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POISSON PROCESSES AND A BESSEL FUNCTION INTEGRAL*

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Abstract. The probability of winning a simple game of competing Poisson processes turns out to be equal to the well-known Bessel function integral $J(x,y)$ (cf. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962). Several properties of J, some of which seem to be new, follow quite easily from this probabilistic interpretation. The results are applied to the random telegraph process as considered by Kac [Rocky Mountain J. Math., 4 (1974), pp. 497–509].

Key words. Poisson process, Bessel function, random telegraph

1. Competing Poisson processes. Several problems can be described as follows: An object has to travel a distance x; it does so at unit speed, but it is obstructed at random moments and then held for a random period of time before it is allowed to continue. The object may be a particle moving between two electrodes, a person walking to a bus stop, or, as in [5, Problem 147], a book being read with random interruptions. The question is: What is the probability that the object reaches its destination at a moment not exceeding $x+y$? The situation may be modelled as a game of two competing (Poisson) renewal processes in the following way (see Fig. 1):

Let $X_1, Y_1, X_2, Y_2, \cdots$ be independent, exponentially distributed random variables with expectation one. Two persons, X and Y, take turns drawing lengths X_j and Y_j. Person X starts, and wins if the sum of his X_j exceeds x before the sum of Y's Y_j exceeds y.

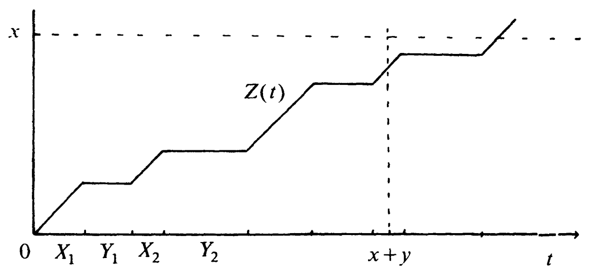

FIG 1. $N_x=5$, $N_y=3$; *X loses*.

More formally, if N_x and N_y are random variables defined by

$$N_x=\min\{n; X_1+\cdots+X_n>x\},$$
$$N_y=\min\{n; Y_1+\cdots+Y_n>y\},$$

then (remember that X starts)

$$X \text{ wins} \Leftrightarrow N_x \leqq N_y \Leftrightarrow X_1+\cdots+X_{N_y}>x. \tag{1}$$

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Remark. For our purposes the assumption that $EX_j = EY_j = 1$ for $j = 1, 2, \cdots$, is no restriction: replacing X_j and Y_j by X_j/λ and Y_j/μ, respectively, is equivalent to replacing x and y by λx and μy, respectively. The *process* $Z(t)$ depicted in Fig. 1, representing the distance travelled by the object at time t, would, of course, be changed by a transformation of the X_j and Y_j.

We shall use the following two well-known facts: $N_y - 1$ has a Poisson distribution with mean y, i.e.,

$$P(N_y = n) = e^{-y}\frac{y^{n-1}}{(n-1)!} \qquad (n = 1, 2, \cdots), \tag{2}$$

and $X_1 + \cdots + X_n$ has a gamma distribution with density

$$\frac{d}{dx}P(X_1 + \cdots + X_n \leqq x) = e^{-x}\frac{x^{n-1}}{(n-1)!} \qquad (x > 0). \tag{3}$$

Now, let $J(x,y)$ be defined by (cf. Luke [4, p. 271])

$$J(x,y) = 1 - e^{-y}\int_0^x I_0(2\sqrt{yt})e^{-t}dt, \tag{4}$$

where I_0 is the modified Bessel function of order zero:

$$I_0(z) = \sum_0^\infty \frac{(z/2)^{2n}}{(n!)^2}. \tag{5}$$

Then we easily obtain

PROPOSITION 1.

$$P(N_x \leqq N_y) = J(x,y). \tag{6}$$

Proof. By (1)–(5) we have

$$\begin{aligned} P(N_x \leqq N_y) &= 1 - P(N_x > N_y) = 1 - P(X_1 + \cdots + X_{N_y} \leqq x) \\ &= 1 - \sum_{n=1}^{\infty} P(N_y = n, X_1 + \cdots + X_n \leqq x) \\ &= 1 - \sum_{n=1}^{\infty} e^{-y}\frac{y^{n-1}}{(n-1)!}\int_0^x e^{-t}\frac{t^{n-1}}{(n-1)!}dt \\ &= 1 - e^{-y}\int_0^x I_0(2\sqrt{yt})e^{-t}dt = J(x,y). \end{aligned}$$

Remark. Srivastava and Kashyap [6, pp. 77, 78] consider an equivalent interpretation, in the context of a randomized random walk; there the interpretation remains implicit and is not pursued.

2. Properties of $J(x,y)$. Several properties of $J(x,y)$ follow immediately from (6). We list the following six together with their simple proofs.

(i) $J(0,y) = P(X_1 > 0) = 1$,

(ii) $J(x,0) = P(X_1 > x) = e^{-x}$.

From (2) and its counterpart for N_x (independent of N_y) it follows that

$$P(N_x=N_y)=\sum_1^\infty P(N_x=n,N_y=n)$$

$$=\sum_1^\infty e^{-x}\frac{x^{n-1}}{(n-1)!}e^{-y}\frac{y^{n-1}}{(n-1)!}=e^{-x-y}I_0(2\sqrt{xy}).$$

From this we conclude using (6) that

(iii) $J(x,y)+J(y,x)=1+P(N_x=N_y)=1+e^{-x-y}I_0(2\sqrt{xy})$,

and especially

(iv) $J(x,x)=\frac{1}{2}+\frac{1}{2}e^{-2x}I_0(2x)$.

Conditioning on $X_1=u$, with density e^{-u}, we have

$$P(N_x\leqq N_y)=\int_0^x\left(1-P(N_y\leqq N_{x-u})\right)e^{-u}du+\int_x^\infty e^{-u}du,$$

or in view of (5)

(v) $$J(x,y)=1-\int_0^x J(y,x-u)e^{-u}du,$$

which seems to be new. Rewriting (v) as

$$e^xJ(x,y)=e^x-\int_0^x J(y,v)e^v dv,$$

and differentiating with respect to x, using (4) we recover (iii):

(vi) $$\frac{\partial}{\partial x}J(x,y)=1-J(x,y)-J(y,x)=-e^{-x-y}I_0(2\sqrt{xy}).$$

Several other relations given in [4] are easily obtained from (i)–(vi). In §3 we collect some asymptotic results.

3. Asymptotics. From the probabilistic interpretation the following limit relations are quite obvious (it is easy to give estimates; also compare (v)):

$$\lim_{x\to\infty}J(x,y)=\lim_{x\to\infty}P(N_x\leqq N_y)=0,$$
$$\lim_{y\to\infty}J(x,y)=\lim_{y\to\infty}P(N_x\leqq N_y)=1.$$

For both x and y large we have the following very simple relation, which seems related to expansions in [2] involving the error function, but which seems to be new in this form. Its proof is a simple consequence of the asymptotic normality of Poisson random variables with large means.

PROPOSITION 2. *For $x\to\infty$ and $y\to\infty$*

$$J(x,y)=\Phi\left(\frac{y-x+1/2}{\sqrt{x+y}}\right)+O\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\right), \tag{7}$$

where Φ is the standard normal distribution function defined as

$$\Phi(u)=(2\pi)^{-1/2}\int_{-\infty}^u e^{-v^2/2}dv.$$

Proof.

$$J(x,y)=P(N_x-N_y\leqq 0)=P(N_x-N_y<\tfrac{1}{2}),$$

where the $\frac{1}{2}$ is the usual "continuity correction". As N_x-N_y is asymptotically normal with mean $x-y$ and variance $x+y$, it follows that

$$J(x,y)=P\left(\frac{N_x-N_y-x+y}{\sqrt{x+y}}\leqq\frac{y-x+1/2}{\sqrt{x+y}}\right)\approx\Phi\left(\frac{y-x+1/2}{\sqrt{x+y}}\right). \tag{8}$$

That $J(x,y)$ actually satisfies (7) follows easily from the Berry–Esseen version of the central limit theorem (Feller [1, p. 542]).

Remark. Relation (7), of course, also holds without the term $\frac{1}{2}$. In practice the approximation (8) is much better than is suggested by (7). For values of x and y of 10 and higher it yields a result correct to about three decimal places. Two examples: $x=10$ and $y=20$ yields $J(10,20)=0.974206$ and $\phi(10.5\sqrt{30})=\Phi(1.917)=0.972$. For $x=y=50$ we find $J(50,50)=0.519972$ and $\Phi(0.5/10)=\Phi(0.05)=0.5199$. The abundance of tables of Φ makes the approximation (8) quite practical. To obtain good (proven) bounds is not so easy.

4. Relation with Kac's random telegraph model. In [3] Kac considers an (integrated) telegraph process $X(t)$ (in his formula (25) denoted by $x(t)$) that is closely related to the process $Z(t)$ of Fig. 1. The process $X(t)$ is constructed from the same X_j and Y_j as $Z(t)$; its graph is sketched in Fig. 2. Evidently, the processes $Z(t)$ and $X(t)$ are related by

$$Z(t)=\tfrac{1}{2}(X(t)+t). \tag{9}$$

From Fig. 1 we immediately see that

$$Z(x+y)>x\Leftrightarrow N_x\leqq N_y,$$

and therefore by Proposition 1 we have, in view of (9),

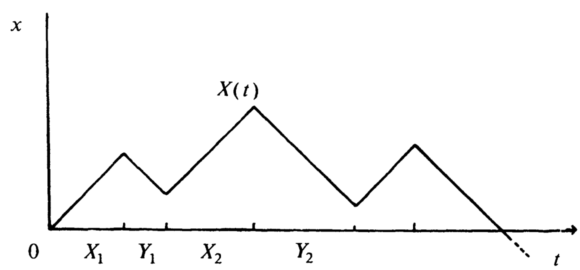

FIG. 2

PROPOSITION 3. *Let $F(x,t)=P(X(t)\leqq x)$ be the distribution function of $X(t)$. Then for $0\leqq x\leqq t$*

$$F(x,t)=1-J\left(\frac{t+x}{2},\frac{t-x}{2}\right). \tag{10}$$

From Proposition 2 we then obtain, not very surprisingly,
COROLLARY.

$$F(x,t) \sim \Phi\left(\frac{x-1/2}{\sqrt{t}}\right) \qquad (t \to \infty),$$

i.e., *$X(t)$ is asymptotically normal with mean $\frac{1}{2}$ and variance t.*

Remark 1. Of course, $X(t)$ is also asymptotically normal with mean zero and variance t; the $\frac{1}{2}$ will improve the approximation, though.

Remark 2. Since by (vi) (see also [4, p. 272]) J satisfies $J_{xy}+J_x+J_y=0$, from (10) it follows that F satisfies the "telegrapher's" equation: $F_{tt}=F_{xx}-2F_t$ as is proved in [3] for a more general F.

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