frac{\|Du\|(\tau B)}{\mu(B)} + t
\end{aligned}$$

for $0 < r \le r_\varepsilon$. The claim follows by taking the limes superior on both sides as $r \to 0$ and then letting $t \to 0$. □

Lemma 3.2. *Let $u \in BV(X)$. Then*

$$-\infty < u^\wedge(x) \le u^\vee(x) < \infty$$

for $\mathcal{H}$-almost every $x \in X$.

Proof. Since the question is of local nature, without loss of generality, we may assume that u has compact support. First we will show that

$$\mathcal{H}(\{u^\wedge = \infty\}) = 0.$$

For $t \in \mathbb{R}$, let

$$F_t = \{u^\wedge > t\} \quad \text{and} \quad E_t = \{u > t\}.$$

The definitions of $u^\wedge$ and F_t imply that $D(E_t, x) = 1$ for every $x \in F_t$. Since μ-almost every point is a Lebesgue point of u, we have $u^\wedge(x) = u(x)$ for μ-almost every x and therefore

$$D(F_t, x) = D(E_t, x) = 1$$

when x is a Lebesgue point of u. The boxing inequality ([12, Remark 3.3.(1)] and [16]) gives disjoint balls $B_i = B(x_i, \tau r_i)$, $i = 1, 2, \ldots$, such that $F_t \subset \cup_{i=1}^{\infty} 5B_i$ and

$$\sum_{i=1}^{\infty} \frac{\mu(5B_i)}{5\tau r_i} \le cP(F_t, X).$$

Since u has compact support, we have $r_i \le R$, $i = 1, 2, \ldots$, for $R > 0$ large enough.

This implies that

$$\mathcal{H}_R(F_t) \le cP(F_t, X)$$

and

$$\begin{aligned} \mathcal{H}_R(\{u^\wedge = \infty\}) = \mathcal{H}_R\Big(\bigcap_{t>0} F_t\Big) &\le \liminf_{t\to\infty} \mathcal{H}_R(F_t) \\ &\le \liminf_{t\to\infty} cP(F_t, X). \end{aligned} \tag{3.1}$$

As $u^\wedge = u$ μ-almost everywhere, we also have

$$P(E_t, X) = P(F_t, X)$$

and, by the coarea formula,

$$\int_{-\infty}^{\infty} P(F_t, X)\, dt = \int_{-\infty}^{\infty} P(E_t, X)\, dt = \|Du\|(X) < \infty.$$

From this we conclude that

$$\liminf_{t\to\infty} P(F_t, X) = 0.$$

By (3.1) we have $\mathcal{H}_R(\{u^\wedge = \infty\}) = 0$ and consequently also

$$\mathcal{H}(\{u^\wedge = \infty\}) = 0,$$

see for example the proof of [12, Lemma 7.6]. A similar argument shows that

$$\mathcal{H}(\{u^\vee = -\infty\}) = 0.$$

The first part of the proof shows that $u^\vee - u^\wedge$ is well defined $\mathcal{H}$-almost everywhere. Next we show that

$$\mathcal{H}(\{u^\vee - u^\wedge = \infty\}) = 0.$$

It follows from the definitions of the approximate limits and the measure theoretic boundary that

$$G_t = \{u^\wedge < t < u^\vee\} \subset \partial^* E_t$$

for every $t \in \mathbb{R}$. Then, by the Fubini theorem,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{H}(G_t)\, dt &= \int_{-\infty}^{\infty} \int_X \chi_{G_t}(x)\, d\mathcal{H}(x)\, dt \\
&= \int_X \int_{-\infty}^{\infty} \chi_{G_t}(x)\, dt\, d\mathcal{H}(x) \\
&= \int_X \int_{u^\wedge(x)}^{u^\vee(x)} 1\, dt\, d\mathcal{H}(x) \qquad (3.2)\\
&= \int_{S_u} (u^\vee(x) - u^\wedge(x))\, d\mathcal{H}(x).
\end{aligned}
$$

By (3.2), Theorem 2.3, and the coarea formula, we have

$$
\begin{aligned}
\int_{S_u} (u^\vee - u^\wedge)\, d\mathcal{H} &= \int_{-\infty}^{\infty} \mathcal{H}(\{u^\wedge < t < u^\vee\})\, dt \\
&\le \int_{-\infty}^{\infty} \mathcal{H}(\partial^* E_t)\, dt \\
&\le c \int_{-\infty}^{\infty} P(E_t, X)\, dt \\
&= c\|Du\|(X) < \infty.
\end{aligned}
$$

Since $u^\wedge = u^\vee$ outside S_u, the claim follows from this. □

The next example shows that (1.1) does not even hold for BV functions in weighted Euclidean spaces.

Example 3.3. Let $X = \mathbb{R}^2$ with the Euclidean distance and the measure with the derivative

$$D_{\mathcal{L}^2}\mu = \omega, \quad \text{where} \quad \omega = 1 + \chi_{B(0,1)}.$$

Let $u = \chi_{B(0,1)}$. Then for every $x \in \partial B(0,1)$, we have

$$\frac{1}{2}(u^\wedge(x) + u^\vee(x)) = \frac{1}{2} \neq \frac{2}{3} = \lim_{r\to 0} \fint_{B(x,r)} u\, d\mu.$$

The above example also shows that we cannot always take $\gamma = \frac{1}{2}$ in Theorem 1.1, for in this example we have $\gamma = \frac{1}{3}$.

We will start the proof of our main result by showing that the claim of Theorem 1.1 holds outside the approximate jump set if u is bounded. Later we remove the extra assumption on boundedness by a limiting argument.

Lemma 3.4. *Let $u \in BV(X) \cap L^\infty(X)$. Then*

$$\lim_{r\to 0} \fint_{B(x_0,r)} |u - u(x_0)|^{Q/(Q-1)}\, d\mu = 0.$$

for all $x_0 \in X \setminus S_u$.

Proof. Let $x_0 \in X \setminus S_u$. Since u is approximately continuous at x_0, there is a measurable set E such that $x_0 \in E$, $D(E, x_0) = 1$ and

$$\lim_{x \to x_0, x \in E} u(x) = u(x_0).$$

Let $r > 0$ and denote $B = B(x_0, r)$. Then we have

$$\begin{aligned}\fint_B |u - u(x_0)|^{Q/(Q-1)}\, d\mu = \frac{1}{\mu(B)} \int_{B \cap E} |u - u(x_0)|^{Q/(Q-1)}\, d\mu \\ + \frac{1}{\mu(B)} \int_{B \setminus E} |u - u(x_0)|^{Q/(Q-1)}\, d\mu.\end{aligned} \tag{3.3}$$

The second term on the right hand side of (3.3) has an upper bound

$$\frac{1}{\mu(B)} \int_{B \setminus E} |u - u(x_0)|^{Q/(Q-1)}\, d\mu \le 2 \frac{\mu(B \setminus E)}{\mu(B)} \|u\|_{L^\infty(X)}^{Q/(Q-1)},$$

which tends to zero as $r \to 0$ because $D(X \setminus E, x_0) = 0$.

Then we estimate the first term on the right hand side of (3.3). Let $0 < \varepsilon < 1$. There is $r_\varepsilon > 0$ such that $|u(x) - u(x_0)| < \varepsilon$ when $d(x, x_0) < r_\varepsilon$ and $x \in E$. Hence, we obtain

$$\frac{1}{\mu(B)} \int_{B \cap E} |u - u(x_0)|^{Q/(Q-1)}\, d\mu \le \varepsilon^{Q/(Q-1)} \frac{\mu(B \cap E)}{\mu(B)} \le \varepsilon^{Q/(Q-1)}$$

for $0 < r < r_\varepsilon$. The claim follows by letting $\varepsilon \to 0$. □

Now we generalize the previous lemma for unbounded BV functions. The proof is similar to the Euclidean argument of [20, Theorem 4.14.3].

Theorem 3.5. *Let $u \in BV(X)$. Then*

$$\lim_{r \to 0} \fint_{B(x_0, r)} |u - u(x_0)|^{Q/(Q-1)}\, d\mu = 0.$$

for $\mathcal{H}$-almost all $x_0 \in X \setminus S_u$.

Proof. Let $0 < \varepsilon < 1$ and denote

$$W_k = \{-k \le u^\wedge \le u^\vee \le k\}, \quad k = 1, 2, \dots$$

By Lemma 3.2, we have

$$\mathcal{H}\Big(X \setminus \bigcup_{k=1}^{\infty} W_k\Big) = 0,$$

and hence it is enough to prove the claim for $x_0 \in (\bigcup_{k=1}^\infty W_k) \setminus S_u$.

For $k = 1, 2, \dots$, let u_k be a truncation of u defined as

$$u_k(x) = \begin{cases} k, & \text{if } u(x) > k, \\ u(x), & \text{if } |u(x)| \le k, \\ -k, & \text{if } u(x) < -k. \end{cases}$$

By the Minkowski inequality, we have

$$\begin{aligned}
&\Big(\fint_{B(x_0,r)} |u-u(x_0)|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\qquad \le \Big(\fint_{B(x_0,r)} |u-u_k|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\qquad\quad + \Big(\fint_{B(x_0,r)} |u_k-u(x_0)|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q}.
\end{aligned} \tag{3.4}$$

Let k be large enough so that $x_0 \in W_k$. Then the approximate limit of $u-u_k$ at x_0 is zero. Therefore Lemma 3.1 implies that

$$\begin{aligned}
&\limsup_{r\to 0} \Big(\fint_{B(x_0,r)} |u-u_k|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\qquad \le c \limsup_{r\to 0} r \frac{\|D(u-u_k)\|(B(x_0,\tau r))}{\mu(B(x_0,r))}.
\end{aligned} \tag{3.5}$$

Let

$$Z_k = \Big\{ x\in X : \limsup_{r\to 0} r \frac{\|D(u-u_k)\|(B(x,\tau r))}{\mu(B(x,r))} \le \varepsilon \Big\}.$$

We begin by estimating the first term on the right hand side of (3.4) in the case that $x_0 \in Z$, where $Z = \cup_{k=1}^{\infty} Z_k$. Observe that $Z_k \subset Z_{k+1}$, which follows from the coarea formula using a similar argument as in the end of this proof. Then the definition of Z_k together with (3.5) shows that for k large enough, we have

$$\limsup_{r\to 0} \Big(\fint_{B(x_0,r)} |u-u_k|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \le c\varepsilon.$$

For the second term on the right hand side of (3.4), we have

$$\begin{aligned}
&\Big(\fint_{B(x_0,r)} |u_k-u(x_0)|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} \\
&\quad \le \Big(\fint_{B(x_0,r)} |u_k-u_k(x_0)|^{Q/(Q-1)}\,d\mu\Big)^{(Q-1)/Q} + |u_k(x_0)-u(x_0)|.
\end{aligned}$$

Since $k \ge |u(x_0)|$, we have $|u_k(x_0)-u(x_0)| = 0$. On the other hand,

$$\lim_{r\to 0} \fint_{B(x_0,r)} |u_k-u_k(x_0)|^{Q/(Q-1)}\,d\mu = 0$$

by Lemma 3.4 because u_k is bounded and $\widetilde{u}_k(x_0) = u_k(x_0)$.

Finally, we will show that $\mathcal{H}(X\setminus Z) = 0$. By Lemma 2.6,

$$\mathcal{H}(X\setminus Z_k) \le \frac{c}{\varepsilon}\|D(u-u_k)\|(X). \tag{3.6}$$

For $t<0$ we have $u-u_k>t$ if and only if $u>t-k$, and for $t>0$ we have $u-u_k>t$ if and only if $u>t+k$. Using the coarea formula

(2.5) and the fact that a complement of a set has the same perimeter as the set itself, we obtain

$$
\begin{aligned}
\|D(u-u_k)\|(X) &= \int_{-\infty}^{\infty} P(\{u-u_k>t\},X)\,dt \\
&= \int_{-\infty}^{0} P(\{u-u_k\le t\},X)\,dt + \int_{0}^{\infty} P(\{u-u_k>t\},X)\,dt \\
&= \int_{-\infty}^{0} P(\{u\le -k+t\},X)\,dt + \int_{0}^{\infty} P(\{u>k+t\},X)\,dt \\
&= \int_{\{|t|>k\}} P(\{u>t\},X)\,dt.
\end{aligned}
\tag{3.7}
$$

Since $u\in BV(X)$, the previous estimate implies that

$$\|D(u-u_k)\|(X)\to 0$$

as $k\to\infty$. Consequently, estimate (3.6) shows that

$$\mathcal{H}(X\setminus Z)\le \mathcal{H}(X\setminus Z_k)\to 0$$

as $k\to\infty$, and the result follows. □

Now we are ready to complete the proof of our main result.

Proof of Theorem 1.1. As before, denote $E_t=\{u>t\}$. Let

$$N=\bigcup_{t\in T}\big(\partial^* E_t\setminus \Sigma_\gamma(E_t)\big),$$

where Σ_γ and γ are as in Theorem 2.4 and T is a countable dense subset of $\mathbb{R}$ so that the set E_t is of finite perimeter when $t\in T$. Theorem 2.4 then implies that $\mathcal{H}(N)=0$.

Fix $x_0\in S_u\setminus N$ so that $-\infty<u^\wedge(x_0)\le u^\vee(x_0)<\infty$, see Lemma 3.2. We set

$$u^*=\frac{u-u^\wedge(x_0)}{u^\vee(x_0)-u^\wedge(x_0)}$$

and

$$v=(u^*-1)_+-u^*_-.$$

Then $x_0\in X\setminus S_v$ and $v(x_0)=0$.

Now take $t\in(0,1)$ such that $t^*=(u^\vee(x_0)-u^\wedge(x_0))t+u^\wedge(x_0)\in T$. By the definition of N, we have $x_0\in\Sigma_\gamma(E_{t^*})$ for each such t. This implies that

$$\liminf_{r\to 0} \fint_{B(x_0,r)} \chi_{E_{t^*}}\,d\mu\ge\gamma.$$

As $u^*\ge v+t\chi_{E_{t^*}}$, we have

$$\liminf_{r\to 0}\fint_{B(x_0,r)} u^*\,d\mu\ge \liminf_{r\to 0}\fint_{B(x_0,r)} v\,d\mu + t\liminf_{r\to 0}\fint_{B(x_0,r)}\chi_{E_{t^*}}\,d\mu\ge t\gamma.$$

Here we used Theorem 3.5 for v. By letting $t \to 1$ we arrive at

$$\begin{aligned}\gamma &\le \liminf_{r\to 0} \fint_{B(x_0,r)} u^*\,d\mu \\ &\le \frac{1}{u^\vee(x_0)-u^\wedge(x_0)}\Big(\liminf_{r\to 0}\fint_{B(x_0,r)} u\,d\mu - u^\wedge(x_0)\Big)\end{aligned}$$

and consequently

$$(1-\gamma)u^\wedge(x_0) + \gamma u^\vee(x_0) \le \liminf_{r\to 0}\fint_{B(x_0,r)} u\,d\mu.$$

The upper bound follows from a similar argument. Outside of the approximate jump set, the result follows from Theorem 3.5. □

The next example shows that, unlike in the classical Euclidean setting, we cannot hope to get the existence of the limit of the integral averages $\mathcal{H}$-almost everywhere in X in Theorem 1.1.

Example 3.6. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric and the measure with the derivative $D_{\mathcal{L}^2}\mu = \omega$ with the weight ω constructed as follows. For $k = 1, 2, \dots$ let

$$A_k = \{(x_1, x_2) \in \mathbb{R}^2 : 4^{-k} < |x_2| \le 4^{1-k}\},$$

and define

$$\omega = 1 + \chi_{\{x_2<0\}} \sum_{k \text{ even}} \chi_{A_k} + \chi_{\{x_2>0\}} \sum_{k \text{ odd}} \chi_{A_k}.$$

Then X is a metric measure space with a doubling measure, which is even Ahlfors 2-regular, and supports a $(1,1)$-Poincaré inequality. However, the characteristic function u of the set

$$\{(x_1,x_2) \in \mathbb{R}^2 : |x_1| \le 1, -1 \le x_2 \le 0\}$$

is in $BV(X)$, and for every $x = (x_1, 0)$ with $|x_1| < 1$, we have

$$\limsup_{r\to 0}\fint_{B(x,r)} u\,d\mu = \frac{1}{2} + \alpha$$

and

$$\liminf_{r\to 0}\fint_{B(x,r)} u\,d\mu = \frac{1}{2} - \alpha,$$

for some constant $0 < \alpha < \frac{1}{2}$.

Taking into account the pointwise behavior of BV functions, it is natural to ask what type of behavior a BV function has on the set where its total variation measure is absolutely continuous with respect to the underlying measure μ. In particular, the following lemma tells us that the total variation measure behaves like a weak derivative on the subset where the function vanishes. In the Euclidean setting the lemma follows from [6, page 232, Theorem 3 and page 233, Theorem 4 remark (i)]. The proof we give here is more direct.

Lemma 3.7. *Let $u \in BV(X)$ and $E \subset X$ a Borel set such that $\|Du\|$ is absolutely continuous with respect to μ on E and $u = 0$ on E. Then $\|Du\|(E) = 0$.*

Proof. Without loss of generality we may assume that $u \geq 0$ on X. Since $u = 0$ on E, we see that $\|Du\|(\mathrm{int}(E)) = 0$. Consequently, if ∂E has μ-measure zero, the conclusion follows. Hence we may assume that $\mu(\partial E) > 0$. Let E_0 be the collection of points $x \in E$ for which

$$\liminf_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0.$$

Note that by the Lebesgue differentiation theorem we have $\mu(E \setminus E_0) = 0$ and hence $\|Du\|(E \setminus E_0) = 0$. Therefore it suffices to prove that $\|Du\|(E_0) = 0$. In order to see this, we apply the coarea formula.

Denote $E_t = \{u > t\}$. Let $t > 0$ be such that E_t has finite perimeter in X. This is possible by the coarea formula. If $\mathcal{H}(E_0 \cap \partial^* E_t) > 0$, then because $E_0 \cap \partial^* E_t \subset S_u$, the measure $\|Du\|$ cannot be absolutely continuous with respect to μ on E_0 and hence on E, see for example [3, Theorem 5.3 and Theorem 4.4]. Therefore for all such $t > 0$ we have $\mathcal{H}(E_0 \cap \partial^* E_t) = 0$. Finally, by the coarea formula again,

$$0 \leq \|Du\|(E_0) = \int_0^\infty P(E_t, E_0)\, dt \leq c \int_0^\infty \mathcal{H}(E_0 \cap \partial^* E_t)\, dt = 0. \quad \square$$

It would be interesting to know the answer to the following question: If E is a Borel set on which $\|Du\|$ is absolutely continuous with respect to μ, is it then true that $u|_E \in N^{1,1}(E)$ with the weak upper gradient

$$g = \lim_{r\to 0} \frac{\|Du\|(B(x,r))}{\mu(B(x,r))}?$$

4. Approximations for BV functions and the Leibniz rule

As an application of Theorem 1.1, we study approximations of BV functions. By definition, for every $u \in BV(X)$ there is a sequence of locally Lipschitz continuous functions u_k, $k = 1, 2, \ldots$, which converges to u in $L^1(X)$ and the sequence $\|Du_k\|$, $k = 1, 2, \ldots$, converges weakly to $\|Du\|$ as $k \to \infty$. The problem with this sequence is that we do not have control over how well it converges to u pointwise. In this section we give two different constructions of approximating sequences that converge to $u \in BV(X)$; unfortunately, we lose the precise control over the weak convergence of the measures. However, in the event that the goal is to study capacities of sets, the needed control is over pointwise convergence, and in this case either of these two propositions would be useful. We continue using the approximately continuous representative $\widetilde{u}$ of u, and we denote it by u.

Proposition 4.1. *Let $u \in BV(X)$. Then there is a sequence of locally Lipschitz functions u_k, $k = 1, 2, \dots$, such that $u_k \to u$ $\mathcal{H}$-almost everywhere in $X \setminus S_u$, $u_k \to u$ in $L^1(X)$ as $k \to \infty$, and*

$$\limsup_{k\to\infty} \|Du_k\|(X) \le c\|Du\|(X).$$

Moreover, we have

$$\begin{aligned}(1-\widetilde{\gamma})u^\wedge(x) + \widetilde{\gamma}u^\vee(x) &\le \liminf_{k\to\infty} u_k(x) \\ &\le \limsup_{k\to\infty} u_k(x) \le \widetilde{\gamma}u^\wedge(x) + (1-\widetilde{\gamma})u^\vee(x)\end{aligned}$$

for $\mathcal{H}$-almost every $x \in X$. Here $\widetilde{\gamma} = \gamma/c$, where $0 < \gamma \le \frac{1}{2}$ is as in Theorem 2.4, and $c > 1$ depends only on the doubling constant and the constants in the Poincaré inequality.

Proof. Let $\varepsilon > 0$. Because μ is doubling, we can cover X by a countable collection B_i, $i = 1, 2, \dots$, of balls $B_i = B(x_i, \varepsilon)$ such that

$$\sum_{i=1}^\infty \chi_{20\tau B_i} \le c,$$

where c depends solely on the doubling constant and τ is the constant in the Poincaré inequality. Subordinate to this cover, there is a partition of unity φ_i with $0 \le \varphi_i \le 1$, $\sum_{i=1}^\infty \varphi_i = 1$, φ_i is c/ε-Lipschitz $\varphi_i \ge \alpha > 0$ in B_i, and $\operatorname{supp}(\varphi_i) \subset 2B_i$ for every $i = 1, 2, \dots$. We set

$$u_\varepsilon = \sum_{i=1}^\infty u_{5B_i}\varphi_i.$$

The function u_ε is sometimes called the discrete convolution of u. For $x \in B_j$, we note that

$$|u_\varepsilon(x) - u(x)| \le \sum_{i=1}^\infty |u_{5B_i} - u(x)|\varphi_i(x) \le \sum_{\{i:2B_i\cap B_j\ne\emptyset\}} |u(x) - u_{5B_i}|.$$

Hence

$$\begin{aligned}\int_{B_j} |u_\varepsilon - u|\,d\mu &\le \sum_{\{i:2B_i\cap B_j\ne\emptyset\}} \int_{B_j} |u - u_{5B_i}|\,d\mu \\ &\le c \sum_{\{i:2B_i\cap B_j\ne\emptyset\}} \int_{10B_j} |u - u_{10B_j}|\,d\mu \\ &\le c\int_{10B_j} |u - u_{10B_j}|\,d\mu \le c\varepsilon\|Du\|(10\tau B_j),\end{aligned}$$

where we used the bounded overlap property of the collection $20\tau B_i$, $i = 1, 2, \ldots$ and the Poincaré inequality. This implies that

$$\begin{aligned}\int_X |u_\varepsilon - u|\,d\mu &\le \sum_{j=1}^{\infty} \int_{B_j} |u_\varepsilon - u|\,d\mu \\ &\le c\varepsilon \sum_{j=1}^{\infty} \|Du\|(10\tau B_j) \le c\varepsilon \|Du\|(X).\end{aligned}$$

Thus $u_\varepsilon \to u$ in $L^1(X)$ as $\varepsilon \to 0$. The fact that $u_\varepsilon \to u$ $\mathcal{H}$-almost everywhere in $X \setminus S_u$ as $\varepsilon \to \infty$ follows from Theorem 3.5.

For $x, y \in B_j$ we have

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(y)| &\le \sum_{i=1}^{\infty} |u_{5B_i} - u_{5B_j}|\,|\varphi_i(x) - \varphi_i(y)| \\ &\le \frac{c}{\varepsilon} d(x, y) \sum_{\{i:2B_i \cap B_j \neq \emptyset\}} |u_{5B_i} - u_{5B_j}| \\ &\le \frac{c}{\varepsilon} d(x, y) \fint_{10B_j} |u - u_{10B_j}|\,d\mu \\ &= c d(x, y) \frac{\|Du\|(10\tau B_j)}{\mu(B_j)},\end{aligned}$$

and so u_ε is locally Lipschitz continuous with the upper gradient

$$g_\varepsilon = c \sum_{j=1}^{\infty} \frac{\|Du\|(10\tau B_j)}{\mu(B_j)} \chi_{B_j}. \tag{4.1}$$

Hence by the bounded overlap property of the covering and by the fact that if B_j intersects B_i then $\mu(B_j) \approx \mu(B_i)$, we have

$$\begin{aligned}\|Du_\varepsilon\|(X) &\le \int_X g_\varepsilon\,d\mu \le \sum_{i=1}^{\infty} \int_{B_i} g_\varepsilon\,d\mu \\ &\le c \sum_{i=1}^{\infty} \sum_{\{j:B_j \cap B_i \neq \emptyset\}} \|Du\|(10\tau B_j) \\ &\le c \sum_{i=1}^{\infty} \|Du\|(20\tau B_i) \le c\|Du\|(X).\end{aligned}$$

Finally, we show estimates for the pointwise limes superior and limes inferior. The idea in the proof is similar to the proof of Theorem 1.1. As in the proof of Theorem 1.1, denote $E_t = \{u > t\}$ and let

$$N = \bigcup_{t \in T} \big(\partial^* E_t \setminus \Sigma_\gamma(E_t)\big),$$

where Σ_γ and γ are as in Theorem 2.4 and T is a countable dense subset of $\mathbb{R}$ so that the set E_t is of finite perimeter when $t \in T$. Fix

$x_0 \in S_u \setminus N$ so that $-\infty < u^\wedge(x_0) \le u^\vee(x_0) < \infty$, see Lemma 3.2. We set

$$u^* = \frac{u - u^\wedge(x_0)}{u^\vee(x_0) - u^\wedge(x_0)}$$

and

$$v = (u^* - 1)_+ - u^*_-.$$

Then $x_0 \in X \setminus S_v$ and $v(x_0) = 0$.

Now take $t \in (0,1)$ such that $t^* = (u^\vee(x_0) - u^\wedge(x_0))t + u^\wedge(x_0) \in T$. We notice that we can write the function v_ε as

$$\begin{aligned} v_\varepsilon(x) &= \sum_{i=1}^\infty v_{5B_i}\varphi_i(x) \\ &= \int_X v \sum_{i=1}^\infty \frac{\chi_{5B_i}\varphi_i(x)}{\mu(5B_i)}\, d\mu = \int_X v a_x^\varepsilon \, d\mu, \end{aligned}$$

where the function a_x^ε is defined as

$$a_x^\varepsilon(y) = \sum_{i=1}^\infty \frac{\chi_{5B_i}(y)\varphi_i(x)}{\mu(5B_i)}$$

and satisfies

$$\frac{\chi_{B(x,\varepsilon)}}{c\mu(B(x,\varepsilon))} \le a_x^\varepsilon \le \frac{c\chi_{B(x,7\varepsilon)}}{\mu(B(x,\varepsilon))}.$$

Indeed, the left-hand side inequality holds because there exists i_0 such that $x \in B_{i_0}$ and by the construction φ_{i_0} we have $\varphi_{i_0} \ge \alpha$ for some $\alpha > 0$.

By the definition of N, we have $x_0 \in \Sigma_\gamma(E_{t^*})$ for each such t. This implies

$$\liminf_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} \chi_{E_{t^*}}\, d\mu \ge \gamma$$

and consequently

$$\liminf_{\varepsilon\to 0} \int_{B(x_0,7\varepsilon)} \chi_{E_{t^*}} a_{x_0}^\varepsilon d\mu \ge \frac{\gamma}{c}.$$

As $v \ge t\chi_{E_{t^*}}$, we have

$$\liminf_{\varepsilon\to 0} v_\varepsilon(x_0) \ge t \liminf_{\varepsilon\to 0} \int_X \chi_{E_{t^*}} a_{x_0}^\varepsilon d\mu \ge t\frac{\gamma}{c}.$$

The claim follows by letting $t \to 1$. By symmetry, we also have the upper bound. From this we conclude that

$$\frac{\gamma}{c} \le \liminf_{\varepsilon\to 0} v_\varepsilon(x_0) \le \liminf_{\varepsilon\to 0} \frac{u_\varepsilon(x_0) - u^\wedge(x_0)}{u^\vee(x_0) - u^\wedge(x_0)}$$

and consequently

$$\frac{\gamma}{c}(u^\vee(x_0) - u^\wedge(x_0)) \le \liminf_{\varepsilon\to 0} u_\varepsilon(x_0) - u^\wedge(x_0).$$

This implies that

$$\Big(1-\frac{\gamma}{c}\Big)u^\wedge(x_0)+\frac{\gamma}{c}u^\vee(x_0)\le\liminf_{\varepsilon\to 0}u_\varepsilon(x_0).$$

The upper bound follows from a similar argument. □

As a consequence of the previous approximation result we obtain a version of Leibniz rule for nonnegative BV functions. Observe, that for the upper gradients of bounded u and v belonging to $N^{1,1}(X)\cap L^\infty(X)$, we have

$$g_{uv}\le ug_v+vg_u,$$

and consequently,

$$d\|D(uv)\|\le u\,d\|Dv\|+v\,d\|Du\|,$$

but for BV functions a multiplicative constant appears.

Proposition 4.2. *Let $u,v\in BV(X)\cap L^\infty(X)$ be nonnegative functions. Then $uv\in BV(X)\cap L^\infty(X)$ and there is a constant $c\ge 1$ such that*

$$d\|D(uv)\|\le cu^\vee d\|Dv\|+cv^\vee d\|Du\|.$$

The constant c depends only on the doubling constant and the constants in the Poincaré inequality.

Proof. For $\varepsilon>0$ let v_ε be the approximation of v as in Proposition 4.1 and let g_ε be an upper gradient of v_ε as in (4.1). Moreover, let u_k, $k=1,2,\dots$, be an approximation of u in the sense that u_k are nonnegative Lipschitz functions uniformly bounded by $\|u\|_{L^\infty(X)}$, $u_k\to u$ in $L^1(X)$ and $\|Du_k\|$ converges weakly to $\|Du\|$ as $k\to\infty$. Now, it is obvious that u_kv_ε forms an approximation of uv in $L^1(X)$ and, consequently, we have

$$\|D(uv)\|\le\liminf_{\varepsilon\to 0}\liminf_{k\to\infty}\|D(u_kv_\varepsilon)\|.$$

By the Leibniz rule for functions in $N^{1,1}_{\mathrm{loc}}(X)$, we have

$$d\|D(u_kv_\varepsilon)\|\le v_\varepsilon d\|Du_k\|+u_kg_\varepsilon\,d\mu.$$

Let us first consider the term $v_\varepsilon\,d\|Du_k\|$. Since v_ε is continuous on X and $\|Du_k\|$ converge weakly to $\|Du\|$, we conclude that the measures with $v_\varepsilon d\|Du_k\|$ converge weakly to the measure with the derivative $v_\varepsilon d\|Du\|$ as $k\to\infty$. Passing $\varepsilon\to 0$, by Proposition 4.1 we have

$$\limsup_{\varepsilon\to 0}v_\varepsilon\le cv^\vee$$

$\mathcal{H}$-almost everywhere in X and hence $\|Du\|$-almost everywhere in X. If ϕ is a compactly supported Lipschitz function on X, then by the dominated convergence theorem we have

$$\liminf_{\varepsilon\to 0}\int_X\phi v_\varepsilon\,d\|Du\|\le c\int_X\phi v^\vee\,d\|Du\|,$$

that is, a weak limit of the sequence of measures with the derivatives $v_\varepsilon d\|Du\|$ is dominated by the measure with $cv^\vee d\|Du\|$.

Next we consider the term $u_k g_\varepsilon$ by first letting $k \to \infty$ and then $\varepsilon \to 0$. Since $u_k \to u$ μ-almost everywhere in X and in $L^1(X)$, we see that

$$\lim_{k\to\infty} \int_X \phi u_k g_\varepsilon \, d\mu = \int_X \phi u g_\varepsilon \, d\mu$$

whenever ϕ is a compactly supported Lipschitz function in X. By the definition of g_ε (see (4.1)) and using the covering B_i, $i = 1, 2, \dots$, as in the proof of Proposition 4.1, we see that

$$\begin{aligned}
\int_X \phi u \, g_\varepsilon \, d\mu &\le c \sum_{i=1}^\infty \int_{B_i} \phi u g_\varepsilon \, d\mu \\
&\le c \sum_{i=1}^\infty \sum_{\{j: B_j \cap B_i \neq \emptyset\}} \fint_{2B_i} \phi u \, d\mu \, \|Du\|(10\tau B_j) \\
&\le c \sum_{i=1}^\infty \fint_{20\tau B_i} \phi u \, d\mu \, \|Dv\|(20\tau B_i)
\end{aligned}$$

for nonnegative ϕ. Let ψ_ε be a function defined by

$$\psi_\varepsilon(x) = \sum_{i=1}^\infty \fint_{B(x,40\tau\varepsilon)} \phi u \, d\mu \, \chi_{10\tau B_i}(x).$$

By the bounded overlap of the collection $10\tau B_i$, $i = 1, 2, \dots$, we have

$$\int_X \phi u g_\varepsilon \, d\mu \le c \int_X \psi_\varepsilon \, d\|Dv\|.$$

As ϕ is continuous, Proposition 4.1 shows that

$$\limsup_{\varepsilon\to 0} \psi_\varepsilon \le c \, \phi u^\vee$$

$\mathcal{H}$-almost everywhere in X and hence $\|Dv\|$-almost everywhere. Consequently, we arrive at

$$\limsup_{\varepsilon\to 0} \lim_{k\to\infty} \int_X \phi u g_\varepsilon \, d\mu \le c \int_X \phi u^\vee \, d\|Dv\|$$

whenever ϕ is a nonnegative compactly supported Lipschitz continuous function in X. If ν is a Radon measure that is obtained as a weak limit of a subsequence of the measures with $u_k g_\varepsilon \, d\mu$, then

$$\begin{aligned}
\nu(X) &= \sup \int_X \phi \, d\nu \\
&= \sup \limsup_{\varepsilon\to 0} \lim_{k\to\infty} \int_X \phi u g_\varepsilon \, d\mu \\
&\le c \sup \int_X \phi u^\vee \, d\|Dv\|,
\end{aligned}$$

where the supremums are taken over all functions $\phi \in \mathrm{Lip}_c(X)$ with $0 \le \phi \le 1$. The above computation can be performed by restricting to functions ϕ with support in a given open set, and so we can conclude that

$$\nu(U) \le c \int_U u^\vee \, d\|Dv\|$$

for all open subsets U of X, and it follows that

$$d\nu \le c u^\vee d\|Dv\|.$$

□

Next we show that Lipschitz functions are dense in $BV(X)$ in the Lusin sense. For the Euclidean case, see [6, page 252, Theorem 2] and for Sobolev functions in metric spaces, we refer to [8] and [12].

Proposition 4.3. *Let $u \in BV(X)$. The for every $\varepsilon > 0$, there exists a Lipschitz function v in X such that*

$$\mu\big(\{x \in X : u(x) \ne v(x)\}\big) < \varepsilon \quad \text{and} \quad \|u - v\|_{L^1(X)} < \varepsilon.$$

In addition, we have

$$\|Dv\|(X) \le c \, \|Du\|(X)$$

for some constant $c > 0$ depending only on the doubling constant and the constants in the Poincaré inequality.

Proof. Let $\lambda > 0$ and define

$$E_\lambda = \Big\{ x \in X : \frac{\|Du\|(B)}{\mu(B)} \le \lambda \text{ for all balls } B \ni x \Big\}.$$

Note that $X \setminus E_\lambda$ is an open set and so E_λ is a Borel set. We begin by showing that there is a constant $c > 0$ such that

$$\mu(X \setminus E_\lambda) \le \frac{c}{\lambda} \|Du\|(X \setminus E_\lambda). \tag{4.2}$$

Let $B_i = B(x_i, r_i) \subset X \setminus E_\lambda$, $i = 1, 2, \dots$, be disjoint balls such that

$$X \setminus E_\lambda \subset \bigcup_{i=1}^{\infty} 5B_i \quad \text{and} \quad \|Du\|(B_i) > \lambda \mu(B_i)$$

for all $i = 1, 2, \dots$. Since μ is doubling and $\|Du\|$ is a measure, we obtain

$$\mu(X \setminus E_\lambda) \le \sum_{i=1}^{\infty} \mu(5B_i) \le \frac{c}{\lambda} \sum_{i=1}^{\infty} \|Du\|(B_i) \le \frac{c}{\lambda} \|Du\|(X \setminus E_\lambda).$$

Hence (4.2) follows.

Next we show that there is a constant $c > 0$ that depends only on the doubling constant and the constants of the Poincaré inequality, such that

$$|u(x) - u(y)| \le c \lambda d(x, y) \tag{4.3}$$

for almost every $x, y \in E_\lambda$. Let $x, y \in E_\lambda$ be Lebesgue points of u. Let $r = d(x, y)$, $B_x = B(x, r)$ and $B_y = B(y, r)$. Then

$$|u(x) - u(y)| \le |u(x) - u_{B_x}| + |u_{B_x} - u_{B_y}| + |u(y) - u_{B_y}|,$$

where, by a standard telescoping argument, using the doubling property and the Poincaré inequality, we conclude that

$$\begin{aligned}
|u(x) - u_{B_x}| &\le \sum_{i=0}^{\infty} |u_{2^{-i}B_x}(x) - u_{2^{-(i+1)}B_x}| \\
&\le c \sum_{i=0}^{\infty} \fint_{2^{-i}B_x} |u - u_{2^{-i}B_x}| \, d\mu \\
&\le cr \sum_{i=0}^{\infty} 2^{-i} \frac{\|Du\|(\tau 2^{-i} B_x)}{\mu(2^{-i}B_x)} \\
&\le cr\lambda \sum_{i=0}^{\infty} 2^{-i} = cr\lambda.
\end{aligned}$$

Similar estimate holds for $|u(y) - u_{B_y}|$. By the doubling property and the Poincaré inequality, we obtain

$$\begin{aligned}
|u_{B_x} - u_{B_y}| &\le \fint_{B_y} |u - u_{B_x}| \, d\mu \\
&\le c \fint_{B_{2B_x}} |u - u_{2B_x}| \, d\mu \le cr \frac{\|Du\|(\tau 2B_x)}{\mu(2B_x)} \le cr\lambda.
\end{aligned}$$

Since μ-almost every point of u is a Lebesgue point, previous estimates imply that inequality (4.3) holds. Now the first claim of this proposition follows using (4.2), (4.3) and McShane's extension theorem for Lipschitz functions. The extension of u is denoted by v_λ. Observe that $v_\lambda = u$ in E_λ and v_λ is $c\lambda$-Lipschitz continuous in X.

We next provide a control for $\|u - v_\lambda\|_{L^1(X)}$. First we assume that u is bounded on X. We may also assume that $|v_\lambda| \le \|u\|_{L^\infty(X)}$ by truncating the function, if necessary. By (4.2), we may conclude that

$$\begin{aligned}
\int_X |u - v_\lambda| \, d\mu &= \int_{X \setminus E_\lambda} |u - v_\lambda| \, d\mu \\
&\le 2\|u\|_{L^\infty(X)} \mu(X \setminus E_\lambda) \\
&\le \frac{c}{\lambda} \|u\|_{L^\infty(X)} \|Du\|(X \setminus E_\lambda).
\end{aligned}$$

Note that $\|Du\|$ is absolutely continuous with respect to μ on E_λ and, by Lemma 3.7, we have

$$\begin{aligned}\|D(u-v_\lambda)\|(X) &= \|D(u-v_\lambda)\|(X\setminus E_\lambda)\\ &\le \|Du\|(X\setminus E_\lambda)+\|Dv_\lambda\|(X\setminus E_\lambda)\\ &\le \|Du\|(X\setminus E_\lambda)+c\lambda\mu(X\setminus E_\lambda)\\ &\le c\|Du\|(X\setminus E_\lambda).\end{aligned}$$

Here we also used the fact that v_λ is $c\lambda$-Lipschitz and (4.2). This implies that

$$\begin{aligned}\|D(v_\lambda)\|(X) &\le \|D(u-v_\lambda)\|(X)+\|Du\|(X)\\ &\le c\|Du\|(X\setminus E_\lambda)+\|Du\|(X)\\ &\le c\|Du\|(X).\end{aligned}$$

We can remove the assumption on boundedness of u by approximating u by bounded functions

$$u_k=\min\{k,\max\{u,-k\}\}\in BV(X),\quad k=1,2,\dots,$$

and approximate u_k by Lipschitz functions as above. Observe that

$$\int_X |u-u_k|\,d\mu \le \int_{\{|u|>k\}}|u|\,d\mu \to 0$$

as $k\to\infty$. By the same argument as in (3.7), we also have

$$\|D(u-u_k)\|(X)\to 0$$

as $k\to\infty$. □

Remark 4.4. By considering χ_B for a ball B in $\mathbb{R}^n$, we see that the result above is optimal.

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