

## Poisson processes and a Bessel function integral

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## POISSON PROCESSES AND A BESSEL FUNCTION INTEGRAL\*

F. W. STEUTEL†

**Abstract.** The probability of winning a simple game of competing Poisson processes turns out to be equal to the well-known Bessel function integral  $J(x, y)$  (cf. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962). Several properties of  $J$ , some of which seem to be new, follow quite easily from this probabilistic interpretation. The results are applied to the random telegraph process as considered by Kac [Rocky Mountain J. Math., 4 (1974), pp. 497–509].

**Key words.** Poisson process, Bessel function, random telegraph

**1. Competing Poisson processes.** Several problems can be described as follows: An object has to travel a distance  $x$ ; it does so at unit speed, but it is obstructed at random moments and then held for a random period of time before it is allowed to continue. The object may be a particle moving between two electrodes, a person walking to a bus stop, or, as in [5, Problem 147], a book being read with random interruptions. The question is: What is the probability that the object reaches its destination at a moment not exceeding  $x + y$ ? The situation may be modelled as a game of two competing (Poisson) renewal processes in the following way (see Fig. 1):

Let  $X_1, Y_1, X_2, Y_2, \dots$  be independent, exponentially distributed random variables with expectation one. Two persons,  $X$  and  $Y$ , take turns drawing lengths  $X_j$  and  $Y_j$ . Person  $X$  starts, and wins if the sum of his  $X_j$  exceeds  $x$  before the sum of  $Y$ 's  $Y_j$  exceeds  $y$ .

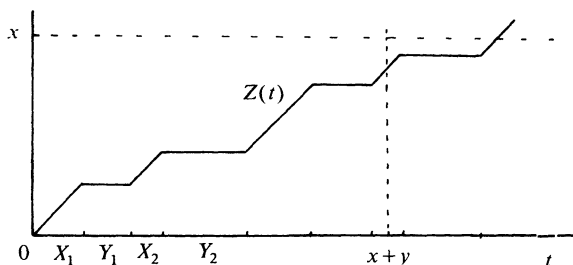


FIG 1.  $N_x = 5, N_y = 3; X$  loses.

More formally, if  $N_x$  and  $N_y$  are random variables defined by

$$N_x = \min\{n; X_1 + \dots + X_n > x\},$$

$$N_y = \min\{n; Y_1 + \dots + Y_n > y\},$$

then (remember that  $X$  starts)

$$(1) \quad X \text{ wins} \Leftrightarrow N_x \leq N_y \Leftrightarrow X_1 + \dots + X_{N_y} > x.$$

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*Remark.* For our purposes the assumption that  $EX_j = EY_j = 1$  for  $j = 1, 2, \dots$ , is no restriction: replacing  $X_j$  and  $Y_j$  by  $X_j/\lambda$  and  $Y_j/\mu$ , respectively, is equivalent to replacing  $x$  and  $y$  by  $\lambda x$  and  $\mu y$ , respectively. The process  $Z(t)$  depicted in Fig. 1, representing the distance travelled by the object at time  $t$ , would, of course, be changed by a transformation of the  $X_j$  and  $Y_j$ .

We shall use the following two well-known facts:  $N_y - 1$  has a Poisson distribution with mean  $y$ , i.e.,

$$(2) \quad P(N_y = n) = e^{-y} \frac{y^{n-1}}{(n-1)!} \quad (n = 1, 2, \dots),$$

and  $X_1 + \dots + X_n$  has a gamma distribution with density

$$(3) \quad \frac{d}{dx} P(X_1 + \dots + X_n \leq x) = e^{-x} \frac{x^{n-1}}{(n-1)!} \quad (x > 0).$$

Now, let  $J(x, y)$  be defined by (cf. Luke [4, p. 271])

$$(4) \quad J(x, y) = 1 - e^{-y} \int_0^x I_0(2\sqrt{yt}) e^{-t} dt,$$

where  $I_0$  is the modified Bessel function of order zero:

$$(5) \quad I_0(z) = \sum_0^{\infty} \frac{(z/2)^{2n}}{(n!)^2}.$$

Then we easily obtain

PROPOSITION 1.

$$(6) \quad P(N_x \leq N_y) = J(x, y).$$

*Proof.* By (1)–(5) we have

$$\begin{aligned} P(N_x \leq N_y) &= 1 - P(N_x > N_y) = 1 - P(X_1 + \dots + X_{N_y} \leq x) \\ &= 1 - \sum_{n=1}^{\infty} P(N_y = n, X_1 + \dots + X_n \leq x) \\ &= 1 - \sum_{n=1}^{\infty} e^{-y} \frac{y^{n-1}}{(n-1)!} \int_0^x e^{-t} \frac{t^{n-1}}{(n-1)!} dt \\ &= 1 - e^{-y} \int_0^x I_0(2\sqrt{yt}) e^{-t} dt = J(x, y). \end{aligned}$$

*Remark.* Srivastava and Kashyap [6, pp. 77, 78] consider an equivalent interpretation, in the context of a randomized random walk; there the interpretation remains implicit and is not pursued.

**2. Properties of  $J(x, y)$ .** Several properties of  $J(x, y)$  follow immediately from (6). We list the following six together with their simple proofs.

- (i)  $J(0, y) = P(X_1 > 0) = 1$ ,
- (ii)  $J(x, 0) = P(X_1 > x) = e^{-x}$ .

From (2) and its counterpart for  $N_x$  (independent of  $N_y$ ) it follows that

$$\begin{aligned} P(N_x = N_y) &= \sum_1^{\infty} P(N_x = n, N_y = n) \\ &= \sum_1^{\infty} e^{-x} \frac{x^{n-1}}{(n-1)!} e^{-y} \frac{y^{n-1}}{(n-1)!} = e^{-x-y} I_0(2\sqrt{xy}). \end{aligned}$$

From this we conclude using (6) that

(iii)  $J(x, y) + J(y, x) = 1 + P(N_x = N_y) = 1 + e^{-x-y} I_0(2\sqrt{xy})$ ,  
and especially

(iv)  $J(x, x) = \frac{1}{2} + \frac{1}{2} e^{-2x} I_0(2x)$ .

Conditioning on  $X_1 = u$ , with density  $e^{-u}$ , we have

$$P(N_x \leq N_y) = \int_0^x (1 - P(N_y \leq N_{x-u})) e^{-u} du + \int_x^{\infty} e^{-u} du,$$

or in view of (5)

(v)  $J(x, y) = 1 - \int_0^x J(y, x-u) e^{-u} du$ ,

which seems to be new. Rewriting (v) as

$$e^x J(x, y) = e^x - \int_0^x J(y, v) e^v dv,$$

and differentiating with respect to  $x$ , using (4) we recover (iii):

(vi)  $\frac{\partial}{\partial x} J(x, y) = 1 - J(x, y) - J(y, x) = -e^{-x-y} I_0(2\sqrt{xy})$ .

Several other relations given in [4] are easily obtained from (i)–(vi). In §3 we collect some asymptotic results.

**3. Asymptotics.** From the probabilistic interpretation the following limit relations are quite obvious (it is easy to give estimates; also compare (v)):

$$\begin{aligned} \lim_{x \rightarrow \infty} J(x, y) &= \lim_{x \rightarrow \infty} P(N_x \leq N_y) = 0, \\ \lim_{y \rightarrow \infty} J(x, y) &= \lim_{y \rightarrow \infty} P(N_x \leq N_y) = 1. \end{aligned}$$

For both  $x$  and  $y$  large we have the following very simple relation, which seems related to expansions in [2] involving the error function, but which seems to be new in this form. Its proof is a simple consequence of the asymptotic normality of Poisson random variables with large means.

**PROPOSITION 2.** For  $x \rightarrow \infty$  and  $y \rightarrow \infty$

(7)  $J(x, y) = \Phi\left(\frac{y-x+1/2}{\sqrt{x+y}}\right) + O\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right),$

where  $\Phi$  is the standard normal distribution function defined as

$$\Phi(u) = (2\pi)^{-1/2} \int_{-\infty}^u e^{-v^2/2} dv.$$

*Proof.*

$$J(x, y) = P(N_x - N_y \leq 0) = P(N_x - N_y < \frac{1}{2}),$$

where the  $\frac{1}{2}$  is the usual “continuity correction”. As  $N_x - N_y$  is asymptotically normal with mean  $x - y$  and variance  $x + y$ , it follows that

$$(8) \quad J(x, y) = P\left(\frac{N_x - N_y - x + y}{\sqrt{x + y}} \leq \frac{y - x + 1/2}{\sqrt{x + y}}\right) \approx \Phi\left(\frac{y - x + 1/2}{\sqrt{x + y}}\right).$$

That  $J(x, y)$  actually satisfies (7) follows easily from the Berry–Esseen version of the central limit theorem (Feller [1, p. 542]).

*Remark.* Relation (7), of course, also holds without the term  $\frac{1}{2}$ . In practice the approximation (8) is much better than is suggested by (7). For values of  $x$  and  $y$  of 10 and higher it yields a result correct to about three decimal places. Two examples:  $x = 10$  and  $y = 20$  yields  $J(10, 20) = 0.974206$  and  $\phi(10.5/\sqrt{30}) = \Phi(1.917) = 0.972$ . For  $x = y = 50$  we find  $J(50, 50) = 0.519972$  and  $\Phi(0.5/\sqrt{10}) = \Phi(0.05) = 0.5199$ . The abundance of tables of  $\Phi$  makes the approximation (8) quite practical. To obtain good (proven) bounds is not so easy.

**4. Relation with Kac’s random telegraph model.** In [3] Kac considers an (integrated) telegraph process  $X(t)$  (in his formula (25) denoted by  $x(t)$ ) that is closely related to the process  $Z(t)$  of Fig. 1. The process  $X(t)$  is constructed from the same  $X_j$  and  $Y_j$  as  $Z(t)$ ; its graph is sketched in Fig. 2. Evidently, the processes  $Z(t)$  and  $X(t)$  are related by

$$(9) \quad Z(t) = \frac{1}{2}(X(t) + t).$$

From Fig. 1 we immediately see that

$$Z(x + y) > x \Leftrightarrow N_x \leq N_y,$$

and therefore by Proposition 1 we have, in view of (9),

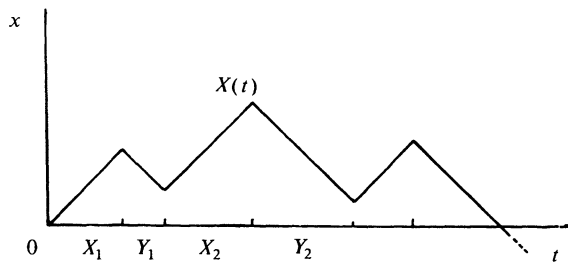


FIG. 2

**PROPOSITION 3.** Let  $F(x, t) = P(X(t) \leq x)$  be the distribution function of  $X(t)$ . Then for  $0 \leq x \leq t$

$$(10) \quad F(x, t) = 1 - J\left(\frac{t+x}{2}, \frac{t-x}{2}\right).$$

From Proposition 2 we then obtain, not very surprisingly,  
COROLLARY.

$$F(x, t) \sim \Phi\left(\frac{x-1/2}{\sqrt{t}}\right) \quad (t \rightarrow \infty),$$

i.e.,  $X(t)$  is asymptotically normal with mean  $\frac{1}{2}$  and variance  $t$ .

*Remark 1.* Of course,  $X(t)$  is also asymptotically normal with mean zero and variance  $t$ ; the  $\frac{1}{2}$  will improve the approximation, though.

*Remark 2.* Since by (vi) (see also [4, p. 272])  $J$  satisfies  $J_{xy} + J_x + J_y = 0$ , from (10) it follows that  $F$  satisfies the “telegrapher’s” equation:  $F_{tt} = F_{xx} - 2F_t$  as is proved in [3] for a more general  $F$ .

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