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Pointwise radial minimization: Hermite interpolation on arbitrary domains

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Abstract

In this paper we propose a new kind of Hermite interpolation on arbitrary domains, matching derivative data of arbitrary order on the boundary. The basic idea stems from an interpretation of mean value interpolation as the pointwise minimization of a radial energy function involving first derivatives of linear polynomials. We generalize this and minimize over derivatives of polynomials of arbitrary odd degree. We analyze the cubic case, which assumes first derivative boundary data and show that the minimization has a unique, infinitely smooth solution with cubic precision. We have not been able to prove that the solution satisfies the Hermite interpolation conditions but numerical examples strongly indicate that it does for a wide variety of planar domains and that it behaves nicely.

Categories and Subject Descriptors (according to ACM CCS): G.1.1 [Numerical Analysis]: Interpolation – Interpolation formulas; I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling – Curve, surface, solid, and object representations; J.6 [Computer-Aided Engineering]: Computer-Aided Design (CAD)
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1. Introduction

Mean value interpolation has emerged as a simple and robust way to smoothly interpolate a function defined on the boundary of an arbitrary domain in $\mathbb{R}^n$, and provides an alternative to solving a PDE such as the Laplace equation. This kind of interpolation started from a generalization of barycentric coordinates in the plane [Flo03], and was later extended to general polygons [HF06] and to triangular meshes [FKR05, JSW05]. These constructions were further generalized to continuous boundaries in [JSW05, SJW07]. More work on mean value coordinates and related topics can be found in [Bel06, FHK06, FH07, JSWD05, LBS06, LKCOL07, WSHD07].

Mean value interpolation only matches the values of a function on the boundary, not its derivatives. However, there are many applications in geometric modeling and scientific computing in which it is also desirable to match certain derivatives at the boundary, and again it is worth looking for methods that avoid solving a PDE such as the biharmonic equation. An early approach was that of Gordon and Wixom [GW74], which is based on averaging the values of interpolatory univariate polynomials across the domain. This method is simple conceptually but only applies to convex domains and requires computing intersections between lines and the domain boundary.

A more recent approach to higher order interpolation by Langer and Seidel [LS08] is an extension of barycentric coordinates to incorporate derivative data given at a set of vertices. Generalized barycentric coordinates such as Wachspress, Sibson, and mean value coordinates are composed with a modifying function and then used to interpolate first order Taylor expansions at the vertices. This method is easy to compute and was applied successfully to give greater control when deforming polygonal meshes. However, even in the univariate case this method lacks cubic precision.

Another recent idea, developed in [DF07, BF], was to build on properties of the mean value weight function, i.e., the reciprocal of the denominator in the rational expression for the interpolant, in order to construct C^1 Hermite interpolants on arbitrary domains. This method has the advantage that it reduces to cubic polynomial interpolation in the univariate case, and in $\mathbb{R}^2$ and $\mathbb{R}^3$ it can be evaluated approximately through explicit formulas for polygons and triangular

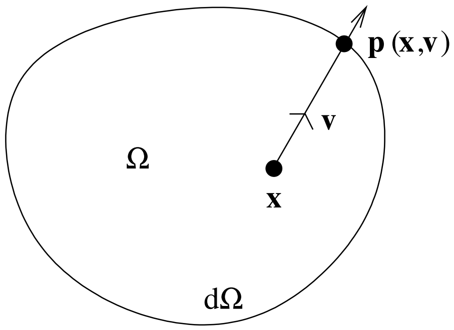

Figure 1: *Definition of* **p**.

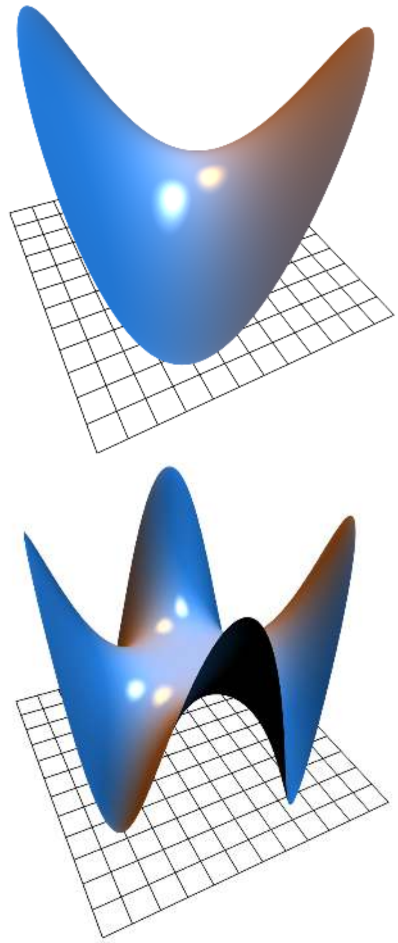

Figure 2: *Mean value interpolants.*

meshes. However, the method does not have cubic precision in $\mathbb{R}^n$ for $n > 1$.

In this paper we propose a new kind of Hermite interpolation on arbitrary domains in $\mathbb{R}^n$. The method has cubic precision for C^1 boundary data for all n, and in $\mathbb{R}^2$ the method appears to perform very well in practice. We start by viewing mean value interpolation as the pointwise minimization of a radial energy function involving first derivatives of linear polynomials. We then generalize this to match boundary derivatives up to order k by minimizing a radial energy function involving $(k+1)$-st derivatives of polynomials of degree $2k+1$. The case $k=0$ is simply mean value interpolation. We analyze the cubic case, $k=1$, and show that the minimization has a unique solution which has cubic precision. We cannot prove that the solution satisfies the Hermite interpolation conditions but numerical examples show that the solution interpolates the derivative boundary data in $\mathbb{R}^2$ for a wide variety of domain shapes.

2. A new look at mean value interpolation

We start by showing that mean value interpolation can be expressed as the solution to a radial minimization problem.

2.1. Convex domains

Consider the case that $\Omega \subset \mathbb{R}^n$ is a bounded, open, convex domain and that $f : \partial\Omega \to \mathbb{R}$ is a continuous function. For any point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in Ω and any unit vector $\mathbf{v}$ in the unit sphere $S \subset \mathbb{R}^n$, let $\mathbf{p}(\mathbf{x},\mathbf{v})$ be the unique point of intersection between the semi-infinite line $\{\mathbf{x}+r\mathbf{v} : r \geq 0\}$ and the boundary $\partial\Omega$; see Figure 1. Let $\rho(\mathbf{x},\mathbf{v})$ be the Euclidean distance $\rho(\mathbf{x},\mathbf{v}) = \|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|$. As developed in the papers [Flo03,JSW05,DF07,BF], the mean value interpolant $g : \Omega \to \mathbb{R}$ is given by the formula

$$g(\mathbf{x}) = \int_S \frac{f(\mathbf{p}(\mathbf{x},\mathbf{v}))}{\rho(\mathbf{x},\mathbf{v})}\, d\mathbf{v} \Big/ \phi(\mathbf{x}),$$
$$\phi(\mathbf{x}) = \int_S \frac{1}{\rho(\mathbf{x},\mathbf{v})}\, d\mathbf{v}, \qquad \mathbf{x} \in \Omega.$$

It was shown in [DF07,BF] that under mild conditions on the shape of the boundary $\partial\Omega$, the function g interpolates f when f is continuous. Figure 2 shows two examples of the mean value interpolant g on a circular domain. In previous papers, g was derived from the mean value property of harmonic functions. In this paper we take a different viewpoint in order to generalize to Hermite interpolation. We claim that at a fixed point $\mathbf{x} \in \Omega$, the value $g(\mathbf{x})$ is the unique minimizer $a = g(\mathbf{x})$ of the local 'energy' function

$$E(a) = \int_S \int_0^{\rho(\mathbf{x},\mathbf{v})} \left(\frac{\partial}{\partial r} F(\mathbf{x}+r\mathbf{v}) \right)^2 dr\, d\mathbf{v},$$

where $F : \overline{\Omega} \to \mathbb{R}$ is the radially linear function,

$$F(\mathbf{x}+r\mathbf{v}) = \frac{\rho(\mathbf{x},\mathbf{v}) - r}{\rho(\mathbf{x},\mathbf{v})} a + \frac{r}{\rho(\mathbf{x},\mathbf{v})} f(\mathbf{p}(\mathbf{x},\mathbf{v})),$$
$$\mathbf{v} \in S, \quad 0 \leq r \leq \rho(\mathbf{x},\mathbf{v}).$$

To see this observe that

$$\frac{\partial}{\partial r}F(\mathbf{x}+r\mathbf{v}) = \frac{f(\mathbf{p}(\mathbf{x},\mathbf{v}))-a}{\rho(\mathbf{x},\mathbf{v})},$$

and therefore,

$$E(a) = \int_S \frac{(f(\mathbf{p}(\mathbf{x},\mathbf{v}))-a)^2}{\rho(\mathbf{x},\mathbf{v})}\,d\mathbf{v}.$$

Thus, setting the derivative of E with respect to a to zero gives

$$\int_S \frac{f(\mathbf{p}(\mathbf{x},\mathbf{v}))-a}{\rho(\mathbf{x},\mathbf{v})}\,d\mathbf{v} = 0,$$

and solving this for a gives the solution $a = g(\mathbf{x})$.

2.2. Non-convex domains

An interpretation in terms of functional minimization can also be made for non-convex domains. Let $\Omega \subset \mathbb{R}^n$ be an open, not necessarily convex domain. Recall that the intersection between a line and a surface is said to be *transversal* if the line does not lie in the tangent space of the surface at the point of intersection. We will say that a unit vector $\mathbf{v}$ is *transversal* with respect to $\mathbf{x}$ in Ω if all the intersections between $\{\mathbf{x}+r\mathbf{v} : r \geq 0\}$ and $\partial\Omega$ are transversal. For example, in the $\mathbb{R}^2$ case of Figure 3 all unit vectors are transversal at $\mathbf{x}$ except for $\mathbf{v}_1$ and $\mathbf{v}_2$. If $\mathbf{v}$ is transversal, let $\mu(\mathbf{x},\mathbf{v})$ be the number of intersections of $\{\mathbf{x}+r\mathbf{v} : r \geq 0\}$ with $\partial\Omega$ which will be an odd number, assumed finite, and let $\mathbf{p}_j(\mathbf{x},\mathbf{v})$, $j = 1,2,\ldots,\mu(\mathbf{x},\mathbf{v})$, be the points of intersection, ordered so that their distances $\rho_j(\mathbf{x},\mathbf{v}) = \|\mathbf{p}_j(\mathbf{x},\mathbf{v})-\mathbf{x}\|$ are increasing,

$$\rho_1(\mathbf{x},\mathbf{v}) < \rho_2(\mathbf{x},\mathbf{v}) < \cdots < \rho_{\mu(\mathbf{x},\mathbf{v})}(\mathbf{x},\mathbf{v}).$$

For example, for $\mathbf{v}$ in Figure 3b, there are three such intersections and so $\mu(\mathbf{x},\mathbf{v}) = 3$. We make the assumption that the set $\{\mathbf{v} \in S : \mathbf{v} \text{ is non-transversal}\}$ has measure zero, so that non-transversal $\mathbf{v}$ can be ignored when integrating over S. The mean value interpolant [DF07, BF] is then

$$g(\mathbf{x}) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})} f(\mathbf{p}_j(\mathbf{x},\mathbf{v}))\,d\mathbf{v} \Big/ \phi(\mathbf{x}),$$

$$\phi(\mathbf{x}) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})}\,d\mathbf{v}.$$

We claim that $g(\mathbf{x})$ is now the unique minimizer $a = g(\mathbf{x})$ of

$$E(a) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1} \int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(q'_{\mathbf{v},j}(r)\right)^2 dr\,d\mathbf{v},$$

where, for $\mathbf{v} \in S$, $0 \leq r \leq \rho_j(\mathbf{x},\mathbf{v})$, and $j = 1,\ldots,\mu(\mathbf{x},\mathbf{v})$, $q_{\mathbf{v},j}$ is the linear polynomial

$$q_{\mathbf{v},j}(r) = \frac{\rho_j(\mathbf{x},\mathbf{v})-r}{\rho_j(\mathbf{x},\mathbf{v})}a + \frac{r}{\rho_j(\mathbf{x},\mathbf{v})} f(\mathbf{p}_j(\mathbf{x},\mathbf{v})).$$

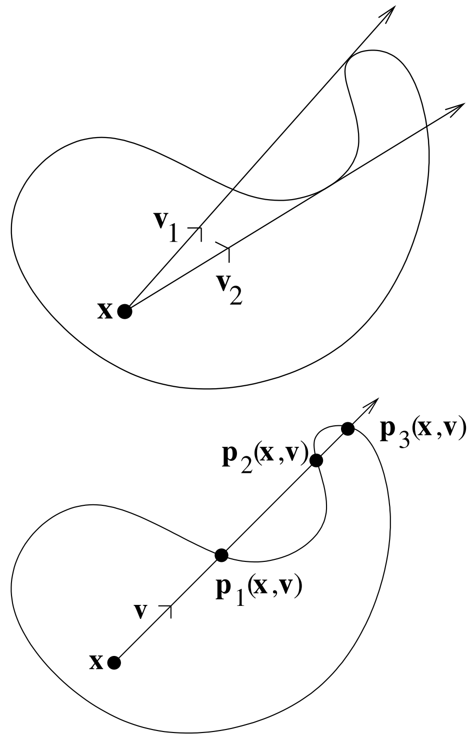

Figure 3: *(a) Two non-transversal vectors. (b) A transversal vector with three intersections.*

Indeed, similar to the convex case, we find

$$E(a) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1} \frac{(f(\mathbf{p}_j(\mathbf{x},\mathbf{v}))-a)^2}{\rho_j(\mathbf{x},\mathbf{v})}\,d\mathbf{v},$$

and setting $E'(a) = 0$ gives $a = g(\mathbf{x})$.

3. Hermite interpolation

Having now recast mean value interpolation as the minimization of a radial energy function, we are ready to explore a possible generalization in which the interpolant also matches derivative boundary data. Instead of minimizing over first derivatives of linear polynomials, we now minimize over $(k+1)$-st derivatives of polynomials of degree $2k+1$, for any $k = 0,1,2,\ldots$. Specifically, we want to interpolate in Ω a C^k continuous real function f, given its values and partial derivatives up to order k at the boundary $\partial\Omega$. Fix $\mathbf{x} \in \Omega$ and let τ be some polynomial in $\pi_k(\mathbb{R}^n)$, the linear space of n-variate polynomials of degree $\leq k$. We think of τ as its own Taylor series at $\mathbf{x}$ because we are only interested in the derivatives of τ up to order k at $\mathbf{x}$. Then, for each transversal $\mathbf{v} \in S$ and for $j = 1,\ldots,\mu(\mathbf{x},\mathbf{v})$, let $q_{\mathbf{v},j}$ be the univariate polynomial of degree $\le 2k+1$ such that

$$q_{\mathbf{v},j}^{(i)}(0) = D_{\mathbf{v}}^i \tau(\mathbf{x}), \qquad i = 0,1,\ldots,k,$$
$$q_{\mathbf{v},j}^{(i)}(\rho_j(\mathbf{x},\mathbf{v})) = D_{\mathbf{v}}^i f(\mathbf{p}_j(\mathbf{x},\mathbf{v})), \quad i = 0,1,\ldots,k,$$

where $D_{\mathbf{v}} f$ denotes the directional derivative of f in the direction $\mathbf{v}$. Then, defining

$$E_{\mathbf{v},j}(\tau) = \int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(q_{\mathbf{v},j}^{(k+1)}(r)\right)^2 dr,$$

we propose to choose τ in $\pi_k(\mathbb{R}^n)$ to minimize

$$E(\tau) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1} E_{\mathbf{v},j}(\tau)\, d\mathbf{v},$$

and then set

$$g(\mathbf{x}) = \tau(\mathbf{x}). \tag{1}$$

This procedure defines, in a pointwise fashion, a function $g : \Omega \to \mathbb{R}$. In general, the minimization will yield a different polynomial τ at each $\mathbf{x}$ and so g will not itself be a polynomial.

Several questions arise, the main ones being

1. does the minimization have a unique solution τ, so that g is well-defined, and
2. do the derivatives of g up to order k match those of f on $\partial\Omega$?

Considering first the univariate case, $n = 1$, it is well known that among all piecewise C^{k+1} functions q on a real interval (a,b) that match derivatives up to order k of a given function f at a and b, the unique minimizer of the energy

$$\int_a^b (q^{(k+1)}(x))^2\, dx$$

is the polynomial Hermite interpolant to f of degree at most $2k+1$. Thus when $n = 1$ and $\Omega = (a,b)$, we see that g, defined by (1), will simply be the Hermite polynomial interpolant to f of degree $\le 2k+1$. Thus the answer to both questions is 'yes' when $n = 1$. We are not able to answer them for general n, however. Instead, in the rest of the paper, we focus on the case $k = 1$ and we show that there is a unique minimizer for all n. We cannot yet show that g really interpolates the derivatives of f of order 0 and 1, but we can show that if f is a cubic then $g = f$, in which case it trivially interpolates, and numerical examples in $\mathbb{R}^2$ strongly suggest that it interpolates any f when $n = 2$.

In the case $k = 1$, we can express any polynomial $\tau \in \pi_1(\mathbb{R}^n)$ in the form

$$\tau(\mathbf{y}) = a + (\mathbf{y} - \mathbf{x}) \cdot \mathbf{b},$$

for some $a \in \mathbb{R}$ and $\mathbf{b} = (b_1,\ldots,b_n)^T \in \mathbb{R}^n$ and then $q_{\mathbf{v},j}$ is the cubic polynomial such that

$$q_{\mathbf{v},j}(0) = a, \qquad q_{\mathbf{v},j}(\rho_j(\mathbf{x},\mathbf{v})) = f(\mathbf{p}_j(\mathbf{x},\mathbf{v})),$$
$$q'_{\mathbf{v},j}(0) = \mathbf{v}\cdot\mathbf{b}, \qquad q'_{\mathbf{v},j}(\rho_j(\mathbf{x},\mathbf{v})) = D_{\mathbf{v}} f(\mathbf{p}_j(\mathbf{x},\mathbf{v})).$$

Then, $E_{\mathbf{v},j}$ and E can be viewed as functions of a and $\mathbf{b}$, and the task is to find $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ to minimize

$$E(a,\mathbf{b}) = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1} E_{\mathbf{v},j}(a,\mathbf{b})\, d\mathbf{v}, \tag{2}$$

where

$$E_{\mathbf{v},j}(a,\mathbf{b}) = \int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(q''_{\mathbf{v},j}(r)\right)^2 dr, \tag{3}$$

and set $g(\mathbf{x}) = a$. In turns out that E has a unique minimizing pair $(a,\mathbf{b})$. In order to show this and to compute a we begin with a lemma.

Lemma 1 Let q be the cubic polynomial such that

$$q(0) = f_0, \quad q'(0) = m_0, \quad q(h) = f_1, \quad q'(h) = m_1,$$

for some $h > 0$, and let

$$E = \int_0^h (q''(x))^2\, dx. \tag{4}$$

Then, with $\Delta f_0 = f_1 - f_0$,

$$E = 12\frac{(\Delta f_0)^2}{h^3} - 12\frac{(\Delta f_0)(m_0+m_1)}{h^2} + 4\frac{m_0^2 + m_0 m_1 + m_1^2}{h}.$$

Proof Using the Bernstein polynomials

$$B_i^k(u) = \binom{k}{i} u^i (1-u)^{k-i}, \quad k \ge 0, \quad 0 \le i \le k,$$

we can express q in the Bernstein form

$$q(x) = \sum_{i=0}^{3} c_i B_i^3(x/h)$$

where

$$c_0 = f_0, \quad c_1 = f_0 + M_0, \quad c_2 = f_1 - M_1, \quad c_3 = f_1,$$

and $M_i = h m_i/3$. Then

$$q''(x) = \frac{6}{h^2} \sum_{i=0}^{1} \Delta^2 c_i B_i^1(x/h),$$

where

$$\Delta^2 c_i = c_{i+2} - 2c_{i+1} + c_i,$$

and squaring and integrating over $[0,h]$ yields

$$E = \frac{12}{h^3}\left((\Delta^2 c_0)^2 + \Delta^2 c_0 \Delta^2 c_1 + (\Delta^2 c_1)^2\right).$$

Then, since

$$\Delta^2 c_0 = \Delta f_0 - 2M_0 - M_1, \qquad \Delta^2 c_1 = -\Delta f_0 + M_0 + 2M_1,$$

a simple calculation gives

$$E = \frac{12}{h^3}\Big((\Delta f_0)^2 - 3(\Delta f_0)(M_0+M_1) + 3(M_0^2 + M_0 M_1 + M_1^2)\Big),$$

which gives the result. □

We can now apply Lemma 1 to give a formula for $E_{\mathbf{v},j}(a,\mathbf{b})$ in (3), using matrix notation. Setting $f_0 = a$, $m_0 = \mathbf{v}\cdot\mathbf{b}$, $f_1 = f(\mathbf{p}(\mathbf{x},\mathbf{v}))$, $m_1 = D_{\mathbf{v}} f(\mathbf{p}(\mathbf{x},\mathbf{v}))$, and $h = \rho(\mathbf{x},\mathbf{v})$, gives

$$E_{\mathbf{v},j}(\mathbf{a}) = \mathbf{a}^T M_{\mathbf{v},j}\mathbf{a} + N_{\mathbf{v},j}^T\mathbf{a} + P_{\mathbf{v},j}, \tag{5}$$

where, with the shorthand $\mathbf{p} := \mathbf{p}_j(\mathbf{x},\mathbf{v})$ and $\rho := \rho_j(\mathbf{x},\mathbf{v})$,

$$\mathbf{a} = \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}, \qquad M_{\mathbf{v},j} = \frac{2}{\rho^3}\begin{pmatrix} 6 & 3\rho\mathbf{v}^T \\ 3\rho\mathbf{v} & 2\rho^2\mathbf{v}\mathbf{v}^T \end{pmatrix},$$

$$N_{\mathbf{v},j} = \frac{4}{\rho^3}\begin{pmatrix} -6f(\mathbf{p}) + 3\rho D_{\mathbf{v}} f(\mathbf{p}) \\ -3\rho f(\mathbf{p})\mathbf{v} + \rho^2 D_{\mathbf{v}} f(\mathbf{p})\mathbf{v} \end{pmatrix},$$

and

$$P_{\mathbf{v},j} = \frac{4}{\rho^3}\left(3(f(\mathbf{p}))^2 - 3\rho f(\mathbf{p}) D_{\mathbf{v}} f(\mathbf{p}) + \rho^2 (D_{\mathbf{v}} f(\mathbf{p}))^2\right).$$

Substituting (5) into (2) now gives

$$E(\mathbf{a}) = \mathbf{a}^T M\mathbf{a} + N^T\mathbf{a} + P, \tag{6}$$

where M is the matrix

$$M = \int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1} M_{\mathbf{v},j}\, d\mathbf{v}, \tag{7}$$

and similarly for N and P, and integrating a matrix means integrating each element. As the set of non-transversal $\mathbf{v}$ has measure zero for each $\mathbf{x}\in\Omega$, the integration can be achieved by splitting the integral into a sum over integrals over the regions of S where the number of intersections $\mu(\mathbf{x},\mathbf{v})$ is constant.

The matrix M is clearly symmetric and if it is also positive definite then it is non-singular and the unique minimum of E is the unique solution $\mathbf{a}$ to the equation

$$M\mathbf{a} = -\frac{1}{2}N, \tag{8}$$

and a can be computed directly from this using Cramer's rule. To show that M is positive definite we use the following lemma.

Lemma 2 If $f_1 = m_1 = 0$ in Lemma 1 then E in (4) is non-increasing in h.

Proof From Lemma 1 we know that

$$E = 12\frac{f_0^2}{h^3} + 12\frac{f_0 m_0}{h^2} + 4\frac{m_0^2}{h},$$

and computing the derivative with respect to h yields

$$\begin{aligned}\frac{dE}{dh} &= -\left(36\frac{f_0^2}{h^4} + 24\frac{f_0 m_0}{h^3} + 4\frac{m_0^2}{h^2}\right) \\ &= -\left(6\frac{f_0}{h^2} + 2\frac{m_0}{h}\right)^2 \le 0.\end{aligned}$$

□

Theorem 1 The matrix M is positive definite.

Proof We first show that $\mathbf{a}^T M\mathbf{a} \ge 0$. To do this, let $f = 0$. Then since $N = \mathbf{0}$ and $P = 0$, we have

$$\mathbf{a}^T M\mathbf{a} = E(\mathbf{a}).$$

Since

$$q_{\mathbf{v},\mu}(\rho_\mu(\mathbf{x},\mathbf{v})) = q'_{\mathbf{v},\mu}(\rho_\mu(\mathbf{x},\mathbf{v})) = 0, \tag{9}$$

Lemma 2 shows that $E_{\mathbf{v},j}(\mathbf{a})$ in (3) is non-increasing in j. Therefore $E_{\mathbf{v},j}(a,\mathbf{b}) - E_{\mathbf{v},j+1}(a,\mathbf{b}) \ge 0$ in (2) and so

$$E(\mathbf{a}) \ge \int_S E_{\mathbf{v},\mu(\mathbf{x},\mathbf{v})}(a,\mathbf{b})\, d\mathbf{v} \ge 0.$$

Thus $\mathbf{a}^T M\mathbf{a} \ge 0$ for all $\mathbf{a}\in\mathbb{R}^{n+1}$.

Further, suppose that $\mathbf{a}^T M\mathbf{a} = 0$. Then, letting $f = 0$ again, we must have

$$\int_S E_{\mathbf{v},\mu(\mathbf{x},\mathbf{v})}(a,\mathbf{b})\, d\mathbf{v} = 0,$$

and hence for each $\mathbf{v}\in S$, with $\mu = \mu(\mathbf{x},\mathbf{v})$, we must have

$$E_{\mathbf{v},\mu}(a,\mathbf{b}) = \int_0^{\rho_\mu(\mathbf{x},\mathbf{v})} \left(q''_{\mathbf{v},\mu}(r)\right)^2 dr = 0.$$

Therefore, for each $\mathbf{v}\in S$, $q_{\mathbf{v},\mu}$ is linear and by (9), $q_{\mathbf{v},\mu} = 0$. Hence $\mathbf{a} = \mathbf{0}$. □

4. Boundary integrals

If a parametric representation of $\partial\Omega$ is available, the integrals in M and N in (6) can be converted to integrals over the parameters of $\partial\Omega$, in the spirit of [JSW05]. To this end, suppose that $\mathbf{s} = (s_1,\ldots,s_{n-1}) : D \to \partial\Omega$ is a parameterization of the curve or surface $\partial\Omega$ with parameter domain $D\subset\mathbb{R}^{n-1}$. We will assume that $\mathbf{s}$ is a regular parameterization, by which we mean that $\mathbf{s}$ is piecewise C^1 and that at every point of differentiability $\mathbf{t} = (t_1,\ldots,t_{n-1})\in D$, the first order partial derivatives $D_i\mathbf{s} := \partial\mathbf{s}/\partial t_i$, $i = 1,2,\ldots,n-1$, are linearly independent. Thus, following the notation of [JSW05] and [BF], their cross product,

$$\mathbf{s}^\perp(\mathbf{t}) := \mathbf{det}(D_1\mathbf{s}(\mathbf{t}),\ldots,D_{n-1}\mathbf{s}(\mathbf{t}))$$

is non-zero, and is orthogonal to the tangent space at $\mathbf{s}(\mathbf{t})$. We make the convention that $\mathbf{s}^\perp$ points outwards from Ω. Then, considering for example the integrals in M in (7), we make the substitution $\mathbf{p}_j(\mathbf{x},\mathbf{v}) = \mathbf{s}(\mathbf{t})$ in (7), so that

$$\mathbf{v} = \frac{\mathbf{s}(\mathbf{t}) - \mathbf{x}}{\|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|},$$

and $\rho_j(\mathbf{x},\mathbf{v}) = \|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|$, and then (see [BF]),

$$d\mathbf{v} = \frac{(\mathbf{s}(\mathbf{t}) - \mathbf{x})\cdot\mathbf{s}^\perp(\mathbf{t})}{\|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|^n}\, d\mathbf{t}.$$

Substituting these expressions into (7), and similarly for N, gives the boundary representations

$$M = \int_D \hat{M} w(\mathbf{x},\mathbf{t})\, d\mathbf{t}, \qquad N = \int_D \hat{N} w(\mathbf{x},\mathbf{t})\, d\mathbf{t}, \tag{10}$$

where

$$w(\mathbf{x},\mathbf{t}) = \frac{(\mathbf{s}(\mathbf{t})-\mathbf{x})\cdot\mathbf{s}^{\perp}(\mathbf{t})}{\|\mathbf{s}(\mathbf{t})-\mathbf{x}\|^{n+3}},$$

and, with the shorthand $\mathbf{s} := \mathbf{s}(\mathbf{t})$,

$$\hat{M} = 2\begin{pmatrix} 6 & 3(\mathbf{s}-\mathbf{x})^T \\ 3(\mathbf{s}-\mathbf{x}) & 2(\mathbf{s}-\mathbf{x})(\mathbf{s}-\mathbf{x})^T \end{pmatrix},$$

$$\hat{N} = 4\begin{pmatrix} -6f(\mathbf{s})+3D_{\mathbf{s}-\mathbf{x}}f(\mathbf{s}) \\ (-3f(\mathbf{s})+D_{\mathbf{s}-\mathbf{x}}f(\mathbf{s}))(\mathbf{s}-\mathbf{x}) \end{pmatrix}.$$

This provides a way of numerically computing the value of g at a point $\mathbf{x}$ by sampling the surface $\mathbf{s}$ and its first derivatives and applying numerical integration. It also shows that g is C^∞ smooth in Ω, due to differentiation under the integral sign.

5. Cubic precision

We now establish cubic precision (for $k=1$ and all n):

Theorem 2 Suppose that $f:\overline{\Omega}\to\mathbb{R}$ is a cubic polynomial. Then $g=f$ in Ω.

Proof Fix an arbitrary $\mathbf{x}\in\Omega$ and let $\hat{a} := f(\mathbf{x})$ and $\hat{\mathbf{b}} := \nabla f(\mathbf{x})$. We will show that $E(\hat{a},\hat{\mathbf{b}}) \le E(a,\mathbf{b})$ for all $a\in\mathbb{R}$ and $\mathbf{b}\in\mathbb{R}^n$ and hence $g(\mathbf{x}) = \hat{a} = f(\mathbf{x})$.
Let $e_{\mathbf{v},j} : [0,\rho_j(\mathbf{x},\mathbf{v})] \to \mathbb{R}$ be the cubic polynomial defined by $e_{\mathbf{v},j}(r) = q_{\mathbf{v},j}(r) - f(\mathbf{x}+r\mathbf{v})$. Then

$$q''_{\mathbf{v},j}(r) = \frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v}) + e''_{\mathbf{v},j}(r)$$

and so

$$\begin{aligned} E_{\mathbf{v},j}(a,\mathbf{b}) = &\int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(\frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v})\right)^2 dr + \\ &2\int_0^{\rho_j(\mathbf{x},\mathbf{v})} e''_{\mathbf{v},j}(r)\frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v})\,dr + \\ &\int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(e''_{\mathbf{v},j}(r)\right)^2 dr. \end{aligned}$$

Now

$$\int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(\frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v})\right)^2 dr = E_{\mathbf{v},j}(\hat{a},\hat{\mathbf{b}}),$$

and as $e_{\mathbf{v},j}$ is a cubic polynomial and

$$e_{\mathbf{v},j}(\rho_j(\mathbf{x},\mathbf{v})) = e'_{\mathbf{v},j}(\rho_j(\mathbf{x},\mathbf{v})) = 0,$$

it follows from Lemma 2 that

$$\sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1}\int_0^{\rho_j(\mathbf{x},\mathbf{v})} \left(e''_{\mathbf{v},j}(r)\right)^2 dr \ge 0.$$

Thus,

$$E(a,\mathbf{b}) \ge E(\hat{a},\hat{\mathbf{b}}) + K,$$

where

$$K = 2\int_S \sum_{j=1}^{\mu(\mathbf{x},\mathbf{v})} (-1)^{j-1}\int_0^{\rho_j(\mathbf{x},\mathbf{v})} e''_{\mathbf{v},j}(r)\frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v})\,dr\,d\mathbf{v}.$$

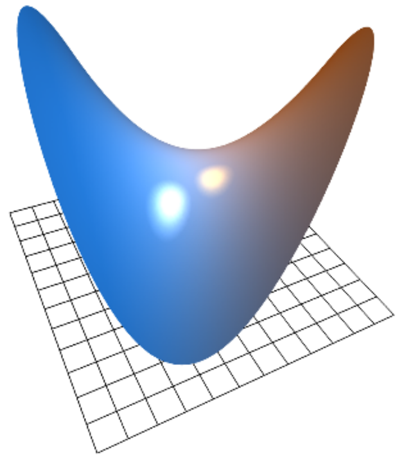

Figure 4: *Reproduction of a quadratic polynomial (maximal absolute error due to numerical integration:* $3.4\cdot 10^{-8}$*).*

We will show that $K=0$, which will complete the proof. We apply integration by parts to the inner integral, giving

$$\begin{aligned} \int_0^{\rho_j(\mathbf{x},\mathbf{v})} & e''_{\mathbf{v},j}(r)\frac{\partial^2}{\partial r^2}f(\mathbf{x}+r\mathbf{v})\,dr \\ &= e_{\mathbf{v},j}(0)D^3_{\mathbf{v}}f(\mathbf{x}) - e'_{\mathbf{v},j}(0)D^2_{\mathbf{v}}f(\mathbf{x}) \\ &= (a-\hat{a})D^3_{\mathbf{v}}f(\mathbf{x}) - (\mathbf{v}\cdot(\mathbf{b}-\hat{\mathbf{b}}))D^2_{\mathbf{v}}f(\mathbf{x}). \end{aligned}$$

Then, since this expression is independent of j, and recalling that $\mu(\mathbf{x},\mathbf{v})$ is odd we deduce that

$$K = 2\left((a-\hat{a})\int_S D^3_{\mathbf{v}}f(\mathbf{x})\,d\mathbf{v} - (\mathbf{b}-\hat{\mathbf{b}})\cdot\int_S D^2_{\mathbf{v}}f(\mathbf{x})\mathbf{v}\,d\mathbf{v}\right).$$

But both integrals in this expression are zero since

$$D^2_{-\mathbf{v}}f(\mathbf{x}) = D^2_{\mathbf{v}}f(\mathbf{x}) \quad \text{and} \quad D^3_{-\mathbf{v}}f(\mathbf{x}) = -D^3_{\mathbf{v}}f(\mathbf{x}).$$

□

6. Numerical Examples

We conclude the paper with some numerical examples of the minimizing C^1 Hermite interpolant g in $\mathbb{R}^2$. All of these examples were computed using Cramer's rule to find the value of a in (8). The elements of M and N were found using the boundary formula (10) and adaptive numerical integration with a relative tolerance.

Figure 4 shows g on the unit disk in the case that $f(x,y) = r^2\sin 2\theta$, with polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. As predicted by Theorem 2, g is numerically equal to f in this case, since f is the quadratic polynomial $f(x,y) = 2xy$. In this example, the Hermite interpolant g is only slightly different to the mean value interpolant to the Lagrange data from f shown in Figure 2.

Figure 5 shows g on a non-convex domain with a hole,

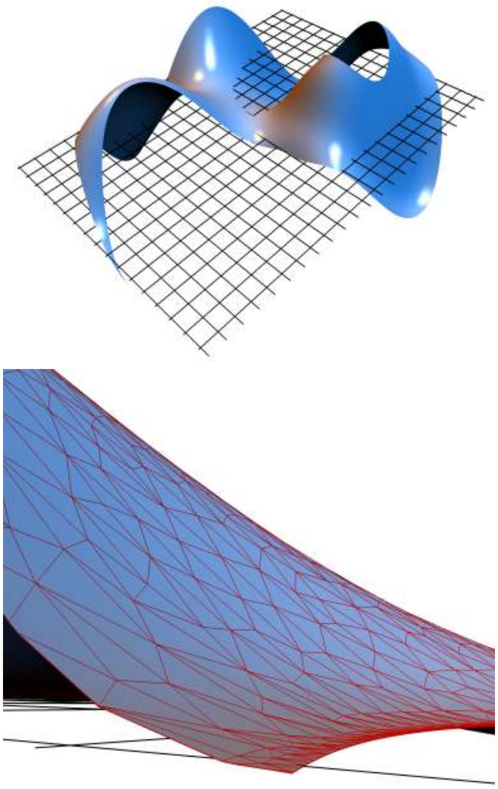

Figure 5: *Interpolant (*top*) and part of the error function (*bottom*) for a non-convex domain with hole and cusp.*

with f given by $f(x,y) = \sin(y)\cos(2x)$. The interpolant g behaves nicely even though there is a cusp at the leftmost point of the boundary. The error between g and f near the boundary of the hole is shown in the lower part of Figure 5. The error function is cut open to demonstrate numerically that both the error and its gradient are zero at the boundary. Figure 9 shows further numerical evidence of the Hermite interpolation property of g. The errors in g along the x-axis are plotted for three different functions f on the unit circle.

Figure 6 compares the minimizing Hermite interpolant g with the Hermite interpolant of [DF07] on the unit circle with $f(x,y) = r^2 \sin 4\theta$. The top left shows the interpolant of [DF07] and the bottom left its error function. The right part of the figure shows g on the top and its error function at the bottom. Note that both error functions are tangentially zero at the boundary.

In Figure 7 we use the interpolant for hole filling, with a circular hole, and with $f(x,y) = \cos r$. In the top f is shown together with the boundary of the hole that is cut out. Then the Hermite interpolant of [DF07] is shown, and at the bottom the minimizing interpolant g.

Figure 8 shows another example of the nice behaviour of

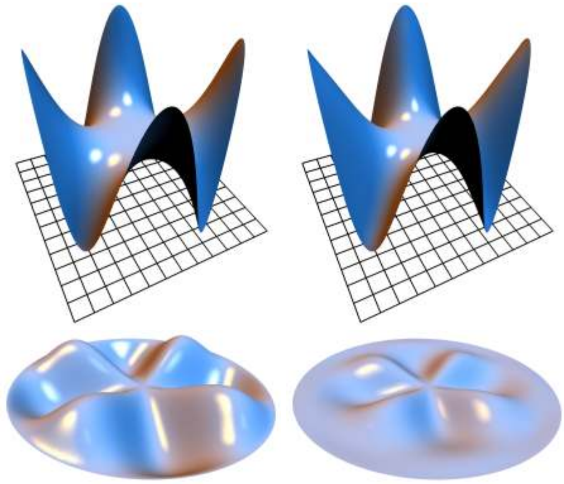

Figure 6: *Comparison between interpolants (*top*) and errors (*bottom*) of the interpolant of [DF07] (*left*) and the minimizing interpolant g (*right*).*

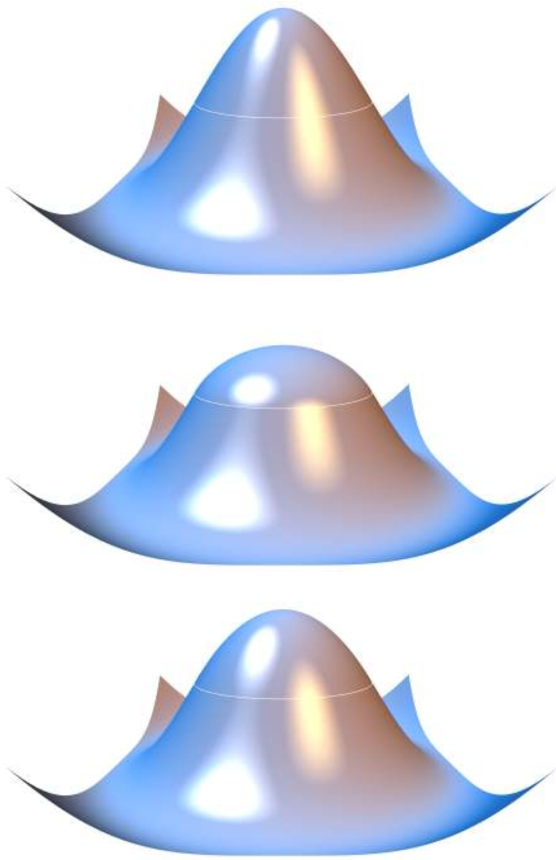

Figure 7: *Hole filling with circular boundary.* From top to bottom: *original function, interpolant of [DF07], minimizing interpolant.*

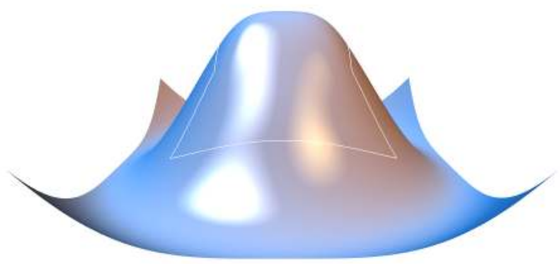

Figure 8: *Hole filling with a piecewise smooth boundary.*

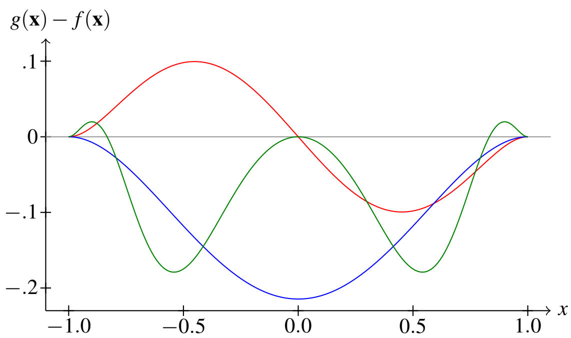

Figure 9: *Errors along the x-axis ($y = 0$) for different f, with the unit circle as domain.*

the minimizing interpolant g for hole filling with the same function f but with a more complicated boundary.

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