Poisson processes with jumps governed by lower incomplete gamma subordinator and its variations

Meena Sanjay Babulal[a], Sunil Kumar Gauttam[a,*], Aditya Maheshwari[b]

[a] Department of Mathematics, The LNM Institute of Information Technology, Rupa ki Nangal, Post-Sumel, Via-Jamdoli Jaipur 302031, Rajasthan, India.

[b] Operations Management and Quantitative Techniques Area, Indian Institute of Management Indore, Indore 453556, Madhya Pradesh, India.

Abstract

In this paper, we study the Poisson process time-changed by independent Lévy subordinators, namely, the incomplete gamma subordinator, the ϵ-jumps incomplete gamma subordinator and tempered incomplete gamma subordinator. We derive their important distributional properties such as probability mass function, mean, variance, correlation, tail probabilities and fractional moments. The long-range dependence property of these processes are discussed. An application in insurance domain is studied in detail. Finally, we present the likelihood plots, the pdf plots and the simulated sample paths for the subordinators and their corresponding subordinated Poisson processes.

Keywords: Incomplete gamma function, Lévy subordinator, compound Poisson process.

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1. Introduction

The stochastic models with random time clock appears in various fields of applications such as finance (see [1, 2]), physics (see [3, 4, 5, 6]), ecology (see [7]), biology (see [8]) and *etc.* As a result, there is ever increasing interest among probabilists into this kind of research problems, and it has given arise to a new field of study called as the stochastic subordination. Stochastic subordination involves investigating the stochastic process where the time variable is replaced by a non-decreasing Lévy process. Its study can be divided into two major classes, namely, the diffusion processes and the counting processes. A pioneer work on the stochastic subordination was first published by Bochner (see [9, 10]) and subsequently many scholars studied various aspect of subordinated stochastic process such as homogeneity (see [11], Markov property (see [12]), long-range depen-

*Corresponding author

Email addresses: 19pmt002@lnmiit.ac.in (Meena Sanjay Babulal), sgauttam@lnmiit.ac.in (Sunil Kumar Gauttam), adityam@iimidr.ac.in (Aditya Maheshwari)

dence (LRD) (see [13, 14]), and infinite divisibility(see [15, 16]), *etc.* A comprehensive coverage can be found in Bertoin (see [17]) and Sato (see [18]). In this paper, we focus on stochastic subordination of a counting process.

The Poisson process is a classical and a widely applicable model for counting phenomenon. Buchak and Sakhno (see [19]) investigated the Poisson process subordinated with gamma subordinator. Kumar *et al.* (see [20]) have discussed various characterstics of the Poisson process subordinated with the stable/inverse stable subordinator and the inverse Gaussian subordinator. Orsingher and Toaldo (see [21]) explored the Poisson process subordinated with a Lévy subordinator. Meerschaert et al.(see [22]) studied the Poisson process subordinated by inverse stable subordinator and found a connection with the fractional Poisson process. Orsinger and Polito (see [23]) studied Poisson process subordinated by stable subordinator in connection with the space-fractional Poisson process. The subordinated Poisson process of order k is studied by Sengar *et. al* ([24]).

Beghin and Ricciuti (see [25]) defined the incomplete gamma (InG) subordinator, the incomplete gamma subordinator with jumps of size greater than or equal ϵ (InG-ϵ) subordinator and tempered incomplete gamma (TInG) subordinator using lower-incomplete gamma function. The InG subordinator is defined as non-decreasing Lévy process with the Laplace exponent $\alpha\gamma(\alpha;\eta)$, where $\gamma(\alpha;\eta)$ is the lower-incomplete gamma function given by

$$\gamma(\alpha;\eta) = \int_0^\eta e^{-y} y^{\alpha-1} dy, \quad \eta > 0, 0 < \alpha \leq 1. \tag{1.1}$$

The InG-ϵ subordinator is a modification of the InG subordinator whose jumps are greater than $\epsilon > 0$ with the Laplace exponent $\frac{\alpha}{\epsilon^\alpha}\gamma(\alpha;\eta\epsilon)$. The TInG subordinator is defined as non-decreasing Lévy process with the Laplace exponent $\alpha\gamma(\alpha;\eta+\theta) - \alpha\gamma(\alpha;\theta)$, where $\theta > 0$ is the tempering parameter. It exhibits finite moment of any integer order. In this paper, we consider the InG, InG-ϵ and TInG subordinators as random clocks for the Poisson process. Our goal is to study important distributional properties, such as, the probability mass function (*pmf*), mean, variance, correlation, tail probabilities and fractional moments.

The LRD property concerns with the memory of stochastic process. A stochastic model having the LRD or long "memory" indicates that it is a non stationary process. This property can provide an alternative explanation to the empirical phenomenon that exhibits memory over a period of time; many interesting application can be found out in (see [26]) and the references therein. The definition of the LRD property is based on the second order property of stochastic processes; more specifically asymptotic behaviour of correlation function. We prove that the TInG subordinator and TInG subordinated Poisson process has the LRD property. We emphasize that these models have a great potential for application.

The Poisson process is used to model risk for an insurance company. We have used the subordinated Poisson process (with the PTInG subordinator) as an alternative to the classical Poisson process in risk model for insurance, and is described as follows

$$Y(t) = ct - \sum_{j=1}^{N(S_{\alpha,\theta}(t))} X_j, t \geq 0, \tag{1.2}$$

where $c > 0$ denotes premium rate, which is assumed to be constant, X_j be non-negative i.i.d. random variables with distribution F, representing the claim size and the Poisson process $\{N(t)\}_{t\geq 0}$ subordinated by the TInG $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$. We derive the governing equation for the joint probability that ruin happens in finite time and the deficit at the time of ruin. We also compute the joint distribution of ruin time and deficit at ruin when the initial capital is zero.

The likelihood plot, the pdf plot and the simulation of sample paths provides a visual aid to understand a stochastic process and it is helpful in estimating parameters in some cases. We present the likelihood plot, the pdf plot and the simulated sample paths for the InG, the InG-ϵ and the TInG subordinators and their corresponding subordinated Poisson processes. We have used the Metropolis algorithm(see [27]) to simulate the sample paths where the candidate density is obtained by truncating the support of the exponential density; this approach is developed for the TInG subordinator and the corresponding subordinated Poisson process.

The paper is organised as follows. In Section 2, we present some preliminary results that are required. Section 3 deals with the Poisson process subordinated with the InG and InG-ϵ subordinator. In Section 4, the LRD property for the Poisson process subordinated by the PTInG subordinator is presented. Section 5 discusses the role of the subordinated Poisson process in insurance. In Section 6, we present the likelihood plots, the pdf plots and simulate the sample paths of the InG, the InG-ϵ and TInG subordinators and the corresponding subordinated Poisson processes.

2. Preliminaries

In this section, we present some preliminary results which are required later in the paper.

Let $\mathbb{Z}^+ := \{0, 1, 2, \cdots\}$ be the set of non-negative integers. Let $\{N(t,\lambda)\}_{t\geq 0}$ be a Poisson process with rate

$\lambda > 0$, so that

$$p(n|t,\lambda) := \mathbb{P}[N(t,\lambda) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n \in \mathbb{Z}^+. \tag{2.1}$$

For simplicity of notation we write $N(t,\lambda)$ as $N(t)$, when no confusion arises.

For $\alpha \in (0,1]$, the InG subordinator $\{S_\alpha(t)\}_{t\geq 0}$ (see [25]) can be represented as a compound Poisson process

$$S_\alpha(t) = \sum_{j=1}^{N_\alpha(t)} Z_j^\alpha, \tag{2.2}$$

where $\{N_\alpha(t)\}_{t\geq 0}$ is a homogeneous Poisson process with the rate $\lambda := \alpha\Gamma(\alpha)$ and the jumps Z_j^α are i.i.d. random variables, taking values in $[1,+\infty)$, with probability density function

$$f_{Z^\alpha}(z) = \frac{(z-1)^{-\alpha} z^{-1} 1_{z\geq 1}}{\Gamma(1-\alpha)\Gamma(\alpha)} = \frac{\sin(\pi\alpha) 1_{z\geq 1}}{\pi(z-1)^\alpha z}, \quad \alpha \in (0,1). \tag{2.3}$$

When $\alpha = 1$, the jumps are unitary, and the process coincides with the Poisson process (see [25]). Note that the subordinator $\{S_\alpha(t)\}_{t\geq 0}$ have jumps of size greater than or equal to 1.

Similarly, the InG-ϵ subordinator $\{S_\alpha^{(\epsilon)}(t)\}_{t\geq 0}$ can be represented as a compound Poisson process

$$S_\alpha^{(\epsilon)}(t) = \sum_{j=1}^{N^\epsilon(t)} Z_j^{(\alpha,\epsilon)}, \tag{2.4}$$

where $N^\epsilon := \{N^\epsilon(t)\}_{t\geq 0}$ is a homogeneous Poisson process with the rate $\lambda := \alpha\Gamma(\alpha)\epsilon^{-\alpha}$ and the jumps $Z_j^{(\alpha,\epsilon)}$ are i.i.d. random variables, taking values in $[\epsilon,+\infty)$, with probability density function

$$f_{Z_j^{(\alpha,\epsilon)}}(z) = \frac{\epsilon^\alpha (z-\epsilon)^{-\alpha} z^{-1} 1_{z\geq\epsilon}}{\Gamma(1-\alpha)\Gamma(\alpha)}, \quad \alpha \in (0,1). \tag{2.5}$$

In contrast to the InG subordinator, the InG-ϵ subordinator $\{S_\alpha^{(\epsilon)}(t)\}_{t\geq 0}$ have jumps of size greater than or equal to ϵ (see [25]).

The TInG subordinator $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$ can be represented as a compound Poisson process

$$S_{\alpha,\theta}(t) = \sum_{j=1}^{N_{\alpha,\theta}(t)} Z_j^{\alpha,\theta}, \tag{2.6}$$

where $N_{\alpha,\theta} := \{N_{\alpha,\theta}(t)\}_{t\geq 0}$ is a homogeneous Poisson process with rate $\lambda := \alpha\Gamma(\alpha;\theta)$, where $\Gamma(\alpha;\theta)$ is the upper incomplete gamma function defined as

$$\Gamma(\alpha;\eta) = \int_{\eta}^{\infty} e^{-y} y^{\alpha-1} dy, \quad \eta > 0, 0 < \alpha \leq 1.$$

We have the following relationship between upper, lower incomplete gamma function and gamma function

$$\Gamma(\alpha;\theta) + \gamma(\alpha;\theta) = \Gamma(\alpha).$$

The jumps $Z_j^{\alpha,\theta}$ are i.i.d. random variables, taking values in $[1,+\infty)$ and with the probability density function

$$f_{Z_j^{\alpha,\theta}} = \frac{e^{-\theta z}(z-1)^{-\alpha} z^{-1} 1_{z\geq 1}}{\Gamma(1-\alpha)\Gamma(\alpha;\theta)}, \quad \alpha \in (0,1). \tag{2.7}$$

Observe that the mean for subordinators the InG and InG-ϵ does not exist, but mean and variance of the PTInG subordinator $S_{\alpha,\theta}(t)$ are given by (see [25])

$$\mathbb{E}S_{\alpha,\theta}(t) = t\alpha\theta^{\alpha-1}e^{-\theta}, \tag{2.8}$$

$$\text{Var}S_{\alpha,\theta}(t) = t\alpha\theta^{\alpha-1}e^{-\theta} + t(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta}. \tag{2.9}$$

The following result (see [28]) is key to our computation for the fractional order moments of the PTInG subordinator and subordinated Poisson processes.

Theorem 2.1. *Let X be a positive random variable with the Laplace transform $\widetilde{f(t)}$. Then its q^{th} order moment, where $q \in (n-1,n)$ is given by*

$$\mathbb{E}(X^q) = \frac{(-1)^n}{\Gamma(n-q)} \int_0^{\infty} \frac{d^n}{du^n}[\widetilde{f(u)}] u^{n-q-1} du. \tag{2.10}$$

3. Poisson process subordinated by the InG subordinator

In this section, we consider the Poisson process subordinated with the InG subordinator. First we define the Poisson process subordinated with the InG subordinator as follows.

Definition 3.1. *The Poisson process subordinated with the InG subordinator (PInG) is defined as*

$$Q(t) := N(S_\alpha(t)), \qquad t \geq 0, \tag{3.1}$$

where $\{N(t)\}_{t\geq 0}$ *is the Poisson process with rate* $\lambda > 0$ *and independent of the InG subordinator* $\{S_\alpha(t)\}_{t\geq 0}$*.*

Next we deduce various characteristics of PInG. First we compute its Laplace exponent.

Proposition 3.2. *The Laplace exponent of subordinated Poisson process the PInG is* $e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}$*.*

Proof. Using a conditioning argument, we obtain the Laplace exponent of $N(S_\alpha(t))$ as

$$\begin{aligned}\mathbb{E}[e^{-\eta N(S_\alpha(t))}] &= \mathbb{E}[\mathbb{E}[e^{-\eta N(S_\alpha(t))}\big|S_\alpha(t)]], \quad \eta \geq 0\\ &= \mathbb{E}[e^{-\lambda S_\alpha(t)(1-e^{-\eta})}]\\ &= e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}.\end{aligned}$$

□

Mimicking the argument as in the proof of Proposition 3.2, the probability generating function (pgf) of PInG can be seen as

$$\begin{aligned}\mathbb{E}[u^{N(S_\alpha(t))}] &= \mathbb{E}[\mathbb{E}[u^{N(S_\alpha(t))}\big|S_\alpha(t)]]\\ &= \mathbb{E}[e^{-\lambda(1-u)S_\alpha(t)}]\\ &= e^{-t\alpha\gamma(\alpha;\lambda(1-u))}.\end{aligned} \tag{3.2}$$

One can calculate pmf of $N(S_\alpha(t))$ using the following relationship with pgf (3.2)

$$\mathbb{P}(N(S_\alpha(t)) = k) = \frac{d^k}{du^k}\frac{(e^{-t\alpha\gamma(\alpha;\lambda(1-u))})}{k!}\bigg|_{u=0}, \qquad k = 0, 1, 2, \cdots .$$

We present values of the pmf $p_k(t)$ for $k = 0, 1, 2, 3$

$$\begin{aligned}\mathbb{P}(N(S_\alpha(t)) = 0) &= e^{-t\alpha\gamma(\alpha;\lambda)},\\ \mathbb{P}(N(S_\alpha(t)) = 1 &= \alpha\lambda^\alpha te^{-\alpha t\gamma(\alpha;\lambda)-\lambda},\\ \mathbb{P}(N(S_\alpha(t)) = 2) &= \frac{1}{2!}\big(\alpha^2\lambda^{2\,\alpha}t^2 - \big(\alpha^2 - \alpha\lambda - \alpha\big)\lambda^\alpha te^\lambda\big)e^{-\alpha t\gamma(\alpha;\lambda)-2\,\lambda},\\ \mathbb{P}(N(S_\alpha(t)) = 3) &= \frac{1}{3!}\,\big[\alpha^3\lambda^{3\,\alpha}t^3 - 3\big(\alpha^3 - \alpha^2\lambda - \alpha^2\big)\lambda^{2\,\alpha}t^2e^\lambda\\ &\qquad + \big(\alpha^3 + \alpha\lambda^2 - 3\,\alpha^2 - 2\,\big(\alpha^2 - \alpha\big)\lambda + 2\,\alpha\big)\lambda^\alpha te^{2\lambda}\big]\,e^{-\alpha t\gamma(\alpha;\lambda)-3\lambda}.\end{aligned}$$

The asymptotic behaviour of tail probability and fractional moments of $S_\alpha(t)$ are discussed in [25] that is, for $\alpha \in (0,1)$ and $t \geq 0$

$$\mathbb{P}(S_\alpha(t) > x) \simeq \frac{tx^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty,$$

and

$$\mathbb{E}S_\alpha^p(t) \simeq \frac{\Gamma(1-\frac{p}{\alpha})}{\Gamma(1-p)} t^{\frac{p}{\alpha}}, \qquad t \to \infty,$$

where $p \in (0,1]$ and $p \leq \alpha$. Now, we discuss the asymptotic behaviour of tail probability and fractional moments of the PInG $N(S_\alpha(t))$.

Theorem 3.3. *Let $\alpha \in (0,1)$ and $t \geq 0$, then*

$$\mathbb{P}(N(S_\alpha(t)) > x) \simeq \frac{t\lambda^\alpha x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty \ . \tag{3.3}$$

Proof. We consider the Laplace transform of tail probability, for $\eta > 0$

$$\begin{aligned}\int_0^\infty e^{-\eta x}\mathbb{P}(N(S_\alpha(t)) > x)dx &= \frac{1-\mathbb{E}e^{-\eta N(S_\alpha(t))}}{\eta} \\ &= \frac{1-e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}}{\eta}.\end{aligned}$$

Using the Taylor series expansion up to first order, for $\eta \to 0$ we obtain

$$\begin{aligned}\frac{1-e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}}{\eta} &\simeq \frac{1-(1-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta})))}{\eta} \\ &\simeq \frac{t\alpha}{\eta}\frac{(\lambda(1-e^{-\eta}))^\alpha}{\alpha} \qquad \left(\gamma(\alpha;\eta) \simeq \frac{\eta^\alpha}{\alpha}, \text{ as } \eta \to 0\right) \\ &\simeq \frac{t\alpha}{\eta}\frac{(\eta\lambda)^\alpha}{\alpha} \\ &= \eta^{\alpha-1}t\lambda^\alpha.\end{aligned}$$

The desired result follows from the Tauberian theorem (see [29, p.446]) for any $t \geq 0$. □

Theorem 3.4. *Let $p \in (0,1)$, then the fractional moment of p^{th} order of the process the PInG $\{N(S_\alpha(t))\}_{t\geq 0}$ exists, is finite for $p < \alpha$ and its asymptotic behaviour is given by*

$$\mathbb{E}[N(S_\alpha(t))]^p \sim \lambda^{p-1}t^p, \qquad t \to \infty.$$

Proof. We first argue the existence of fractional moments. Asymptotic behaviour of the tail probability given by (3.3) allows us to conclude $\mathbb{E}[N(S_\alpha(t))]^p < \infty$ for $p < \alpha$.

Now, we investigate the asymptotic behaviour of the fractional moments. By using (2.10)

$$\begin{aligned}\mathbb{E}[N(S_\alpha(t))]^p &= \frac{-1}{\Gamma(1-p)}\int_0^\infty \frac{d}{d\eta}[e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}]\eta^{-p}d\eta \\ &= \frac{\alpha t\lambda^\alpha}{\Gamma(1-p)}\int_0^\infty (1-e^{-\eta})^{\alpha-1}e^{\lambda(1-e^{-\eta})-\eta}e^{-t\alpha\gamma(\alpha;\lambda(1-e^{-\eta}))}\eta^{-p}d\eta.\end{aligned}$$

Let $h(\eta) = \alpha\gamma(\alpha;\lambda(1-e^{-\eta}))$ and $\phi(\eta) = (1-e^{-\eta})^{\alpha-1}e^{\lambda(1-e^{-\eta})-\eta}\eta^{-p}$.

$$\begin{aligned}h(\eta) &= \alpha\gamma(\alpha;0) + \sum_{k=0}^\infty \frac{(\lambda(1-e^{-\eta}))^{\alpha+k}}{(\alpha+k)k!} \\ &= \alpha\gamma(\alpha;0) + (-\eta)^{\alpha-p}\lambda^\alpha\left[1 - \left(\frac{\alpha^2+(2\lambda+1)\alpha}{2(\alpha+1)}\right)\eta \right. \\ &\qquad \left. +\lambda^2\left(\frac{(3\alpha^3+(12\lambda+7)\alpha^2+2(6\lambda^2+12\lambda+1)\alpha)}{24(\alpha+2)}\eta^2\right) + \cdots\right],\end{aligned}$$

where $h(0) = \alpha\gamma(\alpha;0), a_0 = \lambda^\alpha, a_1 = \left(\frac{\alpha^2+(2\lambda+1)\alpha}{2(\alpha+1)}\right)$ and $\mu = \alpha$. Furthermore

$$\begin{aligned}\phi(\eta) &= (\lambda(1-e^{-\eta}))^{\alpha-1}e^{-\lambda(1-e^{-\eta})-\eta}\eta^{-p} \\ &= -\eta^{\alpha-p-1}\lambda^{\alpha-1}\left[1 - \left(\frac{\alpha-2\lambda+1}{2}\right)\eta + \lambda^2\left(\frac{(-3\alpha^2-(12\lambda+7)\alpha-2(6(\lambda^2+12\lambda+1)))\eta}{24}\right) + \cdots\right],\end{aligned}$$

where $b_0 = \lambda^{\alpha-1}, b_1 = -\lambda^{\alpha-1}\left(\frac{\alpha-2\lambda+1}{2}\right)$ and $\gamma = \alpha - p$.

Now, we apply the Laplace–Erdelyi Theorem (see [30]) to the above integral and we get

$$\mathbb{E}[N(S_\alpha(t))]^p \sim \frac{\alpha t}{\Gamma(1-p)}\sum_{j=0}^\infty \frac{c_j}{t^{1-p+j}}\Gamma(1-p+j).$$

Above series is dominated by first term for large t, which leads to

$$\mathbb{E}[N(S_\alpha(t))]^p \sim \frac{\alpha c_0 t^p\Gamma(1-p)}{\Gamma(1-p)} \sim \lambda^{p-1}t^p,$$

where $c_0 = b_0/(\mu a_0^{\gamma/\mu}) = \lambda^{p-1}/\alpha$. □

The following result for the InG-ϵ subordinator can be obtained by mimicking the arguments as in proof of Theorem 3.3 and 3.4.

Theorem 3.5.

(a) Let $\alpha \in (0,1)$ then, for any $t \geq 0$, we have

$$\mathbb{P}(S_\alpha^{(\epsilon)}(t) > x) \simeq \frac{t x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty.$$

(b) Let $p \in (0,1)$, then fractional moment of order p of the process $S_\alpha^{(\epsilon)}(t)$ exists, finite, for $p \leq \alpha$, and its asymptotic behaviour is given as

$$\mathbb{E}[S_\alpha^{(\epsilon)}(t)]^p \sim \frac{\alpha}{\epsilon} t^p, \qquad t \to \infty.$$

The Poisson process subordinated with the InG-ϵ subordinator have similar properties as PInG, which can be obtained by the same line of arguments.

Definition 3.6. *The Poisson process subordinated with the InG-ϵ subordinator (PInG-ϵ) is defined as*

$$Q(t) = N(S_\alpha^{(\epsilon)}(t)), \qquad t \geq 0, \tag{3.4}$$

where $\{N(t)\}_{t\geq 0}$ is the Poisson process with rate $\lambda > 0$ and independent of subordinator the InG-ϵ.

The following theorem summarizes few results for the PInG-ϵ process. The proof is omitted due to similarity with the proofs of the analogues results associated with the PInG process.

Theorem 3.7.

(a) The Laplace exponent of the PInG-ϵ is $\exp\left(-t\alpha\epsilon^{-\alpha}\gamma(\alpha;\lambda(1-e^{\eta})\epsilon)\right)$.

(b) The pgf of the PInG-ϵ given by $\exp\left(-\frac{1}{\epsilon^\alpha}t\alpha\gamma(\alpha;\lambda(1-u)\epsilon)\right)$.

(c) The pmf of the PInG-ϵ is given by

$$\mathbb{P}(N(S_\alpha^{(\epsilon)}(t)) = 0) = \exp\left(-\frac{t\alpha}{\epsilon^\alpha}\gamma(\alpha;\lambda\epsilon)\right)$$
$$\mathbb{P}(N(S_\alpha^{(\epsilon)}(t)) = 1) = \lambda^\alpha \alpha t \exp\left(-\lambda\epsilon - \frac{\alpha t \gamma\left(\alpha;\lambda\epsilon\right)}{\epsilon^\alpha}\right),$$
$$\mathbb{P}(N(S_\alpha^{(\epsilon)}(t)) = 2) = \frac{1}{2}\left(\lambda^{2\alpha}\alpha^2 t^2 + (\alpha\lambda\epsilon - \alpha^2 + \alpha)\lambda^2 t e^{\lambda\epsilon}\right)\exp\left(-2\lambda\epsilon - \frac{\alpha t \gamma\left(\alpha;\lambda\epsilon\right)}{\epsilon^\alpha}\right),$$
$$\mathbb{P}(N(S_\alpha^{(\epsilon)}(t)) = 3) = \frac{1}{6}\left[\lambda^{3\alpha}\alpha^3 t^3 + 3(\alpha^2\lambda\epsilon - \alpha^3 + \alpha^2)\lambda^{2\alpha}t^2 e^{\lambda\epsilon} + (\alpha\lambda^2\epsilon^2 + \alpha^3 - 2(\alpha^2-\alpha)\lambda\epsilon - 3\alpha^2 + 2\alpha)\lambda^\alpha t e^{2\lambda\epsilon}\right]$$
$$\exp\left(-3\lambda\epsilon - \frac{\alpha t \gamma\left(\alpha;\lambda\epsilon\right)}{\epsilon^\alpha}\right),$$

and so on.

(d) Let $\alpha \in (0,1)$ and $t \geq 0$, then

$$\mathbb{P}(N(S_\alpha^{(\epsilon)}(t)) > x) \simeq \frac{t\lambda^\alpha x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty \ .$$

(e) Let $p \in (0,1)$, then the fractional moment of p^{th} order of the process the PInG exists, is finite for $p < \alpha$ and its asymptotic behaviour is given by

$$\mathbb{E}[N(S_\alpha^{(\epsilon)}(t))]^p \sim \frac{\lambda^{p-1}t^p}{\epsilon^{\alpha-p}}, \qquad t \to \infty.$$

4. Poisson process subordinated by the TInG subordinator

In this section, we investigate the Poisson process subordinated with the TInG subordinator. First we define the Poisson process subordinated with the TInG subordinator as follows.

Definition 4.1. *The Poisson process subordinated with the TInG subordinator (PTInG) is defined as*

$$Q(t) = N(S_{\alpha,\theta}(t)), \qquad t \geq 0, \tag{4.1}$$

where $\{N(t)\}_{t\geq 0}$ is the Poisson process with rate $\lambda > 0$ and independent of subordinator the TInG $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$.

Using a conditioning argument similar to the proof of Proposition 3.2, the Laplace exponent of the PTInG can be obtained as $\exp\left(-t\alpha[\gamma(\alpha;\lambda(1-e^{\eta})+\theta)-\gamma(\alpha;\theta)]\right)$, and the pgf is given by $\exp\left(-t\alpha(\gamma(\alpha;\lambda(1-u)+\theta)-\gamma(\alpha;\theta))\right)$. The pmf for $N(S_{\alpha,\theta}(t))$ is given by

$$\begin{aligned}
\mathbb{P}(N(S_{\alpha,\theta}(t)) = 0) &= \exp\left(-t\alpha\rho(\alpha;\theta)\right),\\
\mathbb{P}(N(S_{\alpha,\theta}(t)) = 1) &= [(\lambda+\theta)^{\alpha-1}\alpha\lambda t]\exp\left(-t\alpha\rho(\alpha;\theta)\right),\\
\mathbb{P}(N(S_{\alpha,\theta}(t)) = 2) &= \frac{1}{2}\Big[(1-\alpha)\alpha(\theta+\lambda)^{\alpha-2}\lambda^2 t + \Big(\alpha(\theta+\lambda)^{\alpha-1}\lambda t\exp\left(-\theta-\lambda\right)+\lambda\Big)\alpha(\theta+\lambda)^{\alpha-1}\lambda t\Big]\, e^{-\alpha t\rho(\alpha;\theta)-\theta-\lambda},\\
\mathbb{P}(N(S_{\alpha,\theta}(t)) = 3) &= \frac{1}{6}\Big[(\alpha-1)(\alpha-2)\alpha(\theta+\lambda)^{\alpha-3}\lambda^3 t - 2\Big(\alpha(\theta+\lambda)^{\alpha-1}\lambda t e^{-\theta-\lambda}+\lambda\Big)(\alpha-1)\alpha(\theta+\lambda)^{\alpha-2}\lambda^2 t\\
&\quad + \Big(\alpha(\theta+\lambda)^{\alpha-1}\lambda t e^{-\theta-\lambda}+\lambda\Big)^2\alpha(\theta+\lambda)^{\alpha-1}\lambda t - \Big((\alpha-1)\alpha(\theta+\lambda)^{\alpha-2}\lambda^2 t e^{-\theta-\lambda}\\
&\quad - \alpha(\theta+\lambda)^{\alpha-1}\lambda^2 t e^{-\theta-\lambda}\Big)\alpha(\theta+\lambda)^{\alpha-1}\lambda\Big]\exp\left(-\alpha t\rho(\alpha;\theta)-\theta-\lambda\right),
\end{aligned}$$

where $\rho(\alpha;\theta) = \gamma\left(\alpha;\theta+\lambda\right) - \gamma\left(\alpha;\theta\right)$ and so on.

Since the PTInG has finite mean and variance, it is expected that the PTInG also have finite mean and variance.

Theorem 4.2. *The mean, variance and covariance of the PTInG are given by*

(a) $\mathbb{E}[N(S_{\alpha,\theta}(t))] = \lambda t\alpha\theta^{\alpha-1}e^{-\theta}$,

(b) $Var[N(S_{\alpha,\theta}(t))] = \lambda^2(t\alpha\theta^{\alpha-1}e^{-\theta} + t(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta}) + \lambda(t\alpha\theta^{\alpha-1}e^{-\theta})$,

(c) $Cov[N(S_{\alpha,\theta}(s)), N(S_{\alpha,\theta}(t))] = Var[N(S_{\alpha,\theta}(s))]$, *for* $0 \leq s < t$.

Proof. Using the conditioning argument, we have

$$\begin{aligned}
\mathbb{E}[N(S_{\alpha,\theta}(t))] &= \mathbb{E}[\mathbb{E}[N(S_{\alpha,\theta}(t))\big|S_{\alpha,\theta}(t)]] \\
&= \mathbb{E}[\lambda S_{\alpha,\theta}(t)] \\
&= \lambda\mathbb{E}[S_{\alpha,\theta}(t)] \\
&= \lambda t\alpha\theta^{\alpha-1}e^{-\theta} \quad \text{(Using (2.8))}. \qquad (4.2)
\end{aligned}$$

Now, we derive expression for variance of the the PTInG .

$$\begin{aligned}
\mathrm{Var}[N(S_{\alpha,\theta}(t))] &= \mathrm{Var}[[\mathbb{E}[N(S_{\alpha,\theta}(t))\big|S_{\alpha,\theta}(t)]]] + \mathbb{E}[\mathrm{Var}[N(S_{\alpha,\theta}(t))\big|S_{\alpha,\theta}(t)]] \\
&= \mathrm{Var}[\lambda S_{\alpha,\theta}(t)] + \mathbb{E}[\lambda S_{\alpha,\theta}(t)] \\
&= \lambda^2\mathrm{Var}[S_{\alpha,\theta}(t)] + \lambda\mathbb{E}[S_{\alpha,\theta}(t)] \\
&= \lambda^2(t\alpha\theta^{\alpha-1}e^{-\theta} + t(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta}) + \lambda(t\alpha\theta^{\alpha-1}e^{-\theta}). \qquad (4.3)
\end{aligned}$$

To obtain covariance of the PTInG, first we compute $\mathbb{E}[N(S_{\alpha,\theta}(s))N(S_{\alpha,\theta}(t))]$ with $s < t$

$$\begin{aligned}
\mathbb{E}[N(S_{\alpha,\theta}(s))N(S_{\alpha,\theta}(t))] &= \frac{1}{2}\left(\mathbb{E}\left[N(S_{\alpha,\theta}(s))\right]^2 + \mathbb{E}\left[N(S_{\alpha,\theta}(t))\right]^2 - \mathbb{E}\left[N(S_{\alpha,\theta}(s)) - N(S_{\alpha,\theta}(t))\right]^2\right) \\
&= \frac{1}{2}\left[\mathbb{E}[\mathbb{E}[[N(S_{\alpha,\theta}(t))]^2\big|S_{\alpha,\theta}(t)]] + \mathbb{E}[\mathbb{E}[N(S_{\alpha,\theta}(s,\lambda))^2\big|S_{\alpha,\theta}(s)]]\right. \\
&\quad \left. +\mathbb{E}[\mathbb{E}[N(S_{\alpha,\theta}(t)) - N(S_{\alpha,\theta}(s))^2\big|S_{\alpha,\theta}(t)]]\right] \\
&= \frac{1}{2}\left[(\mathbb{E}[\lambda S_{\alpha,\theta}(t)) + (\lambda S_{\alpha,\theta}(t))]^2) + (\mathbb{E}[(\lambda S_{\alpha,\theta}(s)) + (\lambda S_{\alpha,\theta}(s))]^2)\right. \\
&\quad \left. -(\mathbb{E}[(\lambda S_{\alpha,\theta}(t-s,\lambda)) + (\lambda S_{\alpha,\theta}(t-s,\lambda))]^2\right] \\
&= \frac{1}{2}\left[(\lambda t\alpha\theta^{\alpha-1}e^{-\theta}) + \lambda^2(t\alpha\theta^{\alpha-1}e^{-\theta} + t(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta}) + ((\lambda t\alpha\theta^{\alpha-1}e^{-\theta})^2)\right] \\
&\quad +[(\lambda s\alpha\theta^{\alpha-1}e^{-\theta}) + \lambda^2(s\alpha\theta^{\alpha-1}e^{-\theta} + s(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})((\lambda s\alpha\theta^{\alpha-1}e^{-\theta})^2)]
\end{aligned}$$

$$
\begin{aligned}
&+[(\lambda(t-s)\alpha\theta^{\alpha-1}e^{-\theta})+\lambda^2((t-s)\alpha\theta^{\alpha-1}e^{-\theta}+(t-s)(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})\\
&+((\lambda(t-s)\alpha\theta^{\alpha-1}e^{-\theta})^2)]\\
=\;&(\lambda s\alpha\theta^{\alpha-1}e^{-\theta})+\lambda^2(s\alpha\theta^{\alpha-1}e^{-\theta}+s(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})+st(\lambda\alpha\theta^{\alpha-1}e^{-\theta})^2. \qquad (4.4)
\end{aligned}
$$

Using (4.2) and (4.4) we get covariance as

$$
\begin{aligned}
\mathrm{Cov}[N(S_{\alpha,\theta}(s)),N(S_{\alpha,\theta}(t))] &= \mathbb{E}[N(S_{\alpha,\theta}(s))N(S_{\alpha,\theta}(t))]-\mathbb{E}[N(S_{\alpha,\theta}(s))]\mathbb{E}[N(S_{\alpha,\theta}(t))]\\
&= [(\lambda s\alpha\theta^{\alpha-1}e^{-\theta})+\lambda^2(s\alpha\theta^{\alpha-1}e^{-\theta}+s(\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})]\\
&= \lambda\mathbb{E}[S_{\alpha,\theta}(s)]+\lambda^2\mathrm{Var}[S_{\alpha,\theta}(s)]\\
&= \mathrm{Var}[N(S_{\alpha,\theta}(s))]. \qquad (4.5)
\end{aligned}
$$

□

The pmf of the the PTInG has the following alternate representation.

Theorem 4.3. *The pmf of the the PTInG is also given by*

$$
\mathbb{P}[N(S_{\alpha,\theta}(t))=n]=\frac{\lambda^n}{n!}\mathbb{E}[e^{-\lambda S_{\alpha,\theta}(t)}(S_{\alpha,\theta}(t))^n], \quad n=0,1,2,3\cdots. \qquad (4.6)
$$

Proof. Let $g(y,t)$ be the probability density function of $S_{\alpha,\theta}(t)$. Then

$$
\begin{aligned}
\mathbb{P}[N(S_{\alpha,\theta}(t))=n] &= \int_0^\infty \mathbb{P}[N(S_{\alpha,\theta}(t))=n\big|S_{\alpha,\theta}(t)]g(y,t)dt\\
&= \int_0^\infty \frac{(\lambda y)^n e^{-\lambda y}}{n!}g(y,t)dt\\
&= \frac{\lambda^n}{n!}\mathbb{E}[e^{-\lambda S_{\alpha,\theta}(t)}(S_{\alpha,\theta}(t))^n].
\end{aligned}
$$

□

The representation (4.6) allows easy verification of the normalizing condition

$$
\sum_{n=0}^{\infty}\mathbb{P}[N(S_{\alpha,\theta}(t))=n]=1.
$$

Consider

$$
\sum_{n=0}^{\infty}\mathbb{P}[N(S_{\alpha,\theta}(t))=n]=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}[e^{-\lambda S_{\alpha,\theta}(t)}(S_{\alpha,\theta}(t))^n]
$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(\lambda y)^n e^{-\lambda y}}{n!} g(y,t)dy$$
$$= \int_0^{\infty} g(y,t)dy = 1.$$

Using simple algebraic calculations, one can see that the transition probabilities of the $N(S_{\alpha,\theta}(t))$ are given by

$$\mathbb{P}[N(S_{\alpha,\theta}(t+h)) = n | N(S_{\alpha,\theta}(t)) = m]$$
$$= \begin{cases} 1 - \alpha f(\lambda) + o(h) & \text{if } n = m \\ -h\left[(-1)^i \frac{\lambda^i}{i!} f^{(i)}(\lambda)\right] + o(h) & \text{if } n = m + i, i = 1, 2, 3 \cdots, \end{cases} \tag{4.7}$$

where $f(\lambda) = \gamma(\alpha; \lambda + \theta) - \gamma(\alpha; \theta)$ is Laplace exponent of $S_{\alpha,\theta}(t)$.

Now, we state results regarding the asymptotic behaviour of tail probability and fractional moments for the TInG subordinator and PTInG, which can be obtained along the similar line as in proof of Theorem 3.3 and 3.4.

Theorem 4.4. *(a) Let $\alpha \in (0,1)$ and $t \geq 0$, we have that*

$$\mathbb{P}(S_{\alpha,\theta}(t) > x) \simeq \frac{tx^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty \ .$$

(b) Let $p \in (0,1]$, then fractional moment of p-th order of the process the TInG exists, it is finite for $p < \alpha$ and is given by

$$\mathbb{E}^p[S_{\alpha,\theta}(t)] \sim \alpha(e^{-\theta}\theta^{\alpha-1}t)^p, \qquad t \to \infty.$$

Theorem 4.5. *(a) Let $\alpha \in (0,1)$ and $t \geq 0$, we have that*

$$\mathbb{P}(N(S_{\alpha,\theta}(t)) > x) \simeq \frac{t\lambda^\alpha x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad x \to \infty \ .$$

(b) Let $p \in (0,1]$, then fractional moment of p-th order of the process the PTInG exists, it is finite for $p < \alpha$ and is given by

$$\mathbb{E}^p[N(S_{\alpha,\theta}(t))] \sim \alpha(e^{-\theta}\theta^{\alpha-1}t)^p, \qquad t \to \infty.$$

Now, we discuss LRD property of the the TInG subordinator which is not shared by the InG and InG-ϵ subordinators. We state the definition as in [14].

Definition 4.6. *Let* $0 < s < t$ *and* s *be fixed. Assume a stochastic process* $\{X(t)\}_{t\geq 0}$ *has the correlation function* $Corr[X(s), X(t)]$ *that satisfies*

$$c_1(s)t^{-d} \leq Corr[X(s), X(t)] \leq c_2(s)t^{-d}$$

for large $t, d > 0$, $c_1(s) > 0$ *and* $c_2(s) > 0$. *That is,*

$$\lim_{t\to\infty} \frac{Corr[X(s), X(t)]}{t^{-d}} = c(s)$$

for some $c(s) > 0$ *and* $d > 0$. *We say* $\{X(t)\}_{t\geq 0}$ *has the long-range dependence (LRD) property if* $d \in (0, 1)$ *and has the short-range dependence (SRD) property if* $d \in (1, 2)$.

Now we show that the TInG $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$ has LRD property.

Theorem 4.7. *The TInG* $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$ *has LRD property.*

Proof. First we compute the covariance function using independent increment property of subordinator. For $0 \leq s < t < \infty$, we have

$$\begin{aligned}
\mathrm{Cov}[S_{\alpha,\theta}(s), S_{\alpha,\theta}(t)] &= \mathrm{Cov}[S_{\alpha,\theta}(s), (S_{\alpha,\theta}(t) - S_{\alpha,\theta}(s)) + S_{\alpha,\theta}(s)] \\
&= \mathrm{Cov}[S_{\alpha,\theta}(s), (S_{\alpha,\theta}(t) - S_{\alpha,\theta}(s))] + \mathrm{Cov}[S_{\alpha,\theta}(s), S_{\alpha,\theta}(s)] \\
&= \mathrm{Var}[S_{\alpha,\theta}(s)]. \qquad (4.8)
\end{aligned}$$

Thus the correlation function is given by

$$\begin{aligned}
\mathrm{Corr}[S_{\alpha,\theta}(s), S_{\alpha,\theta}(t)] &= \frac{\mathrm{Cov}[S_{\alpha,\theta}(s), S_{\alpha,\theta}(t)]}{\mathrm{Var}[S_{\alpha,\theta}(s)]^{1/2}\mathrm{Var}[S_{\alpha,\theta}(t)]^{1/2}} \\
&= \frac{\mathrm{Var}[S_{\alpha,\theta}(s)]^{1/2}}{\mathrm{Var}[S_{\alpha,\theta}(t)]^{1/2}} \\
&= s^{1/2}t^{-1/2}. \qquad (4.9)
\end{aligned}$$

Hence

$$\lim_{t\to\infty} \frac{\mathrm{Corr}[S_{\alpha,\theta}(s), S_{\alpha,\theta}(t)]}{t^{\frac{-1}{2}}} = s^{\frac{1}{2}}.$$

Therefore, the TInG $\{S_{\alpha,\theta}(t)\}_{t\geq 0}$ has LRD property. □

Next, we show that the PTInG has LRD property.

Theorem 4.8. *The PTInG $\{N(S_{\alpha,\theta}(t))\}_{t\geq 0}$ has the LRD property.*

Proof. Using the covariance of the the PTInG from (4.5), we derive expression for correlation function of the PTInG as

$$\begin{aligned}
&\mathrm{Corr}[N(S_{\alpha,\theta}(s)), N(S_{\alpha,\theta}(t))] \\
&= \frac{\mathrm{Cov}[N(S_{\alpha,\theta}(s)), N(S_{\alpha,\theta}(t))]}{(\mathrm{Var}[N(S_{\alpha,\theta}(s))])^{\frac{1}{2}}(\mathrm{Var}[N(S_{\alpha,\theta}(t))])^{\frac{1}{2}}} \\
&= \frac{s\left((\lambda\alpha\theta^{\alpha-1}e^{-\theta}) + \lambda^2(\alpha\theta^{\alpha-1}e^{-\theta} + (\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})\right)}{(ts)^{\frac{1}{2}}(\lambda\alpha\theta^{\alpha-1}e^{-\theta}) + \lambda^2(\alpha\theta^{\alpha-1}e^{-\theta} + (\alpha-1)\alpha\theta^{\alpha-2}e^{-\theta})} \\
&= t^{\frac{-1}{2}}s^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\lim_{t\to\infty}\frac{\mathrm{Corr}[N(S_{\alpha,\theta}(s)), N(S_{\alpha,\theta}(t))]}{t^{\frac{-1}{2}}} = s^{\frac{1}{2}}. \tag{4.10}$$

This completes the proof of LRD property for $\{N(S_{\alpha,\theta}(t))\}_{t\geq 0}$. □

5. Application in insurance ruin

The ruin theory is a branch of actuarial science that deals with the financial modeling of the likelihood of a company or individual becoming insolvent. The classical risk process of insurance defined below models the distribution of claims, balance of assets and liabilities over time

$$Z(t) = ct - \sum_{j=0}^{N(t)} X_j, t \geq 0,$$

where $c > 0$ is fixed premium rate and $\{N(t)\}_{t\geq 0}$ is the homogeneous Poisson process which counts claims arrival till time t. The claim amount X_j with distribution F is independent of $N(t)$.

We here propose to use the PTInG process $\{N(S_{\alpha,\theta}(t)\}_{t\geq 0}$ replacing the Poisson process $\{N(t)\}_{t\geq 0}$ in the classical risk process $\{Z(t)\}_{t\geq 0}$. The number of claims in $\{Z(t)\}_{t\geq 0}$ follows the Poisson distribution which assumes that the arrivals are i.i.d. while our proposal model has the LRD property (see Theorem 4.8). The LRD property assumes some sort of dependence on the successive claims and it is a more closer approximation of a real-life situation.

Consider the risk model

$$Y(t) = ct - \sum_{j=1}^{N(S_{\alpha,\theta}(t))} X_j, t \geq 0, \tag{5.1}$$

where $c > 0$ denotes a constant premium rate and X_j are non-negative i.i.d. random variables with distribution F, representing the claim size.

The joint probability of ruin and deficit is a measure used in actuarial science to assess the financial stability of an insurance company. It describes the probability that an insurance company will not only become insolvent, or "ruined," but also that it will have a deficit in its reserves. This measure is used to evaluate the effectiveness of different risk management strategies, such as adjusting pricing, increasing reserves, or purchasing reinsurance. Actuaries use this measure to evaluate the overall financial stability of the company and to make decisions on how to manage its risks.

In this section, we derive results for the ruin probability, joint distribution of time to ruin and deficit at ruin, and derive its governing differential equation for our proposed model (5.1).

The premium loading factor, denoted by ρ, signifies the profit margin of the insurance firm and is defined as the ratio of $\mathbb{E}[Y(t)]$ and $\mathbb{E}\left[\sum_{j=1}^{N(S_{\alpha,\theta}(t))} X_j\right]$

$$\rho = \frac{\mathbb{E}[Y(t)]}{\mathbb{E}\left[\sum_{j=1}^{N(S_{\alpha,\theta}(t))} X_j\right]} = \frac{ct}{\mu \mathbb{E}[N(S_{\alpha,\theta}(t))]} - 1,$$

where $\mu = \mathbb{E}[X_j]$. Let us denote the initial capital by $u > 0$. Define the surplus process $\{U(t)\}_{t \geq 0}$ by

$$U(t) = u + Y(t), \quad t \geq 0.$$

The insurance company will be called in ruin if the surplus process falls below the zero level. Let T_u be the random variable which denotes the first time to ruin. It is defined as

$$T_u = \inf \{t > 0 : U(t) < 0\}.$$

The probability of ruin is given by $\psi(u) = \mathbb{P}\{T_u < \infty\}$. The joint probability that ruin happens in finite time and the deficit at the time of ruin, denoted as $D = |U(t)|$, is given by

$$G(u, y) = \mathbb{P}\{T_u < \infty, D \leq y\}, \quad y \geq 0. \tag{5.2}$$

Using (4.7), we get

$$
\begin{aligned}
G(u,y) = (1 - hf(\lambda))G(u+ch,y) \\
&- h\frac{(-1)\lambda f'(\lambda)}{1!}\left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right] \\
&- h\frac{(-1)^2\lambda^2 f''(\lambda)}{2!}\left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right] \\
&- \cdots \\
&= (1 - hf(\lambda))G(u+ch,y) - h\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda)\times \\
&\left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right]
\end{aligned}
$$

After rearranging the terms, we have that

$$
\frac{G(u+ch,y) - G(u,y)}{ch} = \frac{1}{c}f(\lambda)G(u+ch,y) + \left(\frac{1}{c}\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda)\left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right]\right).
$$

Now taking limit $h \to 0$, we get

$$
\begin{aligned}
\frac{\partial G}{\partial u} &= \frac{f(\lambda)}{c}G(u,y) + \left(\frac{1}{c}\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda)\left[\int_0^{u} G(u-x,y)dF(x) + (F(u+y) - F(u))\right]\right) \\
&= \frac{f(\lambda)}{c}G(u,y) + \left(\frac{1}{c}\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda)\left[\int_0^{u} G(u-x,y)dF(x) + (F(u+y) - F(u))\right]\right),
\end{aligned}
$$

Using Taylor's series, we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda) &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}f^{(n)}(\lambda) - f(\lambda) \\
&= f(0) - f(\lambda) \\
&= -f(\lambda).
\end{aligned}
$$

Theorem 5.1. *Let $G(u,y)$ defined in (5.2), denote the joint probability distribution of time to ruin and deficit* *at this time of the risk model (*5.1*). Then, it satisfies the following integro-differential equation*

$$\frac{\partial G}{\partial u} = \frac{f(\lambda)}{c}\left[G(u,y) - \int_0^u G(u-x,y)dF(x) + (F(u+y)-F(u))\right]. \tag{5.3}$$

Theorem 5.2. *The joint distribution of ruin time and deficit at ruin when the initial capital is zero, $G(0,y)$ is given by*

$$G(0,y) = \frac{f(\lambda)}{c}\left[\int_0^\infty (F(u+y)-F(u))du\right]. \tag{5.4}$$

Proof. On integrating (5.3) with respect to U on $(0,\infty)$, we obtain

$$G(0,\infty) - G(0.y) = \frac{f(\lambda)}{c}\left[\int_0^\infty G(u,y)du - \int_0^\infty \int_0^u G(u-x,y)dF(x)du \right.$$
$$\left. + \int_0^\infty (F(u+y) - F(u)du)\right].$$

Note that $G(0,\infty) = 0$, then

$$G(0,y) = \frac{f(\lambda)}{c}\left[\int_0^\infty (F(u+y)-F(u))du\right].$$

□

Remark 5.3. *On taking $y \to \infty$ in (*5.4*), we get*

$$\psi(0) = \frac{f(\lambda)}{c}\left[\int_0^\infty (1-F(u))du\right].$$

Remark 5.4. *As $\lim_{y\to\infty} G(u,y) = \psi(u)$, On taking $y \to \infty$ in (*5.3*) we obtain*

$$\frac{\partial \psi}{\partial u} = \frac{f(\lambda)}{c}\left[\psi(u) + \int_0^u \psi(u-x)dF(x) + (1-F(u))\right].$$

6. Plots of sample path simulation, pdf and likelihood function

In this section, we present algorithms to simulate sample paths for the InG, the InG-ϵ, the TInG subordinators and the Poisson process subordinated by these subordinators. Also, we provide the pdf and likelihood plots of the above mentioned subordinators.

6.1. Simulation of subordinators

To simulate the InG subordinator, we first calculate the cdf $F_{Z^\alpha}(\cdot)$ of random variable Z^α with pdf given by (2.3). Let $F_{Z^\alpha}(\cdot)$ be the cdf of random variable Z^α, then

$$\begin{aligned} F_{Z^\alpha}(x) = \int_{-\infty}^{x} f_{Z^\alpha}(z)dz = \int_{-\infty}^{z} \frac{(z-1)^{-\alpha}z^{-1}1_{z\geq 1}}{\Gamma(1-\alpha)\Gamma(\alpha)}dz = \int_{-\infty}^{x} \frac{\sin(\pi\alpha)1_{z\geq 1}}{\pi((z-1)^{-\alpha})z}dz \\ = \int_{1}^{x} \frac{(z-1)^{-\alpha}z^{-1}}{\Gamma(1-\alpha)\Gamma(\alpha)}dz = \frac{\pi\csc(\pi\alpha) - I_{\frac{1}{x}}(\alpha,(1-\alpha))}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ = 1 - B_{\frac{1}{x}}(\alpha,(1-\alpha)), \end{aligned} \tag{6.1}$$

where $B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy$ is the incomplete beta function.

Now, we present algorithm to simulate the InG subordinator using algorithm of compound Poisson process (see [31]).

Algorithm 1: Simulation of the InG subordinator

Input: $\lambda > 0$, $\alpha \in (0,1)$ and $T \geq 0$.

Output: $Y(t)$, simulated sample paths for the InG subordinator.

Initialisation : $t = 0$ and $Y = 0$.

1: **while** $t < T$ **do**
2: generate a uniform random variable $U \sim U(0,1)$.
3: set $t \leftarrow t - U/\lambda$.
4: generate i.i.d. random variable Z^α using the inverse transform method to cdf (6.1).
5: set $Y \leftarrow Y + Z^\alpha$.
6: **end while**
7: **return** Y.

The cdf $F_{Z^{\alpha,\epsilon}}$ of random variable $Z^{(\alpha,\epsilon)}$ can be obtained on same line as (6.1) and by using (2.5)

$$F_{Z^{(\alpha,\epsilon)}}(x) = 1 - B_{\frac{\epsilon}{x}}(\alpha,(1-\alpha)). \tag{6.2}$$

We can simulate the sample paths for InG-ϵ subordinator using Algorithm 1 by replacing equation (6.1) by (6.2) in Step 4:.

It can be noted that we can not simulate the TInG subordinator using Algorithm 1 as the cdf of random variable $Z^{\alpha,\theta}$ does not have closed form and therefore we turn to Markov Chain Monte Carlo (MCMC) simulation schemes. More specifically, we use the Metropolis algorithm as a special case of the MCMC scheme. In this method, we generate Y with pdf f_Y, called as target density, by choosing another random variable V with pdf f_V, called as candidate density, such that f_Y and f_V have common support. The process is repeated for a large number of iterations, and the resulting sequence of accepted values approximates the desired probability distribution.

We next present the Metropolis algorithm (see [27, p.254]) with $f_{Z^{\alpha,\theta}}$ as the target density. We define the candidate density f_V as

$$f_V(v) = \begin{cases} \frac{\lambda e^{-\lambda v}}{e^{-\lambda}}, & v \in [1, \infty) \\ 0, & \text{otherwise} \end{cases}, \tag{6.3}$$

by truncating the exponential density. To generate $Z^{\alpha,\theta} \sim f_{Z^{\alpha,\theta}}$ using $f_V(v)$ we use Metropolis algorithm. Let $Z^{\alpha,\theta} \sim f_{Z^{\alpha,\theta}}(z)$ and $V \sim f_V(v)$, where $f_{Z^{\alpha,\theta}}$ and f_V have common support.

Algorithm 2: Metropolis algorithm

Input: $f_{Z_{\alpha,\theta}}(z)$ and $f_V(v)$ with common support, parameter α, θ, and number of iterations N.

Output: random variable with pdf $f_{Z_{\alpha,\theta}}(z)$.

1: generate an initial sample Z_0 from the distribution $f_V(v)$.

2: **for** $i = 1$ to N **do**

3: generate a random variable $U_i \sim \text{uniform}(0,1)$ and $V_i \sim f_V$.

4: calculate the acceptance probability $\rho_i \leftarrow \min\left\{\frac{f_{Z_{\alpha,\theta}}(V_i)}{f_V(V_i)} \cdot \frac{f_V(Z_{i-1})}{f_{Z_{\alpha,\theta}}(Z_{i-1})}, 1\right\}$.

5: set

$$Z_i \leftarrow \begin{cases} V_i & \text{if } U_i \leq \rho_i \\ Z_{i-1} & \text{if } U_i > \rho_i \end{cases},$$

6: **end for**

7: **return** Z_N

Above algorithm produces random variables Z_i from the pdf (approximately) $f_{Z^{\alpha,\theta}}$. We use this Z_i random variables to simulate the TInG subordinator.

Algorithm 3: Simulation of the TInG subordinator

Input: $\lambda > 0$, $\alpha \in (0,1)$ and $T \geq 0$.

Output: sample paths of $Y(t)$, the TInG subordinator.

Initialisation : $t = 0, Y = 0$.

while $t \leq T$ **do**

  generate a uniform random variable $U \sim U(0,1)$.

  set $t \leftarrow t - (1/\lambda) * (U)$.

  generate i.i.d. random variable Z using Algorithm 2.

  set $Y \leftarrow Y + Z$.

end while

return Y.

Remark 6.1. *The sample paths of the InG and the InG-ϵ subordinators can also be generated using the Metropolis algorithm. To simulate the InG subordinator we use the candidate density* (6.3)*, and for the InG-ϵ subordinator, the candidate density is defined as*

$$f_V(v) = \begin{cases} \frac{\lambda e^{-\lambda v}}{e^{-\lambda \epsilon}}, & v \in [\epsilon, \infty) \\ 0, & otherwise \end{cases}. \tag{6.4}$$

Interpretation of sample paths

We simulate the samples paths of the InG and the InG-ϵ subordinators in Figures 3 and 4 respectively, using Algorithm 1. The sample paths of the TInG subordinator are presented in Figure 6(a) using the Algorithm 3. It can be noted that from Figures 3 and 4 that the size of jumps vary with the parameter α for the InG and InG-ϵ subordinators. The InG subordinator has all the jumps greater than one while we can choose the ϵ-jumps in the InG-ϵ subordinator. It can be observed that, in comparison with other subordinators, the jump activity of the TInG subordinator is quite muted due to the tempering parameter θ. It is clear that one can get an fair idea of the jump activity from the sample paths of these subordinators and it helps choose us a model for jump processes. The parameter estimation of these subordinators will be an important and interesting problem to consider for a future work in this direction.

6.2. Simulation of subordinated Poisson process

Next, we present algorithm for simulating the sample path of subordinated Poisson process. First we reproduce an algorithm for generating the sample paths for the Poisson process (see [2]) with rate $\lambda > 0$.

Algorithm 4: Simulation of Poisson process

Input: $\lambda > 0$ and $T \geq 0$.

Output: the sample paths of $N(t)$, the Poisson process.

Initialisation : $N = 0$ and $t = 0$.

1: **while** $t \leq T$ **do**
2: generate a uniform random variable $U \sim U(0, 1)$.
3: set $dt \leftarrow (1 - U)/\lambda$.
4: $N + 1 \leftarrow N$ and $t \leftarrow t + dt$.
5: **end while**
6: **return** N which denotes the number of events $N(t)$ occurred up to time T.

We now present algorithm to simulate the PInG, PInG-ϵ and PTInG. The following algorithm is common to all three time-changed Poisson processes as we use the corresponding algorithms mentioned above for respective subordinators.

Algorithm 5: Simulation for subordinated Poisson process

Input: Parameter μ for respective subordinator $X(t)$, $\lambda > 0$, $T \geq 0$.

Output: The sample paths of the subordinated Poisson process $N(X(t_i)),\ 1 \leq i \leq n$.

Initialisation :

1: set $h = \frac{T}{n}$ and choose $n + 1$ uniformly spaced time points $0 = t_0, t_1, \ldots, t_n = T$ with $h = t_1 - t_0$.
2: generate the values $X(t_i),\ 1 \leq i \leq n$, (e.g. the InG, InG-ϵ, and TInG) for the subordinator at time points $t_1, t_2, ..., t_n$ using the Algorithms 1 and 3, respectively.
3: use the values $X(t_i),\ 1 \leq i \leq n$, generated in Step 2, as time points and compute the number of events of the subordinated Poisson process $N(X(t_i)),\ 1 \leq i \leq n$, using Algorithm (4).
4: **return** $N(X(t_i))$

Interpretation of sample paths

The sample paths of the PInG and the PInG-ϵ subordinator are presented in Figure 7 while the sample paths of the PTInG are shown in Figure 6 (b). The erratic nature of the count process is more evident in PInG sample paths, less so in PInG-ϵ due to small jump size and even lesser in PTInG due to the tempering parameter θ. As stated in the previous subsection, one can choose the time-changed Poisson process based on the nature of application of the underlying phenomena. A future work on parameter estimation of these processes can add value to this understanding.

6.3. Likelihood and pdf plots

In this subsection, we focus on the nature of subordinators by plotting their pdfs and the likelihood function plot, that is, fixing the jump size z and varying the parameter α. This will enhance our understanding of the relation between parameter α and the jump size z. We will use (2.3), (2.5) and (2.7) to chart the likelihood and pdf plots in Figures 1, 2 and 5, respectively for the InG, the InG-ϵ and the TInG subordinators.

Interpretation of likelihood and pdf plots

It can be noted from Figure 1 (a) and 2 (a) that both the InG and the InG-ϵ subordinator have high pdf value for small size of jumps and value goes on decreasing with increase in jump size. It means that for these subordinators small jumps have more chances to occur comparing to big jumps. As we go on increasing the jumps size than it has more probability of occurrence towards smaller value of α can be seen from Figure 2 and area under the curve start to drift towards smaller value of α. On comparing Figure 1 and 2 we observed that bigger jumps are having less chances of occurrence in InG-ϵ subordinator. Our above claims about jumps occurrence and sizes well supported by Figure 3 and 4, as figure for $\alpha = 0.8$ contains more number of jumps and theirs size is small comparing to figure for $\alpha = 0.2$.

It is observed from Figure 5 that TInG subordinator have similar behaviour to InG and InG-ϵ regarding α and jumps variation, but for it big jumps are too rare to occur for the TInG subordinator can be noted from Figure 6(a).

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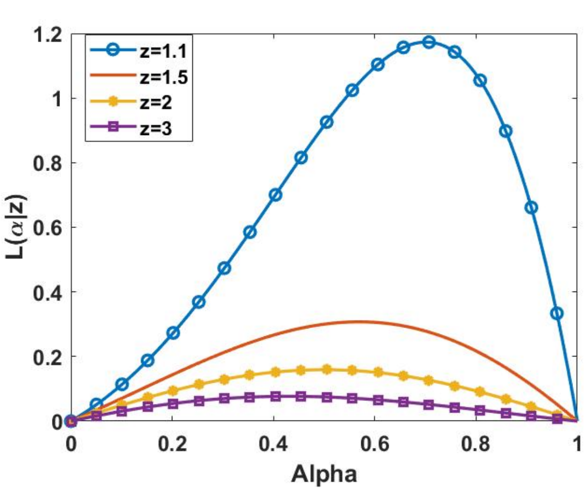
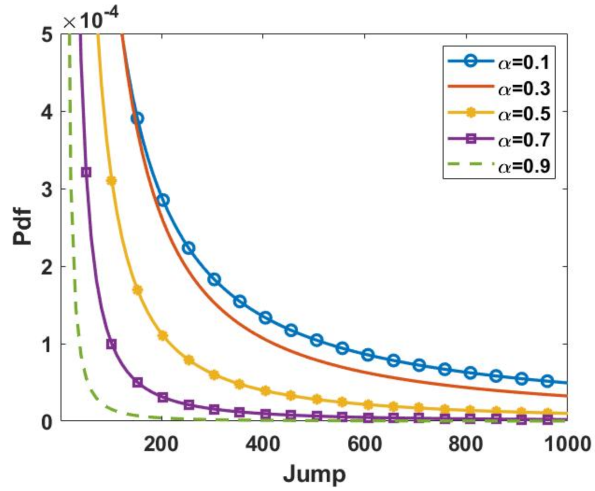
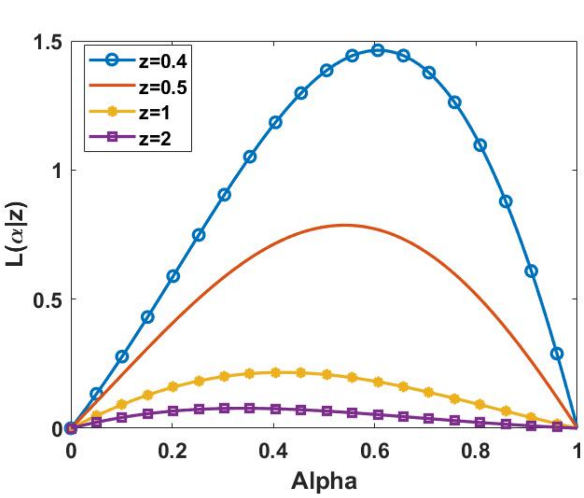


(a) Likelihood L(α|z) plot of InG subordinator

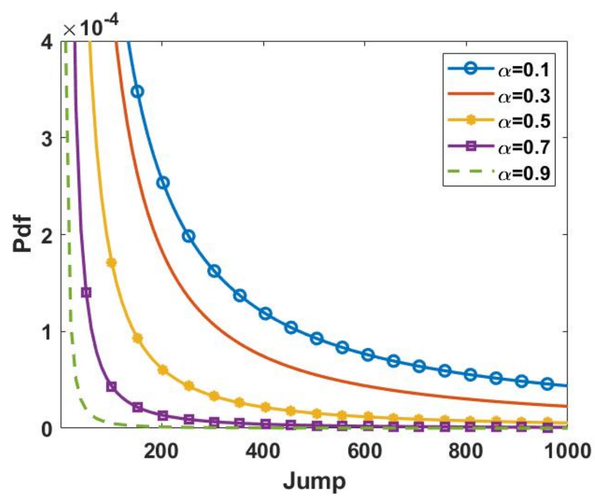


(b) Pdf plot of InG subordinator

Figure 1: Likelihood and pdf plots of the InG subordinator for different fixed (a) jump size z, and (b) value of α.

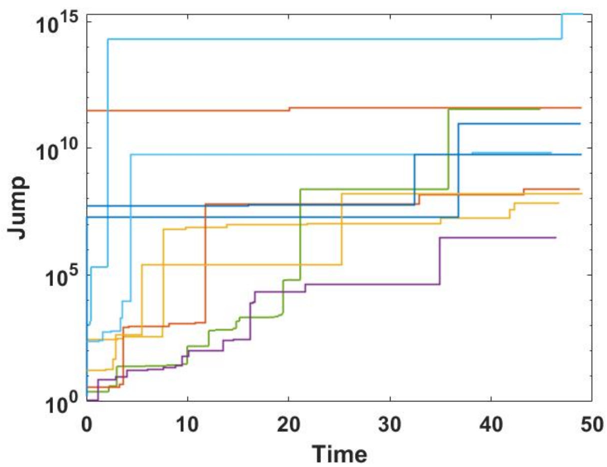


(a) Likelihood L(α|z) plot of the InG-ϵ subordinator

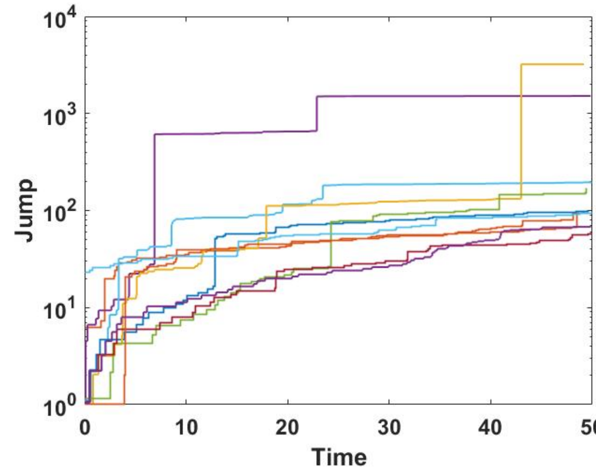


(b) Pdf plot of the InG-ϵ subordinator

Figure 2: Likelihood and pdf plots of the InG-ϵ subordinator for different fixed (a) jump size z, and (b) value of α

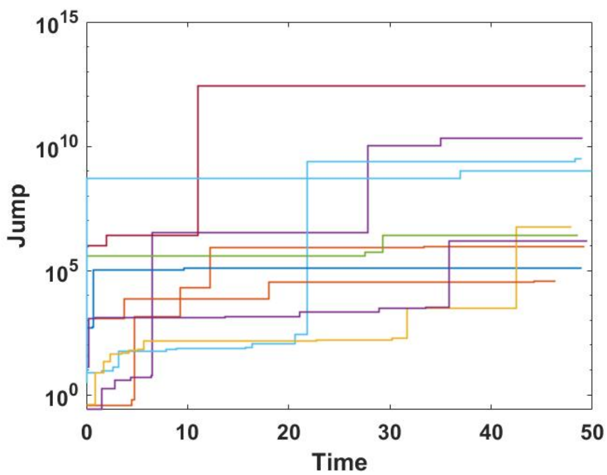


(a) Parameter: $\alpha = 0.2$ and $\lambda = \alpha\Gamma(\alpha) = 0.9182$

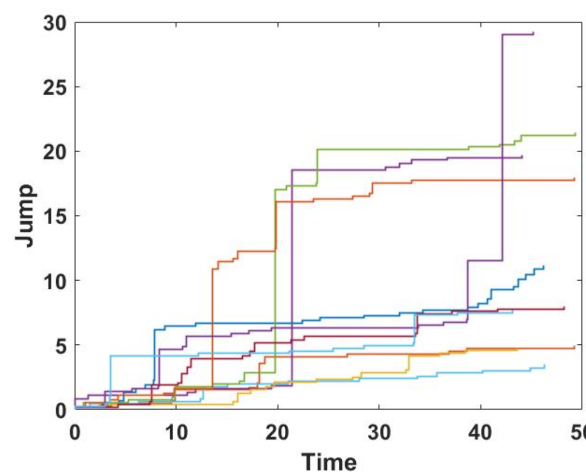


(b) Parameter $\alpha = 0.8$ and $\lambda = \alpha * \Gamma(\alpha) = 0.9314$

Figure 3: Ten simulated sample paths of the InG subordinator for different parameters.

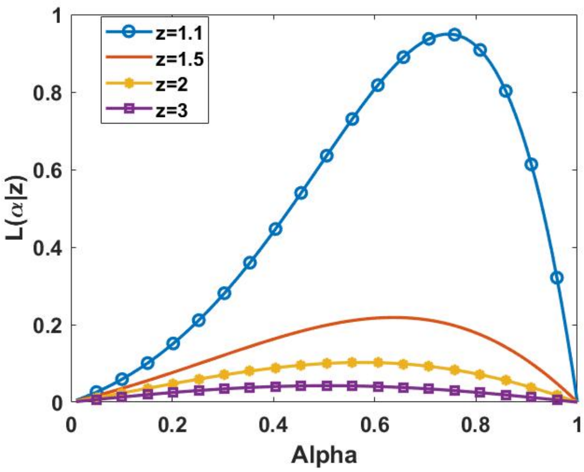


(a) Parameter : $\alpha = 0.1, \epsilon = 0.2$ and $\lambda = \epsilon^{\alpha}\alpha\Gamma(\alpha) = 0.6655$

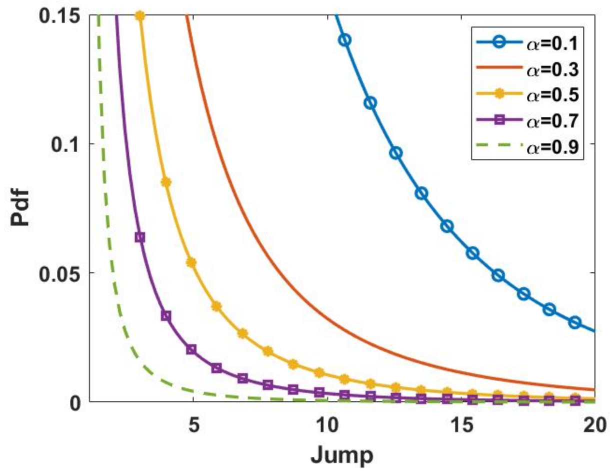


(b) Parameter : $\alpha = 0.8, \epsilon = 0.2$ and $\lambda = \epsilon^{\alpha}\alpha\Gamma(\alpha) = 0.2570$

Figure 4: Ten simulated sample paths of the InG-ϵ subordinator for different parameters.

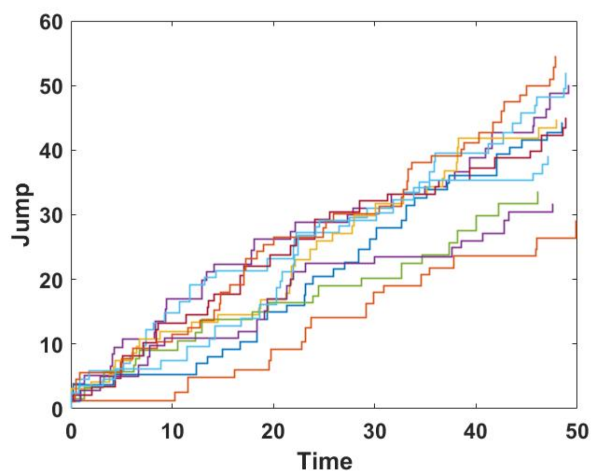


(a) Likelihood L(α|z) plot of the TInG subordinator

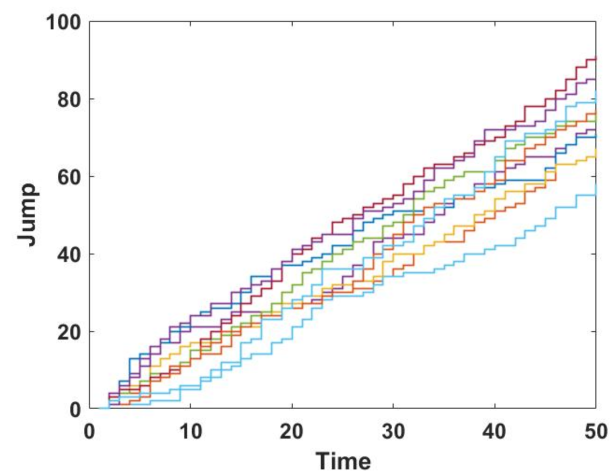


(b) Pdf plot of the TInG subordinator

Figure 5: Likelihood and pdf plots of the TInG subordinator for different fixed (a) jump size z, and (b) value of α

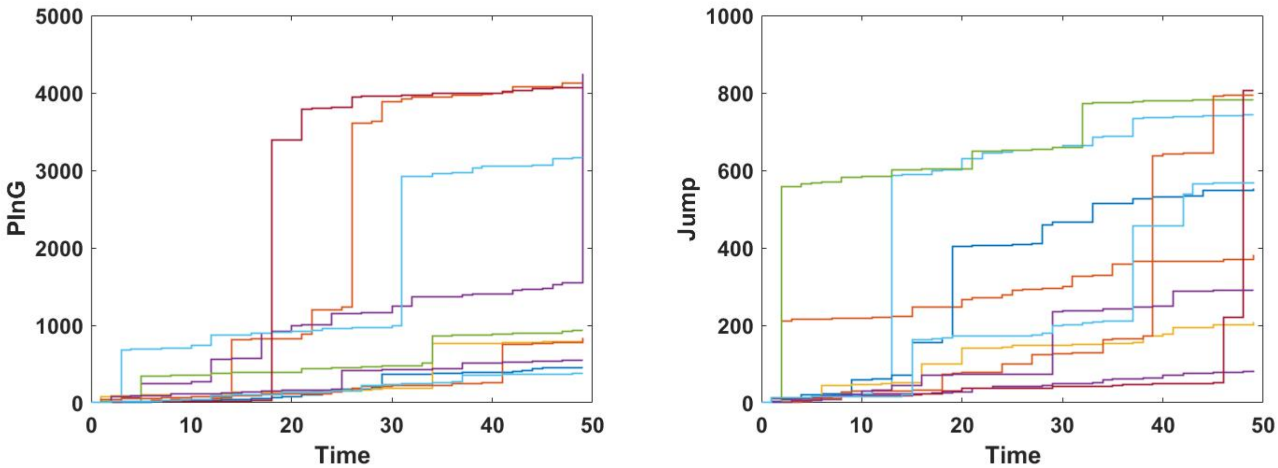


(a) Parameter: $\alpha = 0.2$, $\theta = 0.1$ and $\lambda = \alpha\Gamma[\alpha, \theta] = 0.5802$



(b) Parameter: $\alpha = 0.2$ and $\lambda = 1$

Figure 6: Ten simulated sample paths of the TInG subordinator and PTInG process using the Metropolis algorithm



(a) Parameter: $\alpha = 0.5$ and $\lambda = 1$

(b) Parameter: $\alpha = .5$, $\epsilon = 0.2$ and $\lambda = 1$

Figure 7: Ten simulated sample paths of the PInG and the PInG-ϵ processes.

References

[1] P. K. Clark, A subordinated stochastic process model with finite variance for speculative prices, Econometrica 41 (1973) 135–155.

[2] A. Maheshwari, P. Vellaisamy, Fractional Poisson process time-changed by Lévy subordinator and its inverse, Journal of Theoretical Probability 32 (2019) 1278–1305.

[3] L. Zhang, A. Song, J. He, Stochastic resonance of a subdiffusive bistable system driven by Lévy noise based on the subordination process, Journal of Physics A: Mathematical and Theoretical 42 (47) (2009) 475003. doi:10.1088/1751-8113/42/47/475003.

[4] M. G. Nezhadhaghighi, M. Rajabpour, S. Rouhani, First-passage-time processes and subordinated Schramm-Loewner evolution, Physical Review E 84 (1) (2011) 011134.

[5] D. O. Cahoy, F. Polito, On a fractional binomial process, Journal of Statistical Physics 146 (3) (2011) 646–662. doi:10.1007/s10955-011-0408-3.

[6] A. Stanislavsky, K. Weron, Two-time scale subordination in physical processes with long-term memory, Annals of Physics 323 (3) (2008) 643–653. doi:https://doi.org/10.1016/j.aop.2007.04.011.

[7] H. Scher, G. Margolin, R. Metzler, J. Klafter, B. Berkowitz, The dynamical foundation of fractal stream chemistry: The origin of extremely long retention times, Geophysical Research Letters 29 (5) (2002) 5–1–5–4.
doi:https://doi.org/10.1029/2001GL014123.

[8] I. Golding, E. C. Cox, Physical nature of bacterial cytoplasm, Phys. Rev. Lett. 96 (2006) 098102. doi:10.1103/PhysRevLett.96.098102.

[9] S. Bochner, Diffusion equation and stochastic processes., Proceedings of the National Academy of Sciences of the United States of America 35 7 (1949) 368–70.

[10] J. L. B. Cooper, Harmonic analysis and the theory of probability. by s. bochner pp. 176. 35s. 1955. (california university press and cambridge university press), The Mathematical Gazette 41 (336) (1957) 154–155. doi:10.2307/3610639.

[11] J. W. Woll Jr, Homogeneous stochastic processes., Pacific Journal of Mathematics 9 (1) (1959) 293–325.

[12] A. Miyake, The subordination of lévy system for markov processes, Proc. Japan Acad. 45 (8) (1969) 601–604.

[13] R. Biard, B. Saussereau, Fractional Poisson process: long-range dependence and applications in ruin theory, J. Appl. Probab. 51 (3) (2014) 727–740. `doi:10.1239/jap/1409932670`.

[14] A. Maheshwari, P. Vellaisamy, On the long-range dependence of fractional poisson and negative binomial processes, Journal of Applied Probability 53 (4) (2016) 989–1000.

[15] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Vol. 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999, translated from the 1990 Japanese original.

[16] D. Applebaum, Lévy processes and stochastic calculus, Cambridge university press, 2009.

[17] J. Bertoin, Subordinators: examples and applications, in: Lectures on probability theory and statistics, Springer, 1999, pp. 1–91.

[18] K. Sato, Lévy processes and infinitely divisible distributions (Cambridge studies in advanced mathematics 68, Cambridge university press, Cambridge) (1999).

[19] K. Buchak, L. Sakhno, Compositions of poisson and gamma processes, Modern Stochastics: Theory and Applications 4 (2) (2017) 161–188.

[20] A. Kumar, E. Nane, P. Vellaisamy, Time-changed poisson processes, Statistics & Probability Letters 81 (12) (2011) 1899 – 1910. `doi:http://dx.doi.org/10.1016/j.spl.2011.08.002`.

[21] E. Orsingher, B. Toaldo, Counting processes with Bernstein intertimes and random jumps, Journal of Applied Probability 52 (4) (2015) 1028–1044.

[22] M. Meerschaert, E. Nane, P. Vellaisamy, The fractional Poisson process and the inverse stable subordinator, Electronic Journal of Probability 16 (59) (2011) 1600–1620.

[23] E. Orsingher, F. Polito, The space-fractional Poisson process, Statistics & Probability Letters 82 (4) (2012) 852–858. `doi:10.1016/j.spl.2011.12.018`.

[24] A. S. Sengar, A. Maheshwari, N. Upadhye, Time-changed Poisson processes of order k, Stochastic Analysis and Applications 38 (1) (2020) 124–148.

[25] L. Beghin, C. Ricciuti, Lévy processes linked to the lower-incomplete gamma function, Fractal and Fractional 5 (3) (2021). `doi:10.3390/fractalfract5030072`.

[26] G. Samorodnitsky, et al., Stochastic processes and long range dependence, Vol. 26, Springer, 2016.

[27] G. Casella, R. Berger, Statistical Inference, Thomson Learning, 2002.

[28] A. Kumar, J. Gajda, A. Wyłomańska, R. Połoczański, Fractional Brownian motion delayed by tempered and inverse tempered stable subordinators, Methodology and Computing in Applied Probability 21 (1) (2019) 185–202.

[29] W. Feller, An introduction to probability theory and its applications, vol 2, John Wiley & Sons, 2008.

[30] J. Wojdylo, On the coefficients that arise from Laplace's method, Journal of Computational and Applied Mathematics 196 (1) (2006) 241–266.

[31] S. Abdullah, S. Susilo, I. W. Mangku, F. Ikhsan, S. Ula, Y. Rukmayadi, Algorithm for generating compound Poisson process which has non-homogeneous Poisson process and exponential distribution components, in: 1st International Multidisciplinary Conference on Education, Technology, and Engineering (IMCETE 2019), Atlantis Press, 2020, pp. 241–246.