POISSON'S RATIO FOR ANISOTROPIC ELASTIC MATERIALS CAN HAVE NO BOUNDS

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Summary

Poisson's ratio for isotropic elastic materials is bounded between -1 and $\frac{1}{2}$. It is shown that Poisson's ratio for anisotropic elastic materials can have an arbitrarily large positive or negative value under the prerequisite of positive definiteness of strain energy density. The large Poisson's ratio for cubic materials is physically realistic because the strains are bounded.

1. Introduction

Poisson's ratio $\nu(\mathbf{n}, \mathbf{m})$ of an elastic solid for any two specified orthogonal unit vectors **n** and **m** is the ratio of the lateral contraction in the direction **m** to the axial extension in the direction **n** due to a uniaxial tension of the material along the direction **n**. It is well known that for an isotropic elastic material Poisson's ratio does not depend on the choice of **n** and **m**, and is bounded between -1 and $\frac{1}{2}$ for the material to be stable. In contrast, Poisson's ratio for an anisotropic elastic material depends on the choice of **n** and **m**. There have been considerable fundamental and practical interests in exploring the admissible range of Poisson's ratio that the material response along a certain crystallographic plane can actually attain.

Poisson's ratio for a cubic material was studied by Turley and Sines (**1**) who presented formula for computing Poisson's ratio for any given **n** and **m**. They did not give the extrema of Poisson's ratio. Using the results of (**1**) Baughman *et al.* (**2**) gave the extrema of Poisson's ratio in (**2**, equations (1) and (2)) for **n** at [110] and **m** at $[1\bar{1}0]$ and [001], respectively. From the two equations they obtained numerically that Poisson's ratio is bounded between $(-1, 0)$ and $(0, 2)$, respectively. Recently, Baughman (**3**) stated that 'Although Poisson's ratio for cubic phases must be between -1 and $+2$, there is no theoretical limitation on this ratio for materials with less internal symmetry'. He has been interested mainly in materials with negative Poisson's ratio, called *auxetic materials*, that have some practical applications (see also Lakes (**4**)). Hayes and Shuvalov (**5**) established extreme values of Poisson's ratio for cubic materials with elastic constants satisfying certain inequalities. The original formulation in (**5**) has a flaw in that it does not depend on whether $2(s_{11} - s_{12}) - s_{44}$ is positive or negative. The correct statement is the following (Hayes and

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Shuvalov, private communication). When **n** is at [001] and **m** at [110], $\nu(\mathbf{n}, \mathbf{m}) = -s_{12}/s_{11}$ so that the value of $\nu(\mathbf{n}, \mathbf{m})$ lies between $(-1, \frac{1}{2})$. For crystals with less internal symmetry, Poisson's ratio for transversely isotropic (hexagonal) material was investigated by Li (**6**) but no extrema were presented. Using an example, Boulanger and Hayes (**7**) appear to be the first to show that Poisson's ratio for an orthotropic material can have no bounds. A complete characterization of the admissible sets of Poisson's ratios for orthotropic materials and materials with less internal symmetry was presented by Zheng and Chen (**8**). They showed that points within a three-dimensional closed body can represent the normalized Poisson's ratios, and the body cannot be bounded after transformation back to the physical space.

In this paper we will show that Poisson's ratio for any anisotropic elastic material can have no bounds. It is known that there are eight elastic symmetries (**9** to **12**). They are triclinic, monoclinic, orthotropic (rhombic), trigonal, tetragonal, transversely isotropic (hexagonal), cubic and isotropic. As it turns out, it is easiest to prove that Poisson's ratio can have no bounds for triclinic, monoclinic and orthotropic materials. This is presented in section 2. The trigonal, tetragonal and hexagonal materials are discussed in section 3. The cubic material is the hardest one to prove, and is discussed in section 4. It is shown in the Appendix that the large Poisson's ratio for cubic materials is due to the vanishing of the uniaxial strain in the direction **n** while the strain in the lateral direction **m** is bounded. Thus the large Poisson's ratio is physically admissible within the linear theory of elasticity.

2. Triclinic, monoclinic and orthotropic materials

In a fixed rectangular coordinate system x_i $(i = 1, 2, 3)$, the stress–strain relation can be written as

$$\varepsilon_{ij} = S_{ijkt}\sigma_{kt}, \quad S_{ijkt} = S_{jikt} = S_{ktij} = S_{ijtk}, \tag{2.1}$$

where ε_{ij} and σ_{ij} are the strain and stress and S_{ijkt} is the elastic compliance. The S_{ijkt} is positive definite and possesses the full symmetry shown in $(2.1)_2$. The third equality in $(2.1)_2$ is redundant because the first two imply the third (**13**, p. 32). When the material is subject to a simple tension in the x_1-direction, σ_{ij} vanish except σ_{11} so that

$$\varepsilon_{11} = S_{1111}\sigma_{11}, \quad \varepsilon_{22} = S_{2211}\sigma_{11} = S_{1122}\sigma_{11}. \tag{2.2}$$

Poisson's ratio ν_{12}, which is the ratio of the contraction in the x_2-direction to the extension in the x_1-direction, is

$$\nu_{12} = -\varepsilon_{22}/\varepsilon_{11} = -S_{1122}/S_{1111} = -s_{12}/s_{11}, \tag{2.3}$$

where $s_{\alpha\beta}$ is the contracted notation of S_{ijkt} (see, for example, (**14**, **15**)).

Written as a 6×6 matrix, $s_{\alpha\beta}$ is symmetric. It must be positive definite for the strain energy density to be positive. This means that $\Delta_k > 0$ $(k = 1, 2, \ldots, 6)$, where Δ_k is the determinant of the $k \times k$ leading principal submatrix of $s_{\alpha\beta}$ (**16**, **17**). For triclinic, monoclinic and orthotropic materials, it suffices to consider the 2×2 leading principal submatrix of $s_{\alpha\beta}$, which is positive definite if

$$\Delta_1 = s_{11} > 0, \quad \Delta_2 = \begin{vmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{vmatrix} = s_{11}s_{22} - s_{12}^2 > 0. \tag{2.4}$$

Instead of s_{22}, s_{12} we may prescribe $\Delta_2 > 0$, ν_{12} and let

$$s_{22} = \nu_{12}^2 s_{11} + \Delta_2/s_{11}, \quad s_{12} = -\nu_{12} s_{11}, \quad s_{11} > 0. \tag{2.5}$$

Equation (2.5) satisfies (2.4) for any choice of ν_{12}. Thus Poisson's ratio ν_{12} can have an arbitrarily large positive (or negative) value. This conclusion is independent of the prescription of the rest of $s_{\alpha\beta}$ because they can be prescribed arbitrarily. When the off-diagonal elements of $s_{\alpha\beta}$ are prescribed, one can always find the diagonal elements s_{33}, s_{44}, s_{55} and s_{66}, in that order, such that $\Delta_i > 0$ for $i = 3, 4, 5, 6$ (**18**). Thus Poisson's ratio for triclinic, monoclinic and orthotropic materials can have no bounds. This provides an alternative proof to the one given by Boulanger and Hayes (**7**) for orthotropic materials.

3. Trigonal, tetragonal and hexagonal materials

The 6×6 matrix $s_{\alpha\beta}$ for trigonal, tetragonal and hexagonal materials has the expression (see, for example, (**13**, sections 2.6 and 2.7)),

$$s_{\alpha\beta} = \begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ & s_{22} & s_{23} & 0 & s_{25} & 0 \\ & & s_{22} & 0 & -s_{25} & 0 \\ & & & s_{44} & 0 & 2s_{25} \\ & & & & s_{55} & 0 \\ & & & & & s_{55} \end{bmatrix}, \tag{3.1}$$

where

$$s_{44} = 2(s_{22} - s_{23}) \tag{3.2}$$

for trigonal materials,

$$s_{25} = 0 \tag{3.3}$$

for tetragonal materials, and both (3.2) and (3.3) hold for hexagonal materials. Only the upper triangle of the matrix is shown in (3.1) because $s_{\alpha\beta}$ is symmetric. Equations (2.4) and (2.5) remain valid here. However, not all of the rest of the elastic compliance $s_{\alpha\beta}$ can be prescribed arbitrarily for trigonal, tetragonal and hexagonal materials so that it is not clear if the 6×6 matrix $s_{\alpha\beta}$ can be made positive definite. If it can, then Poisson's ratio for trigonal, tetragonal and hexagonal materials can have no bounds.

The determinants Δ_1, Δ_2 of the first two leading principal submatrices of $s_{\alpha\beta}$ in (3.1) are given in (2.4) while the rest can be shown to be

$$\Delta_3 = (s_{22} - s_{23})K > 0, \tag{3.4}$$

$$\Delta_4 = s_{44}\Delta_3 > 0, \tag{3.5}$$

$$\Delta_5 = s_{44}(s_{55}\Delta_3 - 2s_{25}^2 K) > 0, \tag{3.6}$$

$$\Delta_6 = (s_{55}\Delta_3 - 2s_{25}^2 K)(s_{44}s_{55} - 4s_{25}^2) > 0, \tag{3.7}$$

where

$$K = 2\Delta_2 - s_{11}(s_{22} - s_{23}). \tag{3.8}$$

The inequality (3.4) is satisfied by

$$s_{22} - s_{23} > 0, \quad K > 0, \tag{3.9}$$

which provide the restriction on s_{23} in the range

$$s_{22} - 2(\Delta_2/s_{11}) < s_{23} < s_{22}. \tag{3.10}$$

The conditions (3.5) to (3.7) are satisfied if we choose

$$s_{44} > 0, \quad s_{55}\Delta_3 > 2s_{25}^2 K, \quad s_{44}s_{55} > 4s_{25}^2. \tag{3.11}$$

For trigonal materials for which (3.2) holds, $(3.11)_1$ is automatically satisfied in view of $(3.9)_1$. As to tetragonal materials for which (3.3) holds, $(3.11)_{2,3}$ are satisfied by $s_{55} > 0$. The same is true for hexagonal materials.

Thus the 6×6 matrix $s_{\alpha\beta}$ can be made positive definite, which proves that Poisson's ratio for trigonal, tetragonal and hexagonal materials can have no bounds.

4. Cubic materials

The 6×6 matrix $s_{\alpha\beta}$ for cubic materials has the expression

$$s_{\alpha\beta} = \begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ & s_{11} & s_{12} & 0 & 0 & 0 \\ & & s_{11} & 0 & 0 & 0 \\ & & & s_{44} & 0 & 0 \\ & & & & s_{44} & 0 \\ & & & & & s_{44} \end{bmatrix}. \tag{4.1}$$

The conditions for the strain energy density to be positive definite are

$$s_{11} > 0, \quad \begin{vmatrix} s_{11} & s_{12} \\ s_{12} & s_{11} \end{vmatrix} > 0, \quad \begin{vmatrix} s_{11} & s_{12} & s_{12} \\ s_{12} & s_{11} & s_{12} \\ s_{12} & s_{12} & s_{11} \end{vmatrix} > 0, \quad s_{44} > 0. \tag{4.2}$$

The first two inequalities can be replaced by

$$s_{11} + s_{12} > 0, \quad s_{11} - s_{12} > 0, \tag{4.3}$$

because they satisfy $(4.2)_2$ while the addition of the two inequalities recovers $(4.2)_1$. The third inequality of (4.2) is

$$(s_{11} - s_{12})^2(s_{11} + 2s_{12}) > 0 \tag{4.4}$$

or, in view of $(4.3)_2$,

$$s_{11} + 2s_{12} > 0. \tag{4.5}$$

Since

$$s_{11} + s_{12} = \tfrac{1}{3}(s_{11} - s_{12}) + \tfrac{2}{3}(s_{11} + 2s_{12}), \tag{4.6}$$

$(4.3)_2$ and (4.5) imply $(4.3)_1$. Thus all we need are $(4.3)_2$, (4.5) and $(4.2)_4$. They can be written in a concise form as

$$-\tfrac{1}{2}s_{11} < s_{12} < s_{11}, \quad s_{44} > 0. \tag{4.7}$$

Let **n**, **m**, **t** be mutually orthogonal unit vectors and let x'_i $(i = 1, 2, 3)$ be a new coordinate system in which the x'_1-, x'_2-, x'_3-axes are along the vectors **n**, **m**, **t**, respectively. The elastic compliance S'_{1111} and S'_{1122} referred to the x'_i coordinate system are (**19**)

$$\begin{aligned} S'_{1111} &= n_i n_j n_k n_t S_{ijkt} = s_{11} - 2(s_{11} - s_{12} - \tfrac{1}{2}s_{44})(n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2), \\ S'_{1122} &= n_i n_j m_k m_t S_{ijkt} = s_{12} + (s_{11} - s_{12} - \tfrac{1}{2}s_{44})(n_1^2 m_1^2 + n_2^2 m_2^2 + n_3^2 m_3^2). \end{aligned} \tag{4.8}$$

Poisson ratio $\nu(\mathbf{n}, \mathbf{m})$ is

$$\nu(\mathbf{n}, \mathbf{m}) = -\frac{S'_{1122}}{S'_{1111}} = -\frac{s_{12} + (s_{11} - s_{12} - \tfrac{1}{2}s_{44})(n_1^2 m_1^2 + n_2^2 m_2^2 + n_3^2 m_3^2)}{s_{11} - 2(s_{11} - s_{12} - \tfrac{1}{2}s_{44})(n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2)}. \tag{4.9}$$

Consider the case

$$\begin{gathered} n_1^2 = n_2^2 = \tfrac{1}{3} - \beta, \quad n_3^2 = \tfrac{1}{3} + 2\beta, \\ m_1^2 = m_2^2 = \tfrac{1}{2}, \quad m_3^2 = 0, \end{gathered} \tag{4.10}$$

where $-\frac{1}{6} \leqslant \beta \leqslant \frac{1}{3}$. We show in the Appendix how we discover the **n** in (4.10) and show that the **n** in (4.10) is the only one that can provide an unbounded $\nu(\mathbf{n}, \mathbf{m})$. The signs of n_i, m_i $(i = 1, 2)$ have to be chosen such that $n_1 m_1 + n_2 m_2 = 0$. We then have

$$n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2 = \tfrac{1}{3} - 3\beta^2, \quad n_1^2 m_1^2 + n_2^2 m_2^2 + n_3^2 m_3^2 = \tfrac{1}{3} - \beta, \tag{4.11}$$

and (4.9) becomes

$$\nu(\mathbf{n}, \mathbf{m}) = -\frac{s_{12} + (s_{11} - s_{12} - \tfrac{1}{2}s_{44})(\tfrac{1}{3} - \beta)}{s_{11} - 2(s_{11} - s_{12} - \tfrac{1}{2}s_{44})(\tfrac{1}{3} - 3\beta^2)}. \tag{4.12}$$

When the elastic constants are prescribed, the locations of the extrema of $\nu(\mathbf{n}, \mathbf{m})$ in (4.12) are at

$$\beta = \frac{2\eta - s_{44} \pm \sqrt{D}}{3\,(3s_{11} - \eta - s_{44})}, \tag{4.13}$$

where

$$\eta = s_{11} + 2s_{12}, \quad D = 3\,[s_{11}(\eta + s_{44}) + \eta(\eta - 2s_{44})]. \tag{4.14}$$

Substitution of (4.13) back into (4.12) yields the surprisingly simple expression

$$\nu(\mathbf{n}, \mathbf{m}) = \frac{3s_{11} - \eta - s_{44}}{4(2\eta - s_{44} \pm \sqrt{D})} = \frac{1}{12\beta}. \tag{4.15}$$

The second equality follows from (4.13). The β computed from (4.13) is very small if, and only if, η and s_{44} are very small because η and s_{44} are both positive. Hence, let

$$\eta = s_{11} + 2s_{12} = \delta(\cos\psi)^2 s_{11}, \quad s_{44} = \delta(\sin\psi)^2 s_{11}, s_{11} > 0, \tag{4.16}$$

where $0 < \psi < \pi/2$ and $\delta > 0$ is arbitrarily small. Equation (4.16) satisfies (4.7). Equation (4.13) has the expression

$$\beta = \frac{\delta(3\cos^2\psi - 1) \pm \sqrt{d}}{3\,(3-\delta)}, \tag{4.17}$$

where

$$d = 3\,\delta[1 + \delta\cos^2\psi(1 - 3\sin^2\psi)]. \tag{4.18}$$

When δ is very small,

$$\beta \cong \pm\sqrt{3\delta}/9, \tag{4.19}$$

and (4.15) gives

$$\nu(\mathbf{n}, \mathbf{m}) \cong \pm 3/(4\sqrt{3\delta}). \tag{4.20}$$

Hence Poisson's ratio for cubic materials can have no bounds.

As an illustration, let $s_{12} = -0{\cdot}4999 s_{11}$ and $s_{44} = 0{\cdot}0002 s_{11}$. Equation (4.13) gives $\beta = 0{\cdot}00387$ and $-0{\cdot}00383$ and, from $(4.15)_{1,2}$, $\nu(\mathbf{n}, \mathbf{m}) = 21{\cdot}52$ and $-21{\cdot}77$, respectively. If we let $s_{12} = (-0{\cdot}5 + 10^{-8})s_{11}$ and $s_{44} = 2 \times 10^{-8} s_{11}$, we have $\nu(\mathbf{n}, \mathbf{m}) \cong \pm 2165$ at $\beta \cong \pm 0{\cdot}0000385$.

It is interesting to note that, in the limit $\beta \to 0$, $n_1^2 = n_2^2 = n_3^2 = \frac{1}{3}$ by (4.10) and (4.12) is replaced by

$$\nu(\mathbf{n}, \mathbf{m}) = -\frac{s_{11} + 2s_{12} - \frac{1}{2}s_{44}}{s_{11} + 2s_{12} + s_{44}}. \tag{4.21}$$

When s_{44} is very small (or very large), we obtain $\nu(\mathbf{n}, \mathbf{m}) = -1$ (or $\frac{1}{2}$). Thus while $\nu(\mathbf{n}, \mathbf{m})$ is unbounded when β is very small, it is bounded at $\beta = 0$. The function $\nu(\mathbf{n}, \mathbf{m})$ is not a continuous function of δ and β at $\delta = \beta = 0$.

It is also interesting to see what happens to Poisson's ratio $\nu(\mathbf{n}, \mathbf{t})$ in the other lateral direction when $\nu(\mathbf{n}, \mathbf{m})$ is unbounded. The unit vector $\mathbf{t}$ is mutually orthogonal to $\mathbf{n}$ and $\mathbf{m}$. With the vectors $\mathbf{n}$ and $\mathbf{m}$ given in (4.10), the components of the vector $\mathbf{t}$ are

$$t_1^2 = t_2^2 = \tfrac{1}{6} + \beta, \quad t_3^2 = \tfrac{2}{3} - 2\beta. \tag{4.22}$$

We then have

$$n_1^2 t_1^2 + n_2^2 t_2^2 + n_3^2 t_3^2 = \tfrac{1}{3} + \beta - 6\beta^2, \tag{4.23}$$

and

$$\nu(\mathbf{n}, \mathbf{t}) = -\frac{s_{12} + (s_{11} - s_{12} - \frac{1}{2}s_{44})(\frac{1}{3} + \beta - 6\beta^2)}{s_{11} - 2(s_{11} - s_{12} - \frac{1}{2}s_{44})(\frac{1}{3} - 3\beta^2)}. \tag{4.24}$$

If we ignore $6\beta^2$ in the numerator because it is smaller than β, (4.24) is identical to (4.12) when β is replaced by $-\beta$. Hence (4.15) with β replaced by $-\beta$ applies here, that is,

$$\nu(\mathbf{n}, \mathbf{t}) \cong -\frac{1}{12\beta}. \tag{4.25}$$

It has the opposite sign with the $\nu(\mathbf{n}, \mathbf{m})$ given in (4.15). Hence, when Poisson's ratio has an arbitrarily large positive value in the lateral direction **m**, it has an arbitrarily large negative value in the other lateral direction **t**, and vice versa. The *volume change* δV is

$$\delta V = \varepsilon'_{11} + \varepsilon'_{22} + \varepsilon'_{33} = [1 - \nu(\mathbf{n}, \mathbf{m}) - \nu(\mathbf{n}, \mathbf{t})]\varepsilon'_{11}. \tag{4.26}$$

Substitution of (4.12) and (4.24) into the above and use of the relation $\varepsilon'_{11} = S'_{1111}\sigma'_{11}$ yields

$$\delta V = (s_{11} + 2s_{12})\sigma'_{11} = \delta(\cos^2\psi)s_{11}\sigma'_{11} \cong 0, \tag{4.27}$$

because δ is arbitrarily small. Thus there is practically no volume change even though Poisson's ratio has an arbitrarily large positive number in one lateral direction and has an arbitrarily large negative number in the other lateral direction. We show in the Appendix that the large Poisson's ratio in this case does not involve a large strain in the lateral direction **m**. The large Poisson's ratio is due to the vanishing of uniaxial strain in the direction **n**.

5. Concluding remarks

We have shown that Poisson's ratio $\nu(\mathbf{n}, \mathbf{m})$ for anisotropic elastic materials can have no bounds. With the exception of cubic materials, we proved it by examples in which there is no need to consider a general direction for the vectors **n** and **m**. For cubic materials it is necessary to consider general **n**, **m** other than the crystallographic axes. The specification of **n** and **m** requires three parameters such as the Euler's angles employed in (**1**,**6**). To find the locations of the extrema of $\nu(\mathbf{n}, \mathbf{m})$ we can take the derivatives of $\nu(\mathbf{n}, \mathbf{m})$ with respect to the three parameters and set the results to zero. Since **m** depends on one parameter once **n** is given, it is simpler if we fix the vector **n** and ask what **m** gives extrema for the given **n**. This would involve taking derivative with only one parameter. This is presented in the Appendix. One can then show that equations (1) and (2) presented by Baughman *et al.* in (**2**) are extrema for **n** at [110] and **m** at $[1\bar{1}0]$ and [001], respectively. They are reproduced in (A4) and (A5) in terms of the elastic compliance $s_{\alpha\beta}$. Poisson's ratios obtained from (A4) and (A5) have the bounds $-1 < \nu(\mathbf{n}, \mathbf{m}) < 1$ and $-1 < \nu(\mathbf{n}, \mathbf{m}) < 2$, respectively. These are the 'local' extrema, namely they are extrema in the vicinity of the **n**, **m** selected. In fact (**2**, equations (1) and (2)) are but only two of many local extrema for cubic materials. The 'global' extrema cannot be obtained by exhaustive search of all local extrema because $\nu(\mathbf{n}, \mathbf{m})$ can be unbounded as shown in section 4. The investigation of the local extrema in the Appendix leads us to the discovery of the **n** given in (4.10). We also show in Appendix A that the **n** in (4.10) is the only one that can provide an unbounded $\nu(\mathbf{n}, \mathbf{m})$ for cubic materials. Moreover, the large Poisson's ratio is due to the vanishing of the uniaxial strain in the direction **n**, not due to the large strain in the lateral direction **m**.

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APPENDIX

The local extrema of $\nu(\mathbf{n}, \mathbf{m})$ for cubic materials

It is too complicated to find the locations of the extrema of $\nu(\mathbf{n}, \mathbf{m})$ for cubic materials for all possible choices of the vectors $\mathbf{n}$ and $\mathbf{m}$. Instead, we ask what $\mathbf{m}$ gives the extrema of $\nu(\mathbf{n}, \mathbf{m})$ when the vector $\mathbf{n}$ is given.

When $\mathbf{n}$ is given, let

$$\mathbf{m} = \frac{\cos\theta}{\sqrt{1-n_3^2}}\begin{bmatrix} n_2 \\ -n_1 \\ 0 \end{bmatrix} + \frac{\sin\theta}{\sqrt{1-n_3^2}}\begin{bmatrix} -n_1 n_3 \\ -n_2 n_3 \\ 1-n_3^2 \end{bmatrix}, \quad n_3^2 \neq 1, \tag{A1}$$

where θ is arbitrary. Inserting (A1) into (4.9) and setting the derivative of $\nu(\mathbf{n}, \mathbf{m})$ with θ to zero gives

$$\tan 2\theta = n_1 n_2 n_3 (n_2^2 - n_1^2)/[n_1^2 n_2^2 (1+n_3^2) - n_3^2 (1-n_3^2)^2]. \tag{A2}$$

If θ is a solution of (A2), so is $\theta + \pi/2$. Thus, for each $\mathbf{n}$, there are two $\mathbf{m}$ that provide the extrema.

There are several special **n** and θ that satisfy (A2). The following cases are of particular interest: (i) $n_3 = 0$ and $\theta = 0$, (ii) $n_3 = 0$ and $\theta = \pi/2$, (iii) $n_1 = n_2$ and $\theta = 0$.

For case (i), let $\mathbf{n} = (\cos\psi, \sin\psi, 0)$. Then, by (A1), $\mathbf{m} = (\sin\psi, -\cos\psi, 0)$ and

$$\nu(\mathbf{n}, \mathbf{m}) = -\frac{s_{12} + \frac{1}{2}(s_{11} - s_{12} - \frac{1}{2}s_{44})\sin^2 2\psi}{s_{11} - \frac{1}{2}(s_{11} - s_{12} - \frac{1}{2}s_{44})\sin^2 2\psi}. \tag{A3}$$

It can be shown that the extrema of $\nu(\mathbf{n}, \mathbf{m})$ occur at $\psi = \pm k\pi/4$ $(k = 0, 1, 2\ldots)$. For $\psi = \pm\pi/4$, (A3) reduces to

$$\nu(\mathbf{n}, \mathbf{m}) = \frac{-2(s_{11} + s_{12}) + s_{44}}{2(s_{11} + s_{12}) + s_{44}}. \tag{A4}$$

This is equivalent to (**2**, equation (1)), which gives $-1 < \nu(\mathbf{n}, \mathbf{m}) < 1$.

For case (ii), (A1) gives $\mathbf{m} = (0, 0, 1)$. Again, let $\mathbf{n} = (\cos\psi,\ \sin\psi,\ 0)$ so that

$$\nu(\mathbf{n}, \mathbf{m}) = \frac{-s_{12}}{s_{11} - \frac{1}{2}(s_{11} - s_{12} - \frac{1}{2}s_{44})\sin^2 2\psi}. \tag{A5}$$

The extrema of $\nu(\mathbf{n}, \mathbf{m})$ occur at $\psi = \pm k\pi/4$ $(k = 0, 1, 2\ldots)$. For $\psi = \pm\pi/4$ we have

$$\nu(\mathbf{n}, \mathbf{m}) = \frac{-4s_{12}}{2(s_{11} + s_{12}) + s_{44}}. \tag{A6}$$

This is equivalent to (**2**, equation (2)), which gives $-1 < \nu(\mathbf{n}, \mathbf{m}) < 2$.

For case (iii), $\mathbf{n} = (n_1, n_1, n_3)$ and $\mathbf{m} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ so that

$$\nu(\mathbf{n}, \mathbf{m}) = -\frac{s_{12} + (s_{11} - s_{12} - \frac{1}{2}s_{44})n_1^2}{s_{11} - 2(s_{11} - s_{12} - \frac{1}{2}s_{44})n_1^2(2 - 3n_1^2)}. \tag{A7}$$

By letting $s_{12} \cong -s_{11}/2$ and $s_{44} \cong 0$ (A7) simplifies to

$$\nu(\mathbf{n}, \mathbf{m}) \cong \frac{1}{2(1 - 3n_1^2)} = \frac{1}{3n_3^2 - 1}, \tag{A8}$$

which is very large for n_1^2, n_3^2 near the value of $\frac{1}{3}$. This leads to the **n**, **m** given in (4.10).

The discovery of (A8) is not unexpected. A necessary condition for $\nu(\mathbf{n}, \mathbf{m})$ in (4.9) to be very large is that either its numerator S'_{1122} is very large or its denominator S'_{1111} is very small. It is shown in (**5**) that the maximum of $(n_1^2m_1^2 + n_2^2m_2^2 + n_3^2m_3^2)$ is $\frac{1}{2}$ attained for **n**, **m** at [110], [1$\bar{1}$0], and its minimun is zero attained for **n**, **m** at [110], [001]. Hence the maximum of S'_{1122} in (4.8) is attained for **n**, **m** at [110], [1$\bar{1}$0] or [110], [001]. But these two possibilities are Case (i) and Case (ii) discussed above. They do not give a very large $\nu(\mathbf{n}, \mathbf{m})$. Since S'_{1122} in (4.8) is symmetric with **n** and **m**, the maximum of S'_{1122} is also attained for **n**, **m** at [1$\bar{1}$0], [110] or [001], [110]. They do not give a very large $\nu(\mathbf{n}, \mathbf{m})$. The S'_{1111} in (4.8) is $1/E(\mathbf{n})$ where $E(\mathbf{n})$ is Young's modulus in the direction **n**. It is shown in (**5**) that the extrema of $1/E(\mathbf{n})$ are s_{11} and $(s_{11} + 2s_{12} + s_{44})/3$ for **n** at [100] and $n_1^2 = n_2^2 = n_3^2 = 1/3$, respectively. The former does not provide a very large $\nu(\mathbf{n}, \mathbf{m})$. As to the latter, $(s_{11} + 2s_{12} + s_{44})/3$ at $n_1^2 = n_2^2 = n_3^2 = 1/3$ can be very small if, and only if, $s_{11} + 2s_{12}$ and s_{44} are both very small because $s_{11} + 2s_{12}$ and s_{44} are both positive. This confirms the derivation of (A8) and motivates the assumption (4.10) and (4.16). Thus n_1^2, n_2^2, n_3^2 near $\frac{1}{3}$ is the only **n** that can provide a very large $\nu(\mathbf{n}, \mathbf{m})$ for cubic materials.

Poisson's ratio $\nu(\mathbf{n}, \mathbf{m})$ is $-\varepsilon(\mathbf{m})/\varepsilon(\mathbf{n})$ where $\varepsilon(\mathbf{n})$ and $\varepsilon(\mathbf{m})$ are, respectively, the uniaxial strain in the direction **n** and the lateral strain in the direction **m**. A very large $\nu(\mathbf{n}, \mathbf{m})$ does not necessarily mean that the lateral strain is very large, which would be physically unrealistic for linear theory of elasticity. The lateral strain is S'_{1122} in (4.8), which is bounded, multiplied by the uniaxial stress σ'_{11}. Hence it is bounded. The uniaxial strain is $1/E(\mathbf{n})$ multiplied by σ'_{11}. As pointed out in the previous paragraph, $1/E(\mathbf{n})$ is vanishingly small so that the uniaxial strain is vanishingly small. Thus the large Poisson's ratio for cubic materials presented here is physically realistic.

As to Poisson's ratio for anisotropic elastic materials other than cubic materials discussed in sections 2 and 3, the large $\nu(\mathbf{n}, \mathbf{m})$ is caused by a large lateral strain. However, the **n** in sections 2 and 3 is along a crystallographic axis of the material. If we allow **n** to be arbitrary, it is possible to find an **n** along which $1/E(\mathbf{n})$ is vanishingly small so that the uniaxial strain is vanishingly small while the lateral strain is bounded. We believe that such an **n** exists because it exists for the cubic material, which is a special case of other anisotropic elastic materials. One could perturb a cubic material to other anisotropic elastic materials by introducing small parameters to find an **n** for which $1/E(\mathbf{n})$ is vanishingly small.