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POISSON INTEGRALS OF REGULAR FUNCTIONS

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ABSTRACT. Tangential convergence of Poisson integrals is proved for certain spaces of regular functions which contain the spaces of Bessel potentials of L^p functions, $1 < p < \infty$, and of functions in the local Hardy space h^1, and the corresponding tangential maximal functions are shown to be of strong p type, $p \geq 1$.

1. Introduction. It is well known that for a general L^p function f, $1 \leq p \leq \infty$, its Poisson integral $u(x,y) = P_y * f(x)$ ($P_y(z) = c_n y/(|z|^2 + y^2)^{(n+1)/2}, z \in \mathbf{R}^n, y > 0$) converges nontangentially to $f(x)$ a.e. when y tends to 0. It is also well known [**18**, p. 280] that for general L^p functions this result fails when convergence inside regions with some degree of tangentiality is considered.

However, tangential convergence holds for certain classes of functions: Nagel, Rudin, and Shapiro have recently established [**14**] the existence of tangential limits for a large class of potentials of L^p functions (see also [**14**] for earlier results). A particular instance are the spaces $L^p_a = \{J_a * f \colon f \in L^p\}$, $1 \leq p \leq \infty$, $(J_a)\hat{}(z) = (1 + |z|^2)^{-a/2}$, of Bessel potentials of L^p functions, for which explicit approach regions are given: if $1 \leq p \leq n/a$ and $x \in \mathbf{R}^n$, define $D_{a,p}(x)$ as

(i) $D_{a,p}(x) = \{(z,y) \in \mathbf{R}^{n+1}_+ \colon |z-x| \leq y^{1-ap/n}\}$, $p < n/a$,

(ii) $D_{a,p}(x) = \{(z,y) \in \mathbf{R}^{n+1}_+ \colon |z-x| \leq (\log 1/y)^{-(p-1)/n},\ y \leq 1/e\}$, $p = n/a > 1$,

(iii) $D_{n,1}(x) = \{(x,y) \in \mathbf{R}^{n+1}_+ \colon |z-x| \leq (\log 1/y)^{1/n},\ y \leq 1/e\}$.

Then [**14**, Theorems 2.9, 3.13, and 5.5]

(i) *if* $1 \leq p \leq n/a$ *and* $f \in L^p_a$, $u(x,y) = P_y * f(z)$ *tends to* $f(x)$ *inside* $D_{a,p}(x)$ *for a.e.* $x \in \mathbf{R}^n$;

(ii) *if* $1 < p \leq n/a$, $f \in L^p_a$ *and* $0 < b < a$, $u(z,y)$ *tends to* $f(x)$ *inside* $D_{b,p}(x)$ *for* $B_{a-b,p}$ a.e. $x \in \mathbf{R}^n$($B_{s,t}$ denotes (s,t) Bessel capacity; see §2).

Note that if $a > n/p$ and $f \in L^p_a$, f is continuous.

Furthermore, it is shown in [**14**, Theorem 3.8] that the corresponding maximal operators $T_{a,p}f(x) = \sup\{|u(z,y)| \colon (z,y) \in D_{a,p}(x)\}$ verify $\|T_{a,p}f\|_p \leq C\|f\|_{L^p_a}$, whereas for $p = 1$ Nagel and Stein proved [**15**, Theorem 5] that if F is in the Hardy space H^1, $\|T_{a,1}(J_aF)\|_1 \leq C\|F\|_{H^1}$, $a < n$ ([**15**] also contains results for Bessel potentials of H^p, $p > 0$).

The tangentiality of the approaching regions is shown in [**14**] to depend on the corresponding Bessel kernels J_a; here we will see how it can also be related to the regularity of the L^p_a functions. In fact, similar results (Theorems 1 and 2 below) hold for a larger class of functions, which we now define. If $\mathbf{P}_k$ denotes the set of

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all polynomials of degree $k, x \in \mathbf{R}^n$, $t > 0$, $1 \leq r \leq \infty$ and $f \in L^1_{\text{loc}}$, consider the "polynomial approximation" operator

$$E^k_r f(x,t) = \sup \inf_{P \in \mathbf{P}_k} \left(\fint_Q |f-P|^r \right)^{1/r},$$

the sup taken over all cubes Q with $x \in Q$ and having Lebesgue measure $|Q| = t^n$ (throughout the paper $\fint_E f$ or f_E stand for the mean $\int_E f\,dz/|E|$).

Now, if $a > 0$ and $m = [a]$, its integral part, we define $G_a f(x,t) = \sup_{s \leq t} s^{-a} E^m_1 f(x,s)$, $G_a f(x) = G_a f(x,\infty)$ (in what follows, if $k = m$ and $r = 1$, we will write $Ef(x,t)$ instead of $E^m_1 f(x,t)$); then C^p_a, $1 \leq p \leq \infty$, denotes the space of those L^p functions f such that $G_a f \in L^p$; with the norm $\|f\|_{a,p} = \|f\|_p + \|G_a f\|_p$, C^p_a becomes a Banach space. These spaces were introduced by Calderón and Scott [**4**] and are extensively studied by Devore and Sharpley in [**8**].

Our results are given for a proper subset of C^p_a, the closed subspace F^p_a of those $f \in C^p_a$ such that $G_a f(x,t) = o(1)$ a.e. as t goes to 0 (in fact F^p_a, $p < \infty$, is the closure of C^∞_0, the compactly supported C^∞ functions; see §3). C^p_a and F^p_a can be seen as global versions of the spaces $T^p_a(x)$ and $t^p_a(x)$ of Calderón and Zygmund [**5**]. If $1 < p < \infty$, L^p_a is continuously imbedded in F^p_a; indeed, $f \in L^p_a$ iff $f \in L^p$ and

$$G_{a,2} f(x) = \left(\int_0^\infty Ef(x,t)^2 t^{-2a-1}\,dt \right)^{1/2} \in L^p,$$

and $\|f\|_{L^p_a} \sim \|f\|_p + \|G_{a,2} f\|_p$ (see [**9**]; by $A \sim B$ we mean that $A/C \leq B \leq CA$, for some constant C; in what follows C will stand for any constant independent of sets, points, or functions, and not necessarily the same on each appearance). However, although the imbedding $L^p_a \subset F^p_a$ is proper, the Poisson integrals of functions in F^p_a and L^p_a have the same tangential behavior:

THEOREM 1. *If $1 \leq p < n/a$ or $p = n/a > 1$ and $f \in F^p_a$, then $u(z,y) = P_y * f(z)$ tends to $f(x)$ a.e. when (z,y) tends to x inside $D_{a,p}(x)$.*

The restriction $p \leq n/a$ is due to the fact that functions in F^p_a are continuous when $p > n/a$, and the same is true in F^1_n [**8**, p. 68].

For functions in F^p_a the exceptional set also becomes smaller when the tangentiality of the approach regions is decreased; in fact the results of [**14**] can be slightly improved:

THEOREM 2. (i) *If $f \in F^p_a$, $1 \leq p < n/a$, and $0 < b < a$, then $u(z,y)$ converges to $f(x)$ inside $D_{b,p}(x)$ for all x except a set of zero $H^{n-(a-b)p}$ Hausdorff measure; if moreover $p > 1$, u converges nontangentially to $f(x)$ $B_{a,p}$-a.e.*

(ii) *If $p = n/a > 1$ and $p < r < \infty$, u converges to $f(x)$ inside $D_{n/r,r}(x)$ for $H^{np/r}$-a.a. x, whereas if b is such that $0 \leq b < n/p$, u converges to $f(x)$ inside $D_{b,p}(x)$ for $B_{n/p,p}$-a.a. x.*

Theorem 2 requires some explanation: functions in F^p_a are defined in principle only a.e.; Theorem 2 will be shown to hold after suitably redefining them on a zero measure set.

As could be expected, Theorems 1 and 2 are deduced from weak type estimates for the tangential maximal operators $T_{a,p} f(x) = \sup\{|u(z,y)|\colon (z,y) \in D_{a,p}(x)\}$, but since functions in F^p_a are not representable as potentials of L^p functions, we rely on certain Sobolev and Trudinger type inequalities for them (Theorem 5). However, these weak type inequalities can be strengthened.

THEOREM 3. *If* $f \in C_a^p$, $1 \leq p \leq n/a$, *then* $\|T_{a,p}f\|_p \leq C\|f\|_{a,p}$.

The proof of Theorem 3 is modelled after that of Theorem 3.8 in [**14**], but with an important difference: the key argument in [**14**], Hansson's strong capacitary estimates [**10**], is no longer available here and a strong estimate, valid if $1 \leq p < \infty$, for a certain C_a^p capacity type function, is proved (Theorem 6) along the lines of similar results by Adams [**2**] and Dahlberg [**7**].

Besides L_a^p, the so-called Triebel-Lizorkin spaces $F_a^{p,q}$, $1 \leq p, q < \infty$, $a > 0$ (see [**17**] or §6 for the definition) are also continuously imbedded in F_a^p (Proposition 3) and therefore, the above theorems apply to them; we point out that if $1 < p < \infty$, $F_a^{p,2} = L_a^p$, whereas $F_a^{1,2}$ coincides with the space of Bessel potentials of functions in D. Goldberg's local Hardy space h^1 [**17**, p. 51]. We also remark that Y. Mizuta has recently proved [**13**] results similar to those of Theorems 1 and 2 for functions being locally in the Besov space $B_a^{p,p}$, $0 < a < 1$. Since $B_a^{p,p} = F_a^{p,p}$, Theorems 1 and 2 contain a global version of Mizuta's results.

The paper is organized as follows: §2 contains certain preliminary facts about capacities and Hausdorff measures. The spaces F_a^p are studied in some detail in §3. Theorems 1 and 2 are proved in §4 and Theorem 3 in §5. Finally, in §6 Triebel-Lizorkin spaces $F_a^{p,q}$, $1 \leq p$, $q < \infty$, $a > 0$ are considered.

2. Preliminary results. For $a > 0$ J_a will denote the Bessel kernel of order a, $(J_a)\hat{\ }(z) = (1 + |z|^2)^{-a/2}$, and I_a the Riesz kernel, $I_a(z) = c_{n,a}|z|^{a-n}$, $0 < a < n$; we will also denote by J_a and I_a the corresponding potential operators. The Bessel capacity $B_{a,p}$ and the Riesz capacity $R_{a,p}$ are defined for $E \subset \mathbf{R}^n$ as

$$B_{a,p}(E) = \inf\{\|f\|_p^p \colon f \geq 0, J_a f \geq \chi_E\}, \qquad a > 0,$$
$$R_{a,p}(E) = \inf\{\|f\|_p^p \colon f \geq 0, I_a f \geq \chi_E\}, \qquad 0 < a < n/p$$

(χ_E = characteristic function of E). If $a < n/p$,

$$R_{a,p}(E) \leq B_{a,p}(E) \leq C(R_{a,p}(E) + R_{a,p}(E)^{n/n-ap})$$

[**1**]; thus, both have the same zero sets (see [**12**] for more properties of $R_{a,p}$ and $B_{a,p}$).

If $f \in L^p$ we obviously have

$$(1) \qquad R_{a,p}(\{|I_a f| > t\}) \leq (\|f\|_p/t)^p, \qquad 0 < a < n/p;$$

$$(2) \qquad B_{a,p}(\{|J_a f| > t\}) \leq (\|f\|_p/t)^p;$$

thus, if Mf denotes the Hardy-Littlewood maximal operator, $Mf(x) = \sup\{|f|_Q \colon x \in Q\}$, (1), (2) and the obvious inequalities $M(I_a f) \leq I_a(Mf)$, $M(J_a f) \leq J_a(Mf)$ imply that the complements of the Lebesgue sets of $I_a f$ and $J_a f$ have zero $R_{a,p}$ and $B_{a,p}$ capacity respectively.

Related to $B_{a,p}$ and $R_{a,p}$ is the H^{n-ap} Hausdorff measure: if $0 < r \leq \infty$ and $E \subset \mathbf{R}^n$ we define

$$H_r^{n-ap}(E) = \inf\left\{\sum_0^\infty |Q_i|^{1-ap/n}\right\},$$

the inf taken over all coverings of E by cubes of side $\leq r$; then $H^{n-ap}(E) = \sup_r H_r^{n-ap}(E)$. H^{n-ap} is finer than $B_{a,p}$ in the sense that $B_{a,p}(E) \leq CH_\infty^{n-ap}(E)$ [**12**]. Here we shall use H_∞^{n-ap} rather than H^{n-ap}; both have the same zero sets [**6**].

If $0 < a < n$, $1 \leq p < n/a$, and $f \in L^p$, we define

$$M_a f(x) = \sup\{|Q|^{a/n}|f|_Q : x \in Q\}.$$

LEMMA 1. *For the above a, p, and f, $H_\infty^{n-ap}(\{M_a f > t\}) \leq C(\|f\|_p/t)^p$.*

PROOF. For each $x \in E = \{M_a f > t\}$ there is a cube Q with $x \in Q$ and

$$t < |Q|^{a/n}|f|_Q \leq |Q|^{a/n-1/p}\left(\int_Q |f|^p\right)^{1/p};$$

hence, selecting [**16**, p. 9] a disjoint family $\{Q_i\}$ such that $E \subset \bigcup 5Q_i$ (rQ denotes the cube with same center as Q and side r times side (Q)), we have

$$H_\infty^{n-ap}(E) \leq C\sum |Q_i|^{1-ap/n} \leq Ct^{-p}\sum\int_{Q_i}|f|^p \leq C(\|f\|_p/t)^p.$$

Obviously, the same estimate holds with M_a replaced by $(M_{as}|f|^s)^{1/s}$, $1 < s \leq p$. Also, if we define for $0 < r \leq 1/100$ and $\varphi(t) = (\log 1/t)^{1-p}$, $H_r^\varphi(E) = \inf\{\sum \varphi(|Q_i|) : E \subset \bigcup Q_i,\ Q_i \text{ cubes, side } Q_i \leq r\}$ and the maximal operator $M_\varphi g(x) = \sup\{\int_Q |g|/\varphi(|Q|) : x \in Q, \text{ side } Q \leq 1/1000\}$, the above argument gives the estimate

$$H_{1/100}^\varphi(\{M_\varphi g > t\}) \leq C\|g\|_1/t.$$

LEMMA 2. *If $0 < b \leq a < n$, $1 \leq p < n/a$ and $f \in L^p$, then*

$$H_\infty^{n-(a-b)p}(\{I_a f > t\}) \leq C(\|f\|_p/t)^{p(n-(a-b)p)/(n-ap)}.$$

PROOF. The desired inequality follows from Lemma 1 once we prove

$$|I_a f(x)| \leq C\|f\|_p^{bp/(n-(a-b)p)} M_{a-b}f(x)^{1-bp/(n-(a-b)p)}; \tag{3}$$

now, as in [**11**, Theorem 1], we have for any $r > 0$

$$\begin{aligned} |I_a f(x)| &\leq C\left(\int_{|z|\leq r} + \int_{|z|>r}\right)|f(x-z)|\,|z|^{a-n}\,dz \\ &\leq C\sum_0^\infty (2^{-k}r)^{a-n}\int_{|z|\leq 2^{-k}r}|f(x+z)|\,dz + Cr^{a-n/p}\|f\|_p \\ &\leq C(r^b M_{a-b}f(x) + r^{a-n/p}\|f\|_p) \end{aligned}$$

and (3) follows if we choose $r = (M_{a-b}f(x)/\|f\|_p)^{1/(a-b-n/p)}$

LEMMA 3. *There is a constant C_I such that $M(I_a f) \leq C_I I_a f$ for all positive f. Also, there is a C_J such that $⨍_Q J_a f(x+z)\,dz \leq C_J J_a f(x)$ for all cubes Q centered at 0 with side ≤ 10 and all $f \geq 0$.*

PROOF. If Q has center 0, an easy computation gives $⨍_Q I_a(x+z)\,dz \leq C_I I_a(x)$; if moreover side $(Q) \leq 10$, $⨍_Q J_a(x+z)\,dz \leq C_J J_a(x)$ [**3**, p. 418]. The lemma now follows.

As a consequence, if $g \geq 0$ and $f = J_a g$, $mf(x) \leq Cf(x)$, where m denotes the "local" maximal operator $mf(x) = \sup\{|f|_Q : x \in Q, |Q| \leq 5^n\}$.

3. The spaces F_a^p. We fix $a > 0$, $m = [a]$ and p such that $1 \le p \le \infty$. We first show that Ef can be defined using a minimizing polynomial on each cube Q; in fact, if P_Qf denotes the unique polynomial in $\mathbf{P}_m$ such that for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{N}^n$ with $|\gamma| = \gamma_1 + \cdots + \gamma_n \le m$,

$$\int_Q (f(y) - P_Qf(y))y^\gamma \, dy = 0,$$

then [**8**, p. 17]

$$\text{(4)} \qquad \text{if } D^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_n)^{\gamma_n}, \ \operatorname*{ess\,sup}_Q |D^\gamma P_Q f| \le C|Q|^{-|\gamma|/n}|f|_Q;$$

it now follows that

$$\text{(5)} \qquad \text{for any } R \in \mathbf{P}_k, \qquad ⨍_Q |f - P_Qf| \le C ⨍_Q |f - R|,$$

and therefore, $Ef(x,t) \sim \sup\{ ⨍_Q |f - P_Qf| \colon x \in Q, \ |Q| = t^n\}$;

$$\text{(6)} \qquad \text{if } Q \subset Q', \qquad ⨍_Q |f - P_Qf| \le C(|Q'|/|Q|) ⨍_{Q'} |f - P_{Q'}f|;$$

in particular, if $Q_{x,t}$ denotes the cube with center x and side t,

$$G_af(x) \sim \sup_{t>0} t^{-a} ⨍_{Q_{x,t}} |f - P_{Q_{x,t}}f|;$$

also, balls can be used instead of cubes to define Ef and G_af.

Fix next $x \in Q$, $|Q| = t^n$ and let $Q_1 \subset Q_2 \subset \cdots \subset Q_k = Q$ be a sequence of cubes with $x \in Q_1$ and $|Q_{i+1}| = 2^n|Q_i|$, $i = 1, \ldots, k-1$; writing the polynomials $P_{Q_i}f$ as $P_{Q_i}f(y) = \sum_{|\gamma| \le m} c_\gamma(Q_i)(y-x)^\gamma/\gamma!$, we have by (4)

$$\text{(7)} \qquad \begin{aligned} |c_\gamma(Q_1) - c_\gamma(Q)| &\le \sum_1^{k-1} |c_\gamma(Q_i) - c_\gamma(Q_{i+1})| \\ &\le \sum |D^\gamma(P_{Q_i}f - P_{Q_{i+1}}f)(x)| \\ &\le C \sum (2^{-i}t)^{-|\gamma|} Ef(x, 2^{-i}t) \\ &\le C \int_{2^{-k}t}^t Ef(x,s)s^{-|\gamma|-1} \, ds; \end{aligned}$$

in particular, since $P_Qf(x) = c_0(Q)$ tends to $f(x)$ a.e. [**8**, p. 9], we have

$$\text{(8)} \qquad |f(x) - c_0(Q)| = |f(x) - P_Qf(x)| \le C \int_0^t Ef(x,s) \, ds/s.$$

Next, $C_a^p = \{f \in L^p \colon \|f\|_{a,p} = \|f\|_p + \|G_af\|_p\}$ is a Banach space [**8**, p. 37] and $F_a^p = \{f \in C_a^p \colon G_af(x,t) = o(1)\}$ can also be defined as the subspace of those $f \in C_a^p$ such that $\|G_af(\cdot,t)\|_p = o(1)$: indeed, since $G_af(x,t) \le G_af(x)$, if $f \in F_a^p$, $\|G_af(\cdot,t)\|_p = o(1)$ by dominated convergence; conversely, $\|G_af(\cdot,t)\|_p = o(1)$ implies that $G_af(x,t_j) = o(1)$ for some subsequence t_j, but then $f \in F_a^p$, for $G_af(x,t) \le G_af(x,t_j)$ if $t \le t_j$. Furthermore, it can be easily checked that F_a^p is a closed subspace of C_a^p.

Also, if a is not an integer and $f \in C_a^p$, for a.e. x there is a polynomial $P_x f \in \mathbf{P}_m$ such that [**8**, p. 32]

$$C' G_a f(x,t) \le S_a f(x,t) = \sup_{s \le t} s^{-a} \fint_{Q_{x,s}} |f - P_x f| \le C G_a f(x,t);$$

if $0 < a < 1$, $P_x f$ is the constant polynomial $f(x)$. Furthermore, setting $\chi_t = t^{-n}\chi_{Q_{0,t}}$, (4) gives for $t \ge 1$

$$t^{-a} Ef(x,t) \le C \int_1^\infty |f| * \chi_s(x) s^{-a-1}\, ds$$

and therefore,

$$\|\sup_{t \ge 1} t^{-a} Ef(x,t)\|_p \le C \int_1^\infty \|\,|f| * \chi_s\|_p s^{-a-1}\, ds \le C\|f\|_p;$$

as a consequence, $\|f\|_{a,p} \sim \|f\|_p + \|G_a(\cdot, 1)\|_p$.

THEOREM 4. *For all positive a and b, J_b is an isomorphism from C_a^p and F_a^p onto C_{a+b}^p and F_{a+b}^p respectively; that is, if $f \in F_{a+b}^p$ (C_{a+b}^p) there is a unique $g \in F_a^p$ (C_a^p) such that $f = J_b g$ and $\|f\|_{a+b,p} \sim \|g\|_{a,p}$.*

PROOF. Assuming $b < n$ (the general case follows by the semigroup property of J) we show first that $\|J_b f\|_{a+b,p} \le C\|f\|_{a,p}$. Fix $x \in \mathbf{R}^n$ and Q with $x \in Q$, $|Q| = t^n$; if $T(u,v) = \sum_{|\gamma| \le p} D^\gamma J_b(u) v^\gamma/\gamma!$ denotes the Taylor polynomial of degree $p = [a+b]$ of J_b at u, consider the polynomial in y

$$R_Q(y) = P_Q f * J_b(y) + \int_{c_{2Q}} (f(z) - P_Q f(z)) T(x-z, y-x)\, dz;$$

since $|D^\gamma J_b(u)| \le C(1 + |u|^{b-n-|\gamma|})e^{-|u|}$ [**5**, p. 192], R_Q is well defined and

$$\begin{aligned} |J_b f(y) - R_Q(y)| &\le \int_{2Q} |f(z) - P_Q f(z)| J_b(y-z)\, dz \\ &\quad + \int_{c_{2Q}} |f(z) - P_Q f(z)|\, |J_b(y-z) - T(x-z, y-x)|\, dz \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Clearly,

$$\fint_Q \mathrm{I}\, dy \le \int_{2Q} |f(z) - P_Q f(z)| \fint_Q J_b(y-z)\, dy \le C t^b Ef(x, 2t), \tag{9}$$

and by Taylor's formula and the fact that $|x - z + \theta(y-x)| \ge |x-z|/2$ if $\theta \le 1$ and $|x-z| \ge 2|x-y|$,

$$\begin{aligned} \mathrm{II} &\le C t^{p+1} \int_{c_{2Q}} |f(z) - P_Q f(z)| \cdot |x-z|^{b-n-p-1}\, dz \\ &\le C t^{p+1} \sum_0^\infty (2^k t)^{b-m-1} \left(Ef(x, 2^k t) + \operatorname*{ess\,sup}_{2^k Q} |P_{2^k Q} f - P_Q f| \right); \end{aligned}$$

writing $P_{2^kQ}f(z) = \sum_{|\gamma|\leq m} c_\gamma(2^kQ)(z-x)^\gamma/\gamma!$, (7) gives for $z \in 2^kQ$

$$|P_{2^kQ}f(z) - P_Qf(z)| \leq C\sum_0^m (2^kt)^j \int_t^{2^kt} Ef(x,s)s^{-j-1}\,ds,$$

which, since $a+b-p-1<0$, implies by Fubini's theorem

$$\text{(10)} \qquad \begin{aligned} \mathrm{II} &\leq Ct^{p+1}\left(\int_t^\infty s^{b-p-1}\left(Ef(x,s) + \sum_0^m s^j \int_t^s Ef(x,u)u^{-j-1}\,du\right)ds/s\right) \\ &\leq Ct^{p+1}\int_t^\infty s^{b-p-1}Ef(x,s)\,ds/s. \end{aligned}$$

Now, putting (9) and (10) together,

$$\begin{aligned} EJ_bf(x,t) &\leq C\left(t^bEf(x,2t) + t^{p+1}\int_t^\infty s^{b-p-1}Ef(x,s)ds/s\right) \\ &\leq Ct^{a+b}G_af(x), \end{aligned}$$

and thus, $\|J_bf\|_{a+b,p} \leq C\|f\|_{a,p}$. Also, if $f \in F_a^p$, given $\varepsilon > 0$ and T such that $G_af(x,T) \leq \varepsilon$,

$$\begin{aligned} t^{-a-b}EJ_bf(x,t) &\leq C\left(\varepsilon + t^{p+1-a-b}\left(\int_t^T + \int_T^\infty\right)(s^{b-p-1}Ef(x,s)\,ds/s)\right) \\ &\leq C(\varepsilon + (t/T)^{p+1-a-b}G_bf(x)) \leq C\varepsilon \end{aligned}$$

if t is small enough; hence $J_bf \in F_{a+b}^p$.

Next, if $f \in F_a^p$, $a > 1$, its weak partials $f_i = \partial f/\partial x_i$ verify $\|G_{a-1}f_i\|_p \leq C\|G_af\|_p$ [**8**, p. 42], and also $|f_i(x)| \leq C(G_af(x) + |f|_{Q_{x,1}})$ and

$$Ef_i(x,t) \leq C\left(\int_0^t M(Ef(\cdot,s))(x)s^{-2}\,ds + Ef(x,2t)/t\right)$$

[**9**, Theorem 3 and Lemma 1]; hence $f_i \in F_{a-1}^p$. This and the obvious imbeddings $F_a^p \subset F_{a-\varepsilon}^p$ imply that $I-\Delta$ maps F_a^p, $a>2$, into F_{a-2}^p and $\|(I-\Delta)f\|_{a-2,p} \leq C\|f\|_{a,p}$. Therefore, if $0<b<2$ and $f \in F_{a+b}^p$, $a>0$, $f = J_b(I-\Delta)J_{2-b}f = J_bg$, where $g \in F_a^p$ and $\|f\|_{a+b,p} \sim \|g\|_{a,p}$. The same argument works for the C_a^p and for a general $b>0$. The theorem follows by the semigroup properties of J.

PROPOSITION 1. *C_0^∞ is dense in F_a^p, $1 \leq p < \infty$.*

PROOF. Supposing first $0<a<1$, let $\varphi \geq 0$ be a C^∞ function with $\varphi(x) = 1$ when $|x| \leq 1/10$, $\varphi(x) = 0$ when $|x| \geq 1$ and $\int \varphi\,dx = 1$, and set $\varphi_r(x) = r^{-n}\varphi(x/r)$, $r>0$. If $f \in F_a^p$ and $f_r = f * \varphi_r(x)$, an easy computation yields $G_af_r(x,t) \leq C\varphi_r * G_af(\cdot,t)(x)$. Thus, given ε, if $\|G_af(\cdot,T)\|_p \leq \varepsilon$ and r is small enough, (4) implies

$$\begin{aligned} \|G_a(f-f_r)\|_p &\leq C\|G_af(\cdot,T)\|_p + C\left\|\int_T^\infty t^{-a}|f-f_r| * \chi_t(\cdot)\,dt/t\right\|_p \\ &\leq C\|G_af(\cdot,T)\|_p + C\int_T^\infty \|f-f_r\|_pt^{-a-1}\,dt \\ &\leq C\varepsilon + CT^{-a}\|f-f_r\|_p \leq C\varepsilon. \end{aligned}$$

Next, setting $\varphi^r(x) = \varphi(rx)$ and $f^r(x) = f(x)\varphi^r(x)$, where $f \in C^\infty \cap C_a^p$, it easily follows that

$$\fint_{|y|\le t} |f^r(x+y) - f^r(x)|\,dy \le C\|\varphi^r\|_\infty Ef(x,t) + t\|\nabla\varphi^r\|_\infty |f(x)|;$$

thus, given ε, if $T^{1-a}\|f\|_p \le \varepsilon$, $\|G_a f(\cdot,T)\|_p \le \varepsilon$, and r is small enough, we have

$$\begin{aligned}\|G_a(f-f^r)\|_p &\le C\|G_a f(\cdot,T)\|_p + CT^{1-a}\|f\|_p \\ &\quad + C\left\|\int_T^\infty |f-f^r| * \chi_t(\cdot)t^{-a-1}\,dt\right\|_p \\ &\le C\varepsilon + CT^{-a}\|f-f^r\|_p \le C\varepsilon.\end{aligned}$$

Hence, C_0^∞ is dense in F_a^p, which together with Theorem 4 implies the density of $C^\infty \cap F_a^p$ in F_a^p for all $a > 0$. Finally, the density of C_0^∞ in these F_a^p follows as before.

If $a > n/p$ functions in C_a^p are continuous [**8**, p. 74], whereas if $a \le n/p$ they have a considerable degree of integrability.

PROPOSITION 2. *If $1 \le p < n/a$, $q = np/n - ap$ and $f \in C_a^p$,*

$$\left(\fint_Q |f - P_Q f|^q\right)^{1/q} \le C|Q|^{a/n}\left(\fint_Q (G_a f)^p\right)^{1/p}$$

for any cube Q; if $p > 1$, $a = n/p$ and $p' = p/p-1$, there are constants C, β such that for any cube Q

$$\fint_Q \exp(\beta(|f - P_Q f|/\|G_a f\chi_Q\|_p)^{p'}) \le C.$$

This result, essentially proved in [**8**, Lemma 4.2] also follows easily by the Sobolev and Trudinger inequalities for Riesz potentials [**11**, **16**] from the next theorem.

THEOREM 5. *If $0 < r \le 1$, $a \le n/p$ and $f \in C_a^p$, then for any cube Q and a.e. $y \in Q$,*

$$|f(y) - P_Q f(y)|^r \le CI_{ar}(G_a f\chi_{4Q})^r(y). \tag{11}$$

PROOF. Denoting by $B_{y,s}$ the ball with center y and side s, and by S_{n-1} the unit sphere in $\mathbf{R}^n$, an easy modification of (8) together with (6), polar coordinates and Fubini's theorem give

$$\begin{aligned}|f(y) - P_Q f(y)|^r &\le C\int_0^t Ef(y,s)^r\,ds/s \\ &\le C\int_0^t \left(\fint_{B_{y,s}} Ef(z,s)^r\,dz\right)ds/s \\ &\le C\int_0^{2t} s^{ar}\fint_{B_{y,s}} G_a f(z)^r\,dz\,ds/s \\ &= C\int_0^{2t} s^{az-n}\int_0^s\int_{S_{n-1}} G_a f(x+uy')^r u^{n-1}\,du\,dy'\,ds/s \\ &\le C\int_0^{2t}\int_{S_{n-1}} u^{ar-n} G_a f(x+uy')^r u^{n-1}\,dy'\,du \\ &\le CI_{ar}(G_a f\chi_{4Q})^r(y).\end{aligned}$$

Observe that since $I_a \sim J_a$ near 0, $|f(y) - P_Q f(y)| \le CJ_a(G_a f \chi_{4Q})(y)$; also $\|P_Q f \chi_Q\|_\infty \le C|f|_Q$ tends to 0 if $|Q|$ tends to ∞, and hence, $|f| \le CI_a(G_a f)$ a.e. in $\mathbf{R}^n$, $0 < a < n/p$. Furthermore, if $1 \le s < q$, (11) implies $E_s f(x,t) \le Ct^a(M(G_a f)^r(x))^{1/r}$ for some $r < p$; therefore

$$\left\| \sup_t t^{-a} E_s f(\cdot, t) \right\|_p \sim \|G_a f\|_p, \tag{12}$$

which for the same s and $k \ge m$ extends to [**8**, p. 27]

$$\sup_t t^{-a} E_s^k f(x,t) \sim \sup_t t^{-a} E_s f(x,t). \tag{13}$$

Finally we note that since $||f(x+y)| - |f(x)|| \le |f(x+y) - f(x)|$, if $f \in C_a^p$ or F_a^p, $0 < a < 1$, so does $|f|$ and $\| \, |f| \, \|_{a,p} \le \|f\|_{a,p}$.

4. Tangential boundary values. We derive now Theorems 1 and 2 from pointwise estimates for the corresponding tangential maximal functions.

PROOF OF THEOREM 1. If $f \in F_a^p$ and $u(x,y) = P_y * f(x)$, define $T_{a,p}f(x) = \sup\{|u(z,y)| : (z,y) \in D_{a,p}(x)\}$; we will show

$$T_{a,p}f(x_0) \le C(Mf(x_0) + (M(G_a f)^p(x_0))^{1/p}); \tag{14}$$

obviously, (14) implies that $|\{T_{a,p}f > t\}| \le C(\|f\|_{a,p}/t)^p$, and standard arguments give then Theorem 1.

Suppose $x_0 = 0$; if $(x,y) \in D_{a,p}(0)$ and $Q = Q_{0,2|x|}$, we have

$$|u(x,y)| = \left| \left(\int_Q + \int_{^cQ} \right) f(z) P_y(x-z)\, dz \right| = \mathrm{I} + \mathrm{II};$$

if $z \in {}^cQ$, $|z - x| \ge |z|/2$ and $P_y(x-z) \le P_y(z/2)$; thus,

$$\mathrm{II} \le \int_{\mathbf{R}^n} |f(z)| P_y(z/2)\, dz \le CMf(0).$$

Next, by (4),

$$\begin{aligned} \mathrm{I} &\le \int_Q |f(z) - P_Q f(z)| P_y(x-z)\, dz + \int_Q |P_Q f(z)| P_y(x-z)\, dz \\ &\le \mathrm{III} + CMf(0) \int_Q P_y(x-z)\, dz \le \mathrm{III} + CMf(0). \end{aligned}$$

If $a < n/p$, $q = np/n - ap$ and $q' = q/q-1$, Hölder's inequality and Proposition 2 give

$$\begin{aligned} \mathrm{III} &\le \|P_y\|_{q'} \left(\int_Q |f - P_Q f|^q \right)^{1/q} \\ &\le Cy^{-n/q} |x|^{a+n/q} |x|^{-a} \left(\fint_Q |P_Q f|^q \right)^{1/q} \\ &\le Cy^{-n/q} |x|^{n/p} (M(G_a f)^p(0))^{1/p} \\ &\le C(M(G_a f)^p(0))^{1/p}, \end{aligned}$$

since $|x| < y^{p/q}$; thus, (14) is proved in this case.

If $a = n/p$, $p > 1$ and $p' = p/p-1$, we will use an Orlicz space version of Hölder's inequality: if $\phi(t) = t(\log(1+t))^{1/p'}$ and Ψ is its conjugate Orlicz function, then $\Psi(t) \leq Ce^{\alpha t^{p'}}$ for appropriate C and α, and therefore [**18**, p. 171]

$$\left| \fint_Q gh\,dz \right| \leq \|g\|_\phi \max\left(1, \fint_Q \Psi(h)\,dz\right) \leq C\|g\|_\phi \fint_Q e^{\alpha|h|^{p'}}\,dz,$$

where $\|g\|_\phi$ denotes the ϕ-Orlicz norm of $g\chi_Q$ with respect to $dz/|Q|$. This inequality and Proposition 2 imply

$$\begin{aligned} \mathrm{III} &\leq C\|G_a f\chi_Q\|_p|Q| \fint_Q P_y(x-z)\frac{|f(z)-P_Qf(z)|}{\|G_af\chi_Q\|_p}\,dz \\ &\leq C|x|^{n+n/p}(M(G_af)^p(0))^{1/p}\|P_y(x-\cdot)\|_\phi \fint_Q \exp\left(\beta\left(\frac{|f-P_Qf|}{\|G_af\chi_Q\|_p}\right)^{p'}\right)dz \\ &\leq C|x|^{n+n/p}(M(G_af)^p(0))^{1/p}\|P_y(x-\cdot)\|_\phi. \end{aligned}$$

But $\|P_y(x-\cdot)\|_\phi = \inf\{t\colon \fint_Q \phi(P_y(x-z)/t)\,dz \leq 1\}$ [**18**, p. 173]; thus, if $T = C_0(\log 1/y)^{1/p'}/|Q|$, C_0 to be fixed later, then $T \geq C_0/2^n|x|^n(\log 1/y)^{p/p'} \geq C_0 2^{-n}$, for $(x,y) \in D_{a,p}(0)$, and therefore,

$$\begin{aligned} &\fint_Q (P_y(x-z)/T)(\log(1+P_y(x-z)/T)^{1/p'}\,dz \\ &\qquad \leq \int_{2Q} P_y(z)(\log(1+c_ny^{-n}))^{1/p'}\,dz/T|Q| \\ &\qquad \leq C((\log 1/y)^{1/p'}/T|Q|)\int_{2Q} P_y(z)\,dz \leq 1 \end{aligned}$$

for an appropriate C_0. Hence, $\|P_y(x-\cdot)\|_\phi \leq T$ and we have

$$\begin{aligned} \mathrm{III} &\leq C|x|^{n+n/p}|x|^{-n}(\log 1/y)^{1/p'}(M(G_af)^p(0))^{1/p} \\ &\leq C(M(G_af)^p(0))^{1/p}. \end{aligned}$$

PROOF OF THEOREM 2. If $1 < p < n/a$, Theorem 5 and Lemma 3 imply that $Mf(x) \leq CI_aG_af(x)$, and it easily follows that f can be redefined in a zero measure set so that the complement of the Lebesgue set of the new f has zero $R_{a,p}$, and hence, $B_{a,p}$ capacity; clearly this implies nontangential convergence $B_{a,p}$-a.e. When $p = 1$, the embeddings $F_a^1 \subset F_{a-n/p'}^p \subset L_{a-n/p'-e}^p$, $1 < p < n/n-a$, $e > 0$ [**8**, pp. 72 and 58] tell us that any $f \in F_a^1$ can be redefined in a zero measure so that the complement of its Lebesgue set has zero $B_{a-n/p'-e,p}$ capacity and hence, zero $H^{(n-a)p+pe}$ Hausdorff measure [**12**]. Thus, for any $\varepsilon > 0$, we have nontangential convergence of $P_y * f$ for all x outside a set of zero $H^{n-a+\varepsilon}$ Hausdorff measure.

Next, if $0 < b < a$, fix $x_0 = 0$ and $(x,y) \in D_{b,p}(0)$. Proceeding as in the proof of Theorem 1, we obtain $|u(x,y)| \leq \mathrm{III} + CMf(0)$, and setting $r = np/n-bp$, Hölder's inequality and Theorem 5 imply

$$\begin{aligned}\mathrm{III} &\le Cy^{-n/r}|x|^{a+n/r}|x|^{-a}\left(\fint_Q |f-P_Qf|^q\right)^{1/q}\\ &\le Cy^{-n/r}|x|^{n/p+a-b}\left(\fint_Q (G_af)^p\right)^{1/p} \le C|x|^{a-b}\left(\fint_Q (G_af)^p\right)^{1/p}\\ &\le C(M_{(a-b)p}(G_af)^p(0))^{1/p},\end{aligned}$$

since $|x| \le y^{p/r}$. Also, $Mf \le CI_aG_af$ and therefore

$$T_{b,p}f(0) \le C(M_{(a-b)p}(G_af)^p(0))^{1/p} + CI_a(G_af)(0),$$

which by Lemmas 1 and 2 gives

$$\begin{aligned}H_\infty^{n-(a-b)p}(\{T_{b,p}f > t\}) &\le H_\infty^{n-(a-b)p}(\{M_{(a-b)p}(G_af)^p > t^p/C\})\\ &\quad + H_\infty^{n-(a-b)p}(\{I_aG_af > t/C\})\\ &\le C(\|G_af\|_p/t)^p + C(\|G_af\|_p/t)^{p(n-(a-b)p)/n-ap}\end{aligned}$$

and standard arguments finish now the proof of part (i).

In part (ii) we first divide $\mathbf{R}^n$ into a mesh of disjoint cubes of side $1/1000$. If x is in such a cube Q',

$$u(x,y) = \int P_y(x-z)(f\chi_{4Q'}(z) + f\chi_{c_{4Q'}}(z))\,dz = u_1(x,y) + u_2(x,y),$$

and since $|u_2(x,y)| \le Cy^{1/p'}\|f\|_p$ tends to 0 with y uniformly in $2Q'$, it is enough to study the convergence of u_1. Fix now $x_0 = 0$, $(x,y) \in D_{n/r,r}(0)$, $p < r < \infty$ and $Q = Q_{0,2|x|}$, and assume $0 \in Q'$ with Q' in the above mesh, and y small enough so that side $Q \le 1/1000$. Using again the Orlicz space version of Hölder's inequality, this time with the function $\phi(t) = t(\log(1+t))^{1/r'}$, we obtain as in Theorem 1

$$\begin{aligned}|u_1(x,y)| &\le \mathrm{III} + CM(f\chi_{4Q'})(0)\\ &\le C(\log 1/y)^{1/r'}\|G_{n/p}f\chi_Q\|_p + CM(f\chi_{4Q'})(0);\end{aligned}$$

now, Theorem 5, Lemma 3, and (4) give

$$\begin{aligned}M(f\chi_{4Q'})(0) &\le CM(I_{n/p}(G_{n/p}f\chi_{8Q'})(0) + C|f|_{8Q'}\\ &\le CI_{n/p}(G_{n/p}f\chi_{8Q'})(0) + CJ_{n/p}(|f|\chi_{8Q'})(0)\\ &\le CJ_{n/p}(G_{n/p}f + |f|)(0),\end{aligned}$$

and since $(x,y) \in D_{n/r,r}(0)$,

$$\begin{aligned}|u_1(x,y)| &\le C|x|^{n/r}\left(\int_Q (G_{n/p}f)^p\right)^{1/p} + CJ_{n/p}(G_{n/p}f + |f|)(0)\\ &\le C(M_{n-np/r}(G_{n/p}f)^p(0))^{1/p} + CJ_{n/p}(G_{n/p}f + |f|)(0).\end{aligned}$$

Thus, defining $T'_{s,t}f(x) = \sup\{|u_1(z,y)| \colon (z,y) \in D_{s,t}(x)\}$, we have

$$\begin{aligned}H_\infty^{np/r}(\{T'_{n/r,r}f > t\}) &\le H_\infty^{np/r}(\{M_{n-np/r}(G_{n/p}f)^p > t^p/C\})\\ &\quad + H_\infty^{np/r}(\{J_{n/p}(G_{n/p}f + |f|) > t/C\})\\ &= \mathrm{I} + \mathrm{II},\end{aligned}$$

where $\mathrm{I} \le C\|G_{n/p}f\|_p^p/t^p$, by Lemma 1. Also, since

$$J_{n/p}(G_{n/p}f + |f|) = J_{n/p-e}(J_e(G_{n/p}f + |f|)) \le I_{n/p-e}F,$$

with $F = J_e(G_{n/p}f + |f|)$, then, if $e < n/r$, Lemma 2 implies that

$$\mathrm{II} \le C(\|F\|_p/t)^{np/re} \le C(\|f\|_{n/p,p}/t)^{np/re}.$$

Convergence inside $D_{n/r,r}(x)$ for $H^{np/r}$-a.a. x follows now by standard arguments from these estimates.

Finally, if $0 \le b < n/p$ and $(x,y) \in D_{b,p}(0)$, $\log 1/y \le C \log 1/|x|$, and proceeding as before, we obtain

$$\begin{aligned} |u_1(x,y)| &\le C((\log 1/|x|)^{p-1} \int_Q (G_{n/p}f)^p)^{1/p} + CM(f\chi_{4Q'})(0) \\ &\le C(M_\varphi(G_{n/p}f)^p(0))^{1/p} + CJ_{n/p}(G_{n/p}f + |f|)(0), \end{aligned}$$

with M_φ as in §2, and since $B_{n/p,p} \le CH^\varphi_{1/100}$ [**12**], it follows that

$$\begin{aligned} B_{n/p,p}(\{T'_{b,p}f > t\}) &\le CH^\varphi_{1/100}(\{M_\varphi(G_{n/p}f)^p > t^p/C\}) \\ &\quad + B_{n/p,p}(\{J_{n/p}(G_{n/p}f + |f|) > t/C\}) \\ &\le C(\|f\|_{n/p,p}/t)^p, \end{aligned}$$

and again standard arguments can be used to finish the proof.

5. Strong L^p estimates. The proof of Theorem 3 depends on a strong inequality for a certain capacity type set function associated to C_a^p which we now define. Fix a, b and p with $0 < b < a$ and $1 \le p < \infty$, and for any $E \subset \mathbf{R}^n$ denote by $O^b_{a,p}(E)$ the set $\{g \in C_b^p : g \ge 0, J_{a-b}g \ge \chi_E\}$; we define then

$$U^b_{a,p}(E) = \inf\{(\|g\|_{b,p})^p : g \in O^b_{a,p}(E)\}.$$

Obviously, $U^b_{a,p}(E) \le U^b_{a,p}(E')$ if $E \subset E'$ and $U^b_{a,p}(E \cup F) \le C(U^b_{a,p}(E) + U^b_{a,p}(F))$; furthermore, by Theorem 4, if $g \in C_b^p$,

$$U^b_{a,p}(\{J_{a-b}(g) > t\}) \le (\|g\|_{b,p}/t)^p \le C(\|J_{a-b}g\|_{a,p}/t)^p; \tag{15}$$

it can also be proved that $U^b_{a,p} \sim U^{b'}_{a,p}$ if $0 < b, b' < a$ and that $R_{a,p} \le CU^b_{a,p}$, $a < n/p$; since we clearly have $U^b_{a,p} \le CB_{a,p}$, it follows that $R_{a,p}$, $B_{a,p}$ and $U^b_{a,p}$ have the same zero sets when $a < n/p$ (Lemma 3 and (15) imply that any $f \in F_a^p$ can be modified in a set of measure zero so that the complement of its Lebesgue set has zero $U^b_{a,p}$ "capacity", and hence zero $B_{a,p}$ capacity if $1 < p < \infty$).

$U^b_{a,p}$ satisfies the following strong type inequality.

THEOREM 6. *If $0 < b < a$, $1 \le p < \infty$, and $g \in C_b^p$, $g \ge 0$, then*

$$\int_0^\infty s^{p-1} U^b_{a,p}(\{J_{a-b}g > s\})\, ds \le C(\|J_{a-b}g\|_{a,p})^p.$$

Once this is proved, Theorem 3 is deduced as in [**14**]: given $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^{n+1}_+$ set $S(A) = \mathbf{R}^{n+1}_+ - \bigcup\{C(x) : x \notin A\}$, with $C(x) = \{(z,y) \in \mathbf{R}^{n+1}_+ : |z-x| \le y\}$, and $J(B) = \{x \in \mathbf{R}^n : B \cap D_{a,p}(x) \ne \varnothing\}$; then, if $g \in O^b_{a,p}(E)$, and $g' = J_{a-b}g$, $J(S(E)) \subset \{T_{a,p}g' \ge C_0\}$ for some numerical C_0, and the weak inequalities of

Theorem 1 give $|J(S(E))| \leq C(\|g\|_{b,p})^p$. Thus, taking inf in $O^b_{a,p}(E)$, we obtain $|J(S(E))| \leq CU^b_{a,p}(E)$. Now, if $f \in C^p_a$,

$$\{T_{a,p}f > s\} \subset J(S(\{Nf > s/C\})),$$

where $Nf(x) = \sup\{|u(x,y)| \colon y > 0\}$; writing $f = J_{a-b}(g)$, $g \in C^p_b$, $0 < b < 1$, we have $Nf(x) \leq J_{a-b}(N(|g|))(x)$, but

$$\int |g|(x' - z)P_y(z)\,dz = \int_{|z|\leq 2} + \int_{|z|>2} = \mathrm{I} + \mathrm{II},$$

and standard arguments give $\mathrm{I} \leq Cm(|g|)(x')$, which since $|g| = J_{b-e}h$, $h \in C^p_e$, implies $\mathrm{I} \leq Cm(J_{b-e}|h|)(x') \leq CJ_{b-e}|h|(x')$, by Lemma 3. Also, if $y \leq 1$ and $|z| \geq 2$, $P_y(z) \leq C/(1+|z|)^{n+1} = Q(z)$ and $\mathrm{II} \leq Q * |g|(x')$, which belongs to C^p_b. Therefore $Nf \leq C(J_{a-e}|h| + J_{a-b}(Q * |g|))$, and since $\|Q * |g|\,\|_{b,p} \leq C\|f\|_{a,p}$ and $\|\,|h|\,\|_{e,p} \leq C\|f\|_{a,p}$, Theorem 6 gives

$$\begin{aligned}
\|T_{a,p}f\|^p_p &= p\int_0^\infty s^{p-1}|\{T_{a,p}f > s\}|\,ds \\
&\leq C\int_0^\infty s^{p-1}U^b_{a,p}(\{Nf > s/C\})\,ds \\
&\leq C\int_0^\infty s^{p-1}(U^b_{a,p}(\{J_{a-b}(J_{b-e}|h|) > s\})) \\
&\qquad\qquad + (U^b_{a,p}(\{J_{a-b}(Q * |g|) > s\}))\,ds \\
&\leq C(\|\,|h|\,\|^p_{e,p} + \|Q * |g|\,\|^p_{b,p}) \leq C\|f\|^p_{a,p}.
\end{aligned}$$

Our proof of Theorem 6 is an adaptation and simplification of the one given by Dahlberg in [**7**]; we need a preliminary lemma.

LEMMA 4. *Let h be a C^∞ function with $h(t) = 0$ if $t < 0$, and $|t^{j-1}h^{(j)}(t)| \leq A$, $0 \leq j \leq m+1$, $m = [a]$. Then if $f = J_{a-b}g$, with $g \in C^p_b$, and $g \geq 0$, $h(f) \in C^p_a$ and $\|h(f)\|_{a,p} \leq C\|f\|_{a,p}$.*

PROOF. We will estimate $G_a h(f)(x,1)$ using the centered version of E (see §3). Fix $x \in \mathbf{R}^n$ and write Q for $Q_{x,t}$, where we assume $t \leq 1$. Writing $P_Q f(y) = \sum_{|\gamma|\leq m} c_\gamma(t)(y-x)^\gamma/\gamma!$, define

$$R(y) = f(x) + \sum_{0<|\gamma|\leq m} c_\gamma(t)(y-x)^\gamma/\gamma!;$$

by (4) and Lemma 3 we have

$$\begin{aligned}
|R(y)| &\geq f(x) - C\sum_{0<|\gamma|\leq m} f_Q(|y-x|/t)^{|\gamma|} \\
&\geq f(x)\left(1 - C\sum_{0<|\gamma|\leq m}(|y-x|/t)^{|\gamma|}\right);
\end{aligned}$$

thus, if $|x-y| < \varepsilon t$ with ε small enough, $R(y) > Cf(x)$. Considering now the polynomial

$$S(y) = \sum_0^m h^{(j)}(R(x))(R(y) - R(x))^j/j!,$$

Taylor's formula and (8) give

$$\begin{aligned}|h(f(y)) - S(y)| &\leq |h(f(y)) - h(R(y))| + |h(R(y)) - S(y)| \\ &\leq A|f(y) - R(y)| + C|R(y) - R(x)|^{m+1} \\ &\quad \times |h^{(m+1)}((1-\theta)R(x) + \theta R(y))| \\ &\leq C(|f(y) - P_Q f(y)| + t^a G_a f(x) \\ &\qquad + |R(y) - R(x)|^{m+1}/f(x)^m).\end{aligned}$$

Set now $T = (f(x)/G_a f(x))^{1/a}$ and suppose $T \leq 1$; since

$$(R(y) - R(x))^{m+1} = \sum_{j=m}^{m(m+1)} \sum_{|\gamma|=j} c_\gamma (y-x)^\gamma,$$

where c_γ equals the sum of all terms $c_{\gamma_1}(t) \cdots c_{\gamma_{m+1}}(t)$ with $\gamma_1 + \cdots + \gamma_{m+1} = \gamma$, then (7) and (4) imply for $0 < t \leq T$ that

$$\begin{aligned}|c_{\gamma_i}(t)| &\leq CT^{-|\gamma_i|} f_{Q_{x,T}} + CT^{a-|\gamma_i|} G_a f(x) \\ &\leq CT^{-|\gamma_i|}(f(x) + T^a G_a f(x)) \leq CT^{-|\gamma_i|} f(x),\end{aligned}$$

if $|\gamma_i| < a$, or

$$\begin{aligned}|c_{\gamma_i}(t)| &\leq CT^{-|\gamma_i|} f(x) + C\log(T/t) G_a f(x) \\ &\leq CT^{-|\gamma_i|} \log(T/t) f(x),\end{aligned}$$

if $|\gamma_i| = a$. In any case,

$$|R(y) - R(x)|^{m+1} \leq Cf(x)^{m+1} (\log eT/t)^{m+1} \sum_{m+1}^{m(m+1)} (t/T)^j,$$

which implies

$$\begin{aligned}&E^{m(m+1)} h(f)(x,t) \\ &\quad \leq C(Ef(x,t) + t^a G_a f(x) + f(x)(\log eT/t)^{m+1} \sum_{m+1}^{m(m+1)} (t/T)^j),\end{aligned}$$

and therefore

$$\sup_{t \leq T} t^{-a} E^{m(m+1)} h(f)(x,t) \leq C(G_a f(x) + f(x) T^{-a}) \leq CG_a f(x). \tag{16}$$

If $T \leq t \leq 1$, then $E^{m(m+1)} h(f)(x,t) \leq Cf_Q \leq Cf(x)$, by (4) and Lemma 3, and we have

$$\sup_{T \leq t \leq 1} t^{-a} E^{m(m+1)} h(f)(x,t) \leq CT^{-a} f(x) \leq CG_a f(x). \tag{17}$$

In the case $T > 1$, then $f(x) > G_a f(x)$ and we estimate the coefficients $c_{\gamma_i}(t)$ as $|c_{\gamma_i}(t)| \leq C(f(x) + \log(e/t) G_a f(x)) \leq C \log(e/t) f(x)$, and replace (16) by

$$\begin{aligned}\sup_{t \leq 1} t^{-a} E^{m(m+1)} h(f)(x,t) &\leq C(G_a f(x) + f(x) \sum_{m+1}^{m(m+1)} \sup_{t \leq 1} t^{j-a} \log(e/t)) \\ &\leq Cf(x).\end{aligned} \tag{18}$$

Thus, (16), (17), and (18) yield for a.e. x

$$G_a h(f)(x,1) \le C(G_a f(x) + f(x)) \tag{19}$$

and, since $h(f) \le Af$, we conclude that $\|h(f)\|_{a,p} \le C\|f\|_{a,p}$.

To finish the proof of Theorem 6 fix a, b, p and write U instead of $U^b_{a,p}$. As in [**2** or **7**], if h is a C^∞ function with $h(t) = 0$ if $t < 0$, $h(t) = 1$ if $t > 1$, define for any integer j $h_j(t) = 2^j h(2^{2-j}t - 1)$ and $f_j = h_j(f)$. By Lemma 4 $f_j \in C^p_a$, and since $f_j(x) = 2^j$ if $f(x) > 2^j$, (15) gives

$$\int_0^\infty s^{p-1} U(\{f > s\})\, ds \le C \sum_{-\infty}^{\infty} 2^{jp} U(\{f_j \ge 2^j\}) \le C \sum_{-\infty}^{\infty} (\|f_j\|_{a,p})^p.$$

Now, the h'_j have disjoint support and are uniformly bounded; therefore,

$$\begin{aligned} \sum_{-\infty}^{\infty} (f_j(x))^p &= \sum_{-\infty}^{\infty} \left(\int_0^{f(x)} h'_j(s)\, ds \right)^p \\ &\le f(x)^{p-1} \sum_{-\infty}^{\infty} \int_0^{f(x)} |h'_j(s)|\, ds \le C f(x)^p, \end{aligned} \tag{20}$$

and $\sum \|f_j\|_p^p \le C\|f\|_p^p$. Fix $x \in \mathbf{R}^n$ and denote $Q_{x,t}$ as Q; if $t \le 1$, (4) and Lemma 3 give $f(x) \le |f(x) - P_Q f(x)| + C f_Q \le C t^a G_a f(x) + C' f(y)$ for any $y \in Q$; thus,

$$f(y) \ge (f(x) - C t^a G_a f(x))/C' > f(x)/2C'$$

if $t \le T = (\varepsilon f(x)/G_a f(x))^{1/a}$ with ε small enough. Now $C' \sim 2^K$ for some K independent of f or x, and, when $f(x) > 2^{j+K+1}$, $f(y) > 2^j$ on Q and $f_j(y) = 2^j$. Hence, using again the centered version of E,

$$E f_j(x,t) \le \fint_Q |f_j(y) - f_j(x)|\, dy = 0.$$

If $t \ge \min(1, T)$, then $E f_j(x,t) \le C(f_j)_Q \le C 2^j = C f_j(x)$ and therefore

$$G_a f_j(x,1) \le \sup_{t > \min(1,T)} t^{-a} E f_j(x,t) \le C(f_j(x) + T^{-a} f_j(x)),$$

and (20) gives ($\sim$ means the index set equals the preceding one)

$$\begin{aligned} \sum_{f(x) > 2^{j+K+1}} G_a f_j(x,1)^p &\le C \sum_{\sim} (f_j(x)^p + T^{-ap} f_j(x)^p) \\ &\le C(f(x)^p + G_a f(x)^p). \end{aligned} \tag{21}$$

Suppose next $f(x) < 2^{j-3}$; if $t \le 1$ and we set as before $R(y) = f(x) + P_Q f(x) - c_0(t)$,

$$\begin{aligned} |R(y)| &\le f(x) + C \sum_{0 < |\gamma| \le m} f_Q (|y - x|/t)^{|\gamma|} \le f(x) \left(1 + C \sum_{\sim} (|y - x|/t)^{|\gamma|} \right) \\ &\le 2 f(x) \end{aligned}$$

if $|x-y| \leq \varepsilon t$ with ε small enough. Hence $h_j(R(y)) = 0$ and, setting $S = \{y \in Q\colon f(y) > 2^{j-2}\}$, (4) implies

$$\begin{aligned} Ef_j(x,\varepsilon t) &\leq C \fint_Q f_j(y)\,dy = Ct^{-n}\int_S h_j(f(y))\,dy \\ &\leq Ct^{-n}\int_S |h_j(f(y)) - h_j(R(y))|\,dy \\ &\leq C\left(\fint_Q |f-R|^s\right)^{1/s}(t^{-n}|S|)^{1-1/s} \\ &\leq C(E_sf(x,t) + t^aG_af(x))(2^{-j}f(x))^{1-1/s}, \end{aligned}$$

for, by Lemma 3, $|S| \leq C2^{-j}\int_Q f\,dz \leq C2^{-j}t^nf(x)$. Thus,

$$G_af_j(x,1) \leq C\left(\sup_t t^{-a}E_sf(x,t) + G_af(x)\right)(2^{-j}f(x))^{1-1/s},$$

and since $\sum_{f(x)<2^{j-3}} 2^{-jp(1-1/s)} \leq Cf(x)^{-p(1-1/s)}$,

$$\sum_{f(x)<2^{j-3}} G_af_j(x,1)^p \leq C\left(\sup_t t^{-a}E_sf(x,t)\right). \tag{22}$$

By (19), we estimate the remaining $K+4$ f_j as $G_af_j(x,1) \leq C(G_af(x)+f(x))$, which with (21) and (22) gives

$$\sum_{-\infty}^{\infty} G_af_j(x,1)^p \leq C\left(\left(\sup_t t^{-a}E_sf(x,t)\right)^p + f(x)^p\right);$$

taking now (12) and (20) into account, we obtain

$$\sum_{-\infty}^{\infty} \|G_af_j(\cdot,1)\|_p^p \leq C(\|G_af\|_p^p + \|f\|_p^p) \leq C\|f\|_{a,p}$$

and the proof of Theorem 6 is finished.

6. Further remarks. We discuss here the imbeddings of the Triebel-Lizorkin spaces in F_a^p. These spaces are usually defined as follows [**17**]: let ψ be a function in Schwartz's class $\mathcal{S}$ such that $\Psi = \hat{\psi} \geq 0$ and $\operatorname{supp}\Psi \subset \{z\colon 1/2 \leq |z| \leq 2\}$, and set $\psi_t(z) = t^{-n}\psi(z/t)$; then $F_a^{p,q}$, $a > 0$, $1 \leq p,q \leq \infty$ is the space of those L^p functions such that

$$D_{a,p}f(x) = \left(\int_0^\infty (t^{-a}|f * \psi_t(x)|)^q\,dt/t\right)^{1/q}$$

is in L^p. With the norm $\|f\|_{a,p,q} = \|f\|_p + \|D_{a,q}f\|_p$, $F_a^{p,q}$ becomes a Banach space, and as mentioned before, if $1 < p < \infty$, $F_a^{p,2} = L_a^p$, and $F_a^{1,2} = J_a(h^1)$.

The extension of Theorems 1, 2 and 3 to the $F_a^{p,q}$ is a consequence of

PROPOSITION 3. *If $1 \leq p,q < \infty$, $F_a^{p,q}$ is continuously imbedded in F_a^p.*

PROOF. If $0 < a < 1$, $f \in F_a^{p,q}$ iff

$$S_{a,q}f(x) = \left(\int_0^\infty \left(t^{-a}\int_{|y|\leq 1} |f(x+ty) - f(x)|\,dy\right)^q dt/t\right)^{1/q}$$

is in L^p and $\|D_{a,q}f\|_p \sim \|S_{a,q}f\|_p$ [**17**, p. 108]. But then $G_a f(x) \le CS_{a,q}f(x)$ (see §3) and therefore, $\|f\|_{a,p} \le C\|f\|_{a,p,q}$. The general case is reduced to this one by Theorem 4 and the fact [**17**, p. 58] that the Bessel operator J_b is an isomorphism between $F_a^{p,q}$ and $F_{a+b}^{p,q}$ (in fact it can be shown that $f \in F_a^{p,q}$, $1 \le p, q < \infty$, $a > 0$ iff

$$G_{a,q}f(x) = \left(\int_0^\infty (t^{-a}Ef(x,t))^q \, dt/t\right)^{1/q}$$

is in L^p, and $\|f\|_{a,p,q} \sim \|f\|_p + \|G_{a,q}\|_p$).

As a consequence, Theorems 1, 2, and 3 also hold for certain Besov spaces (see [**16**, **17**] for their definition): indeed, if $1 \le r \le p$, $B_a^{p,r}$ is continuously imbedded in $F_a^{p,r}$ [**17**, p. 47]. If $r > p$, the methods used here do not apply to $B_a^{p,r}$, although the embeddings $B_a^{p,r} \subset L_{a-\varepsilon}^p$ yield convergence of the Poisson integral of $f \in B_a^{p,r}$ inside any region $D_{a-\varepsilon,p}$, $\varepsilon > 0$; since $C_a^p \subset L_{a-\varepsilon}^p$, the same is true of C_a^p.

REFERENCES

1. D. R. Adams, *Quasiadditivity and sets of finite L^p capacity*, Pacific J. Math. **79** (1978), 283–291.
2. ——, *On the existence of capacitary strong type estimates in $\mathbf{R}^n$*, Ark. Mat. **14** (1976), 125–140.
3. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*. I, Ann. Inst. Fourier **11** (1961), 385–475.
4. A. P. Calderón and R. Scott, *Sobolev type inequalities for $p > 0$*, Studia Math. **62** (1978), 75–92.
5. A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171–225.
6. L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, 1967.
7. B. Dahlberg, *Regularity properties of Riesz potentials*, Indiana Univ. Math. J. **28** (1979), 257–268.
8. R. Devore and R. Sharpley, *Maximal operators and smoothness*, Mem. Amer. Math. Soc., No. 293 (1984).
9. J. R. Dorronsoro, *A characterization of potential spaces*, Proc. Amer. Math. Soc. **95** (1985), 21–31.
10. K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. **45** (1979), 77–102.
11. L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.
12. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue spaces*, Math. Scand. **26** (1970), 255–292.
13. Y. Mizuta, *On the boundary limits of harmonic functions with gradient in L^p*, Ann. Inst. Fourier (Grenoble) **34** (1984), 99–109.
14. A. Nagel, W. Rudin and J. Shapiro, *Tangential boundary behavior of functions in Dirichlet type spaces*, Ann. of Math. (2) **116** (1982), 331–360.
15. A. Nagel and E. M. Stein, *On certain maximal functions and approach regions*, Adv. in Math. **54** (1984), 83–106.
16. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.
17. H. Triebel, *Theory of function spaces*, Birkhäuser, 1983.
18. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, 1959.

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