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POISSON SPACES WITH A TRANSITION PROBABILITY

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The common structure of the space of pure states $\mathcal{P}$ of a classical or a quantum mechanical system is that of a Poisson space with a transition probability. This is a topological space equipped with a Poisson structure, as well as with a function $p : \mathcal{P} \times \mathcal{P} \rightarrow [0, 1]$, with certain properties. The Poisson structure is connected with the transition probabilities through unitarity (in a specific formulation intrinsic to the given context).

In classical mechanics, where $p(\rho, \sigma) = \delta_{\rho\sigma}$, unitarity poses no restriction on the Poisson structure. Quantum mechanics is characterized by a specific (complex Hilbert space) form of p, and by the property that the irreducible components of $\mathcal{P}$ as a transition probability space coincide with the symplectic leaves of $\mathcal{P}$ as a Poisson space. In conjunction, these stipulations determine the Poisson structure of quantum mechanics up to a multiplicative constant (identified with Planck's constant).

Motivated by E. M. Alfsen, H. Hanche-Olsen and F. W. Shultz (*Acta Math.* **144** (1980) 267–305) and F.W. Shultz (*Commun. Math. Phys.* **82** (1982) 497–509), we give axioms guaranteeing that $\mathcal{P}$ is the space of pure states of a unital C^*-algebra. We give an explicit construction of this algebra from $\mathcal{P}$.

1. Introduction

Section 1.1 motivates the axiomatic study of state spaces (rather than operator algebras) in the foundations of quantum mechanics. In 1.2 we review the work of Alfsen et al. on the structure of state spaces of C^*-algebras. In 1.3 we discuss the concept of a transition probability space, and in 1.4 it is shown how the pure state space of a C^*-algebra is an example of such a space. Section 1.5 recalls the concept of a Poisson manifold, and introduces (uniform) Poisson spaces generalizing this concept. Poisson structures may be intertwined with transition probabilities, leading to the notion of unitarity, and to the central idea of this paper, a Poisson space with a transition probability.

In Sec. 2 we introduce our axioms on pure state spaces, and formulate the theorem relating these axioms to pure state spaces of C^*-algebras. Section 3 outlines the proof of this theorem, which essentially consists of the reconstruction of a C^*-algebra from its pure state space, endowed with the structure of a uniform Poisson space with a transition probability. This reconstruction is of interest in its own right. Some longer proofs and other technical comments appear in Sec. 4.

In this paper functions and functionals are real-valued, unless explicitly indicated otherwise. Hence $C(X)$ stands for $C(X, \mathbb{R})$, etc. Similarly, vector spaces (including the various algebras appearing in this paper) are generally over $\mathbb{R}$, unless there is an explicit label $\mathbb{C}$ denoting complexification. An exception to this rule is that we use the standard symbols $\mathcal{H}$ for a complex Hilbert space, and $\mathfrak{B}(\mathcal{H})$ ($\mathfrak{K}(\mathcal{H})$) for the set of all bounded (compact) operators on $\mathcal{H}$. The self-adjoint part of a C^*-algebra $\mathfrak{A}_{\mathbb{C}}$ is denoted by $\mathfrak{A}$; we denote the state space of $\mathfrak{A}_{\mathbb{C}}$ by $\mathcal{S}(\mathfrak{A})$ or $\mathcal{S}(\mathfrak{A}_{\mathbb{C}})$, and its pure state space by $\mathcal{P}(\mathfrak{A})$ or $\mathcal{P}(\mathfrak{A}_{\mathbb{C}})$. Here the 'pure state space' is the space of all pure states, rather than its w^*-closure.

1.1. Algebraic aspects of mechanics

At face value, quantum mechanics (Hilbert space, linear operators) looks completely different from classical mechanics (symplectic manifolds, smooth functions). The structure of their respective algebras of observables, however, is strikingly similar. In quantum mechanics, one may assume [46, 22] that the observables $\mathfrak{A}$ form the self-adjoint part of some C^*-algebra $\mathfrak{A}_{\mathbb{C}}$. The associative product does not map $\mathfrak{A}$ into itself, but the anti-commutator $A \circ B = \frac{1}{2}(AB + BA)$ and the (scaled) commutator $[A, B]_\hbar = i(AB - BA)/\hbar$ do; in conjunction, they give $\mathfrak{A}$ the structure of a so-called **Jordan–Lie algebra** [26, 22]. This is a vector space V equipped with two bilinear maps $\circ$ and $[\ ,\ ] : V \times V \to V$, such that $\circ$ is symmetric, $[\ ,\ ]$ is a Lie bracket (i.e., it is anti-symmetric and satisfies the Jacobi identity), and the Leibniz property

$$[A, B \circ C] = [A, B] \circ C + B \circ [A, C] \tag{1.1}$$

holds; in other words, the commutator is a derivation of the Jordan product. Moreover, one requires the associator identity

$$(A \circ B) \circ C - A \circ (B \circ C) = k[[A, C], B] \tag{1.2}$$

for some $k \in \mathbb{R}$. This implies the Jordan identity $A^2 \circ (A \circ B) = A \circ (A^2 \circ B)$ (where $A^2 = A \circ A$), which makes $(V, \circ)$ a Jordan algebra [22, 28]); accordingly, the symmetric product $\circ$ is referred to as the **Jordan product**. Note that for $V = \mathfrak{A}$ and $[A, B] = [A, B]_\hbar$ one has $k = \hbar^2/4$.

Conversely, a Jordan–Lie algebra $\mathfrak{A}$ for which $k > 0$ (cf. [22] for comments on the case $k < 0$), and which in addition is a so-called JB-algebra, is the self-adjoint part of a C^*-algebra $\mathfrak{A}_{\mathbb{C}}$.

Here a JB-**algebra** [9, 28] is a Jordan algebra which is a Banach space, and satisfies $\| A \circ B \| \leq \| A \| \| B \|$, $\| A^2 \| = \| A \|^2$, and $\| A^2 \| \leq \| A^2 + B^2 \|$ for all $A, B \in \mathfrak{A}$; the first axiom can actually be derived from the other two; alternatively, the last two axioms may be replaced by $\| A \|^2 \leq \| A^2 + B^2 \|$.

The associative C^*-product is given by $A \cdot B = A \circ B - i\sqrt{k}[A, B]$ (the $\cdot$ is usually omitted); the associativity follows from the Leibniz property, (1.2), and the Jacobi identity. For the construction of the norm and the verification of the axioms for a C^*-algebra, see [58, 47] and Sec. 3.8 below.

In classical mechanics, one takes the Jordan–Lie algebra to consist of all smooth functions on the phase space, equipped with the operations of pointwise multiplication $f \circ g = fg$ and Poisson bracket $[f, g] = \{f, g\}$ (the latter coming from a symplectic structure, or from a more general abstract Poisson structure [55, 39]). The identity (1.2) is then satisfied with $k = 0$. A Jordan–Lie algebra for which $k = 0$ in (1.2) is called a **Poisson algebra**.

Thus from an algebraic point of view the only difference between classical and quantum mechanics is that in the former the Jordan product $\circ$ is associative, whereas in the latter the more general identity (1.2) is satisfied for some $k > 0$.

From an axiomatic point of view, it is rather difficult to justify (1.2), and it is hard to swallow that the non-associativity of $\circ$ should be the defining property of quantum mechanics. Historically, the commutator hardly played a role in algebraic quantum axiomatics, all attention being focused on the Jordan structure [43, 49, 9, 28, 22]. Whereas the Jordan identity may be justified by the need to have a spectral theory, the step from the Jordan- to the full C^*-structure has had to be justified algebraically by an appeal to the need to combine different physical systems using a well-behaved tensor product [11, 27]. This gives the commutator a different status from its classical counterpart (viz. the Poisson bracket), which describes the way observables lead to flows (i.e., dynamics).

1.2. *State spaces and the work of Alfsen, Shultz, and Hanche-Olsen*

A transparent way of analyzing and justifying algebras of observables is the study of their state spaces. A state on a JB-algebra $\mathfrak{A}$ is defined as a linear functional ω on $\mathfrak{A}$ satisfying $\omega(A^2) \geq 0$ for all $A \in \mathfrak{A}$ and $\|\omega\| = 1$; in case that $\mathfrak{A}$ has an identity $\mathbb{I}$ this implies that $\omega(\mathbb{I}) = 1$. The idea is that the algebraic structure of $\mathfrak{A}$ is encoded in certain (geometric) properties of its state space $\mathcal{S}(\mathfrak{A})$, so that $\mathfrak{A}$ may be reconstructed from $\mathcal{S}(\mathfrak{A})$, equipped with these properties. The most basic property of $\mathcal{S}(\mathfrak{A})$ is that it is a convex set, which is compact in the w^*-topology if $\mathfrak{A}$ is a JB-algebra with unit. The description of quantum mechanics in terms of general compact convex state spaces is closely tied to the so-called operational approach, and is invariably interpreted in terms of laboratory procedures such as filtering measurements [48, 40, 41, 42, 37, 14, 35].

For C^*-algebras (which are special instances of complexified JB-algebras) this type of study culminated in [5], where axioms were given which guarantee that a given compact convex set K (assumed to be embedded in a locally convex Hausdorff vector space) is the state space of a C^*-algebra with unit (also cf. [4, 12, 8]). In order to motivate our own approach, we need to explain these axioms to some extent.

Firstly, a **face** F is defined as a convex subset of K with the property that ρ and σ are in F if $\lambda\sigma + (1 - \lambda)\rho \in F$ for some $\lambda \in (0, 1)$. A face F is called **norm-exposed** [7] if it equals $F = \{\rho \in K | \langle f, \rho \rangle = 0\}$ for some $f \in A_b^+(K)$. Here $A_b(K)$ is the space of all bounded affine functions on K, and $A_b^+(K)$ its subspace of positive functions. $A(K)$ will stand for the space of continuous affine functions on K [6, 12].

A face F is said to be **projective** [6] if there exists another face $F^\sharp$ such that F and $F^\sharp$ are norm-exposed and affinely independent [3], and there exists a map (a so-called affine retraction) : $K \to K$ with image the convex sum of F and $F^\sharp$, leaving its image pointwise invariant, and having the technical property of transversality (cf. [6, 3.8] or [4]) (alternative definitions are possible [6]). The first axiom of [5] is

Axiom AHS1. *Every norm-exposed face of K is projective.*

A face consisting of one point is called a pure state, and the collection of pure states forms the so-called **extreme boundary** $\partial_e K$ of K. The smallest face containing a subset $S \subset K$ is denoted by $F(S)$, and we write $F(\rho, \sigma)$ for $F(\{\rho, \sigma\})$. Two pure states ρ, σ are called inequivalent if $F(\rho, \sigma)$ is the line segment $\{\lambda\sigma + (1-\lambda)\rho \mid \lambda \in [0,1]\}$. Otherwise, they are called **equivalent**. The second axiom is

Axiom AHS2. *If pure states ρ and $\sigma \neq \rho$ are equivalent, then $F(\rho, \sigma)$ is norm-exposed and affinely isomorphic to the state space of the C^*-algebra $\mathfrak{M}_2(\mathbb{C})$ of 2×2 matrices over $\mathbb{C}$. Moreover, each pure state is norm-exposed.*

The state space $\mathcal{S}(\mathfrak{M}_2(\mathbb{C}))$ is affinely isomorphic to the unit ball B^3 in $\mathbb{R}^3$. Concretely, we identify a state on $\mathfrak{M}_2(\mathbb{C})$ with a density matrix on $\mathbb{C}^2$, which may be parametrized as

$$\frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \tag{1.3}$$

where $x, y, z \in \mathbb{R}$. The positivity of this matrix then corresponds to the constraint $x^2 + y^2 + z^2 \leq 1$ (see [5]).

From the point of view of quantum logic (cf. e.g. [54, 14, 31]), Axiom AHS1 allows one to define an orthomodular lattice, whose elements are the projective faces of K [6, §4]. Axiom AHS2 not only allows one to prove that this lattice has the covering property [8, 6.15], but also eventually implies that the co-ordinatizing field of the lattice is $\mathbb{C}$ (cf. Sec. 4.1). In the finite-dimensional case Axioms AHS1 and AHS2 are sufficient to construct a C^*-algebra $\mathfrak{A}_\mathbb{C}$ whose state space is K; as a Banach space $\mathfrak{A} = A(K)$ with the sup-norm. To cover the general case, more axioms are needed.

Axiom AHS3. *The σ-convex hull of $\partial_e K$ is a split face of K.*

Here the σ-**convex hull** in question consists of all sums $\sum_i \lambda_i \rho_i$, where $\rho_i \in \partial_e K$, $\lambda_i \in [0,1]$, $\sum_i \lambda_i = 1$, and the sum converging in the norm topology (regarding K as a subset of the dual of the Banach space $A(K)$). A face F of K is **split** if there exists another face F' such that $K = F \oplus_c F'$ (direct convex sum). Let $C \subset \partial_e K$ consist of all pure states in a given equivalence class, and let $\overline{F}(C)$ be the σ-convex hull of C (this coincides with the smallest split face containing any member of C). Then $A_b(\overline{F}(C))_\mathbb{C}$ can be made into a von Neumann algebra (with predual $\overline{F}(C)_\mathbb{C}$) on the basis of axioms 1–3 [8, §6], [5, §6]. Axiom AHS3 is used to show that this is an atomic (type I) factor, i.e., $\mathfrak{B}(\mathcal{H}_C)$ for some Hilbert space $\mathcal{H}_C$.

The remaining axioms serve to combine all the $A(\overline{F}(C))$ into $A(K)$ in such a way that one obtains the self-adjoint part of a C^*-algebra. The Jordan product $A \circ B$ (or, equivalently, A^2) is constructed using the non-commutative spectral

theory defined by K [6, 7]. This product then coincides with the anti-commutator in $A_b(\overline{F}(C)) \simeq \mathfrak{B}(\mathcal{H}_C)$. In principle this could map $A \in A(K)$ into $A^2 \in A_b(K)$ (that is, not necessarily in $A(K)$). Hence

Axiom AHS4. *If $A \in A(K)$ then $A^2 \in A(K)$.*

This is not the formulation of the axiom given in [8, 5], but by [6, 9.6], [8, 7.2] it is immediately equivalent to the version in the literature. Finally, the commutator, already defined on each $A(\overline{F}(C))$, needs to be well-defined on all of $A(K)$. This is guaranteed by

Axiom AHS5. K *is orientable.*

Roughly speaking, this means that one cannot transport a given face $F(\rho, \sigma) \simeq B^3$ (cf. Axiom AHS2) in a continuous way around a closed loop so that it changes its orientation (cf. [5, §7] for more detail; also Sec. 4.3 below). It is remarkable that $A(K)$ is automatically closed under the commutator, given the axioms. It is proved in [5] that a compact convex set is the state space of a unital C^*-algebra iff Axioms AHS1–AHS5 are satisfied.

Even if one is happy describing quantum mechanics with superselection rules in terms of C^*-algebras, from a physical perspective one should not necessarily regard the above axioms as unique, or as the best ones possible. The notion of a projective face (or, equivalently, a P-projection [6]) is a complicated one (but cf. [11] for a certain simplification in the finite-dimensional case, and [35] for an analogous interpretation in terms of filters in the general case). One would like to replace the concept of orientability by some statement of physical appeal. Most importantly, the comparison of classical and quantum mechanics seems facilitated if one could start from the space of pure states $\partial_e K$ as the basic object. Moreover, from an ontological rather than an epistemological point of view one would prefer a formulation in terms of pure states as well, and the same comment applies if one is interested in an individual (as opposed to a statistical) interpretation of quantum mechanics.

1.3. *Transition probability spaces*

Clearly, the extreme boundary $\partial_e K$ of a given compact convex set K as a topological space does not contain enough information to reconstruct K. However, one can equip $\partial_e K$ with the additional structure of a so-called transition probability, as first indicated by Mielnik [41] (also cf. [50]). Namely, given $\rho, \sigma \in \partial_e K$ one can define p by

$$p(\rho, \sigma) = \inf\{f(\rho)|f \in A_b(K), 0 \le f \le 1, f(\sigma) = 1\}. \tag{1.4}$$

For later use, we notice that it follows that

$$p(\sigma, \rho) = 1 - \sup\{f(\sigma)|f \in A_b(K), 0 \le f \le 1, f(\rho) = 0\}. \tag{1.5}$$

For the moment we denote $\partial_e K$ by $\mathcal{P}$. By construction,

$$p : \mathcal{P} \times \mathcal{P} \to [0, 1] \tag{1.6}$$

satisfies $\rho = \sigma \Rightarrow p(\rho, \sigma) = 1$. Moreover, we infer from (1.5) that

$$p(\rho, \sigma) = 0 \iff p(\sigma, \rho) = 0. \tag{1.7}$$

If K has the property that every pure state is norm-exposed, then, as is easily verified, $p(\rho, \sigma) = 1 \Rightarrow \rho = \sigma$, so that

$$p(\rho, \sigma) = 1 \iff \rho = \sigma. \tag{1.8}$$

Any function p on a set $\mathcal{P}$ with the properties (1.6), (1.7), and (1.8) is called a *transition probability*, and $(\mathcal{P}, p)$ is accordingly called a *transition probability space*. (In its abstract form these concepts are due to von Neumann [44], who in addition required p to satisfy (1.9) below; also cf. [40, 59, 13, 14, 45]). A transition probability is called **symmetric** if

$$p(\rho, \sigma) = p(\sigma, \rho) \, \forall \rho, \sigma \in \mathcal{P}. \tag{1.9}$$

A subset $S \subset \mathcal{P}$ is called orthogonal if $p(\rho, \sigma) = 0$ for all pairs $\rho \neq \sigma$ in S. A basis B of $\mathcal{P}$ is an orthogonal subset for which $\sum_{\rho \in B} p(\rho, \sigma) = 1$ for all $\sigma \in \mathcal{P}$ (here the sum is defined as the supremum of all finite partial sums). A basic theorem is that all bases of a given symmetric transition probability space have the same cardinality [40]; this cardinality is the **dimension** of $\mathcal{P}$.

One imposes the requirement

$$\text{\textit{Every maximal orthogonal subset of} } \mathcal{P} \text{ \textit{is a basis.}} \tag{1.10}$$

A transition probability space is called **irreducible** if it is not the union of two (nonempty) orthogonal subsets. A **component** C is a subset of $\mathcal{P}$ with the property that $p(\rho, \sigma) = 0$ for all $\rho \in C$ and all $\sigma \in \mathcal{P} \backslash C$. Thus a transition probability space is the disjoint union of its irreducible components [13]. An irreducible component of $\mathcal{P}$ is called a **sector**. This agrees with the terminology in algebraic quantum mechanics, where $\mathcal{P}$ is the pure state space of a C^*-algebra (of observables) [46]. If one defines a topology on $\mathcal{P}$ through the metric $d(\sigma, \rho) = \text{l.u.b.}\{|p(\rho, \tau) - p(\sigma, \tau)|, \tau \in \mathcal{P}\}$ [13], then the topological components coincide with the components just defined. However, a different topology may be defined on $\mathcal{P}$, and therefore we shall use the term 'sector' as referring to 'component' in the first (probabilistic) sense. Two points lying in the same sector of $\mathcal{P}$ are called **equivalent** (and **inequivalent** in the opposite case).

Any subset $Q \subset \mathcal{P}$ has an orthoplement $Q^{\perp} = \{\sigma \in \mathcal{P} \mid p(\rho, \sigma) = 0 \, \forall \rho \in Q\}$. One always has $Q \subseteq Q^{\perp\perp}$; a subset Q is called *orthoclosed* if $Q = Q^{\perp\perp}$. Any set of the type $Q^{\perp}$ (hence in particular $Q^{\perp\perp}$) is orthoclosed. In particular, one may choose an orthogonal subset S, in which case [40, 59] $S^{\perp\perp} = \{\rho \in \mathcal{P} | \sum_{\sigma \in S} p(\rho, \sigma) = 1\}$. (Clearly, if $S = B$ is a basis then $B^{\perp\perp} = \mathcal{P}$.) Not every orthoclosed subset is necessarily of this form, however there exist examples of orthoclosed subsets which do not have any basis [59, 14]. To exclude pathological cases, one therefore adds the axiom [59, 14]:

$$\text{\textit{If } } Q \subseteq \mathcal{P} \text{ \textit{is orthoclosed then every maximal orthogonal subset of}}$$
$$Q \text{ \textit{is a basis of} } Q. \tag{1.11}$$

Definition 1. *A well-behaved transition probability space is a pair $(\mathcal{P}, p)$ satisfying* (1.6)–(1.11).

Of course, (1.7) and (1.10) follow from (1.9) and (1.11), respectively.

The simplest example of a well-behaved transition probability space is given by putting the 'classical' transition probabilities

$$p(\rho, \sigma) = \delta_{\rho\sigma} \tag{1.12}$$

on any set $\mathcal{P}$.

One can associate a certain function space $\mathfrak{A}(\mathcal{P})$ with any transition probability space $\mathcal{P}$. Firstly, for each $\rho \in \mathcal{P}$ define $p_\rho \in \ell^\infty(\mathcal{P})$ by

$$p_\rho(\sigma) = p(\rho, \sigma). \tag{1.13}$$

Secondly, the normed vector space $\mathfrak{A}_{00}(\mathcal{P})$, regarded as a subspace of $\ell^\infty(\mathcal{P})$ (with sup-norm), consists of all finite linear combinations of the type $\sum_{i=1}^N c_i p_{\rho_i}$, where $c_i \in \mathbb{R}$ and $\rho_i \in \mathcal{P}$. The closure of $\mathfrak{A}_{00}(\mathcal{P})$ is called $\mathfrak{A}_0(\mathcal{P})$. Thirdly, the double dual of $\mathfrak{A}_0(\mathcal{P})$ will play a central role in what follows, so that we use a special symbol:

$$\mathfrak{A}(\mathcal{P}) = \mathfrak{A}_0(\mathcal{P})^{**}. \tag{1.14}$$

Since $\mathfrak{A}_0(\mathcal{P}) \subseteq \ell_0(\mathcal{P})$, one has $\mathfrak{A}(\mathcal{P}) \subseteq \ell_0(\mathcal{P})^{**} = \ell^\infty(\mathcal{P})$. The space $\mathfrak{A}(\mathcal{P})$ is the function space intrinsically related to a transition probability space $\mathcal{P}$. In the case (1.12) one immediately finds $\mathfrak{A}(\mathcal{P}) = \ell^\infty(\mathcal{P})$.

(Following a seminar the author gave in Göttingen, 1995, A. Uhlmann informed him that in his lectures on quantum mechanics $\mathfrak{A}_{00}(\mathcal{P})$ had long been employed as the space of observables.)

1.4. *Transition probabilities on pure state spaces*

Using the results in [8] (in particular, the so-called 'pure state properties') as well as Theorem 2.17 in [6], it is not difficult to show that the pure state space of a unital JB-algebra (where every pure state is indeed norm-exposed) is a symmetric transition probability space.

If one further specializes to the pure state space $\mathcal{P}(\mathfrak{A})$ of a unital C^*-algebra $\mathfrak{A}_\mathbb{C}$, from (1.4) one may derive the explicit expression

$$p(\rho, \sigma) = 1 - \tfrac{1}{4}\|\rho - \sigma\|^2, \tag{1.15}$$

which coincides with

$$p(\rho, \sigma) = |(\Omega_\rho, \Omega_\sigma)|^2 \tag{1.16}$$

if ρ and σ are equivalent (where Ω_ρ is a unit vector implementing ρ in the corresponding GNS representation, etc.), and equals 0 if they are not; cf. [25, 46, 50]. This will be proved in Sec. 4.2.

The notion of equivalence between pure states used here may refer either to the one defined between Eqs. (1.10) and (1.11) in the context of transition probability

spaces, or to the unitary equivalence of the GNS-representations defined by the states in question in the context of C^*-algebras; these notions coincide. In fact, $\mathcal{P}(\mathfrak{A})$ has the following decomposition into sectors (see [46], which on this point relies on [25]):

$$\mathcal{P}(\mathfrak{A}) = \cup_\alpha \mathbb{P}\mathcal{H}_\alpha \,, \tag{1.17}$$

where $\mathcal{H}_\alpha$ is isomorphic to the irreducible GNS-respresentation space of an arbitrary state in the projective Hilbert space $\mathbb{P}\mathcal{H}_\alpha$. All states in a given subspace $\mathbb{P}\mathcal{H}_\alpha$ are equivalent, and any two states lying in different such subspaces are inequivalent.

We regard the self-adjoint part $\mathfrak{A}$ of $\mathfrak{A}_\mathbb{C}$ as a subspace of $C(\mathcal{P}(\mathfrak{A}))$ (where $\mathcal{P}(\mathfrak{A})$ is equipped with the w^*-topology inherited from $\mathcal{S}(\mathfrak{A})$) through the Gel'fand transform $\hat{A}(\rho) = \rho(A)$, for arbitrary $A \in \mathfrak{A}$ and $\rho \in \mathcal{P}(\mathfrak{A})$. Similarly, an operator $A \in \mathfrak{B}(\mathcal{H})$ is identified with a function $\hat{A} \in C(\mathbb{P}\mathcal{H})$ through the canonical inclusion $\mathbb{P}\mathcal{H} \subset \mathcal{S}(\mathfrak{B}(\mathcal{H}))$ (where $\mathbb{P}\mathcal{H}$ carries the w^*-topology relative to this inclusion). Under these identifications, for each $\rho \in \mathcal{P}(\mathfrak{A})$ the irreducible representation $\pi_\rho(\mathfrak{A})$ is unitarily equivalent to the restriction of $\mathfrak{A}$ to the sector containing ρ; every irreducible representation of $\mathfrak{A}$ is therefore given (up to unitary equivalence) by the restriction of $\mathfrak{A}$ to one of its sectors.

In any case, one recovers the usual transition probabilities of quantum mechanics. If $\mathfrak{A}_\mathbb{C} = \mathfrak{K}(\mathcal{H})$ (or $\mathfrak{M}_N(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^N)$), the pure state space $\mathcal{P}(\mathfrak{K}(\mathcal{H}))$ is the projective Hilbert space $\mathbb{P}\mathcal{H}$ (or $\mathbb{P}\mathbb{C}^N$). One may then equally well interpret Ω_ρ (etc.) in (1.16) as a lift of $\rho \in \mathbb{P}\mathcal{H}$ to the unit sphere $\mathbb{S}\mathcal{H}$ in $\mathcal{H}$.

In particular, it follows that the pure state space of a unital C^*-algebra is a well-behaved transition probability space. The space $\mathfrak{A}(\mathcal{P}(\mathfrak{A}))$ can be explicitly identified. Let π_{ra} be the reduced atomic representation of $\mathfrak{A}_\mathbb{C}$ [30]; recall that π_{ra} is the direct sum over irreducible respresentations $\pi_{\mathrm{ra}} = \oplus_\rho \pi_\rho$ (on the Hilbert space $\mathcal{H}_{\mathrm{ra}} = \oplus_\rho \mathcal{H}_\rho$), where one includes one representative of each equivalence class in $\mathcal{P}(\mathfrak{A})$. For the weak closure one obtains $\pi_{\mathrm{ra}}(\mathfrak{A}_\mathbb{C})^- = \oplus_\rho \mathfrak{B}(\mathcal{H}_\rho)$. The Gel'fand transform maps $\pi_{\mathrm{ra}}(\mathfrak{A})^-$ into a subspace of $\ell^\infty(\mathcal{P}(\mathfrak{A}))$. It will be shown in Sec. 3.4 that this subspace is precisely $\mathfrak{A}(\mathcal{P}(\mathfrak{A}))$; we write this as

$$\mathfrak{A}(\mathcal{P}(\mathfrak{A})) = \hat{\pi}_{\mathrm{ra}}(\mathfrak{A})^- \,. \tag{1.18}$$

The isomorphism between $\pi_{\mathrm{ra}}(\mathfrak{A})^-$ and $\mathfrak{A}(\mathcal{P}(\mathfrak{A}))$ thus obtained is isometric and preserves positivity (since the Gel'fand transform does).

For any well-behaved transition probability space $\mathcal{P}$ one can define a lattice $\mathfrak{L}(\mathcal{P})$, whose elements are the orthoclosed subsets of $\mathcal{P}$ (including the empty set $\emptyset$, and $\mathcal{P}$ itself). The lattice operations are: $Q \leq R$ means $Q \subseteq R$, $Q \wedge R = Q \cap R$, and $Q \vee R = (Q \cup R)^{\perp\perp}$. The zero element 0 is $\emptyset$. Note that the dimension of $\mathfrak{L}(\mathcal{P})$ as a lattice equals the dimension [31] of $\mathcal{P}$ as a transition probability space. It is orthocomplemented by $\perp$, and is easily shown to be a complete atomic orthomodular lattice [59, 13, 14] (cf. [31] for the general theory of orthomodular lattices). In our approach, this lattice plays a somewhat similar role to the lattice $\mathcal{F}(K)$ of projective faces of K (or, equivalently, of P-projections [6]; note that for C^*-algebras $\mathcal{L}(\partial_e K)$

is not necessarily isomorphic to $\mathcal{F}(K)$). However, it seems to us that both the definition and the physical significance of $\mathfrak{L}(\mathcal{P})$ are more direct.

If $\mathcal{P}$ is a classical transition probability space (see 1.12) then $\mathfrak{L}(\mathcal{P})$ is the distributive (Boolean) lattice of subsets of $\mathcal{P}$. If $\mathcal{P} = \mathcal{P}(\mathfrak{A})$ is the pure state space of a C^*-algebra $\mathfrak{A}_{\mathbb{C}}$ then $\mathcal{L}(\mathcal{P}(\mathfrak{A}))$ may be shown to be isomorphic (as an ortho-complemented lattice) to the lattice of all projections in the von Neumann algebra $\pi_{\mathrm{ra}}(\mathfrak{A}_{\mathbb{C}})^-$.

For general compact convex sets it is not clear to what extent $\partial_e K$ as a transition probability space equipped with the w^*-topology determines K. If, however, $K = \mathcal{S}(\mathfrak{A})$ is the state space of a unital C^*-algebra $\mathfrak{A}_{\mathbb{C}}$ (with self-adjoint part $\mathfrak{A}$), then one *can* reconstruct $\mathfrak{A}$ as a JB-algebra, and hence the state space $\mathcal{S}(\mathfrak{A})$, from the pure state space $\mathcal{P}(\mathfrak{A})$ as a transition probability space (with transition probabilities given by (1.15)), equipped with the w^*-uniformity (this is the uniformity [33] $\mathcal{U}$ generated by sets of the form $\{(\rho, \sigma) \in \mathcal{P} \times \mathcal{P} \,|\, |\langle \rho - \sigma, A \rangle| < \varepsilon\}$ for some $\varepsilon > 0$ and $A \in \mathfrak{A}$; the physical interpretation of such uniformities has been discussed by Haag, Kastler, and Ludwig, cf. [57] for a very clear discussion).

The essential step in this reconstruction is the following reformulation of a result of Shultz [50] (whose formulation involved $\pi_{\mathrm{ra}}(\mathfrak{A}_{\mathbb{C}})^-$ rather than $\mathfrak{A}(\mathcal{P}(\mathfrak{A}))$) and Brown [16]: if $\mathfrak{A}$ is the self-adjoint part of a unital C^*-algebra then

$$\mathfrak{A} = \mathfrak{A}(\mathcal{P}(\mathfrak{A})) \cap C_u(\mathcal{P}(\mathfrak{A})), \tag{1.19}$$

where $C_u(\mathcal{P}(\mathfrak{A}))$ is the space of uniformly continuous functions on $\mathcal{P}(\mathfrak{A})$, and, as before, $\mathfrak{A}$ has been identified with a subspace of $C(\mathcal{P}(\mathfrak{A}))$ through the Gel'fand transform. Note that to recover $\mathfrak{A}_{\mathbb{C}}$ as a C^*-algebra from the pure state space $\mathcal{P}(\mathfrak{A})$, one in addition needs an orientation of $\mathcal{P}(\mathfrak{A})$, see [5,50] and Sec. 4.3.

For certain C^*-algebras (called **perfect**, cf. [50, 2]) one can replace $C_u(\mathcal{P}(\mathfrak{A}))$ by $C(\mathcal{P}(\mathfrak{A}))$ (with respect to the w^*-topology). These include $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{K}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$.

1.5. *Poisson spaces with a transition probability*

Our goal, then, is to give axioms on a well-behaved transition probability space $\mathcal{P}$ which enable one to construct, by a unified procedure, a C^*-algebra *or* a Poisson algebra, which has $\mathcal{P}$ as its space of pure states, and reproduces the original transition probabilities. Moreover, even if one is not interested in these axioms and takes quantum mechanics (with superselection rules) at face value, the structure laid out in this paper provides a transparent reformulation of quantum mechanics, which may prove useful in the discussion of the classical limit [36].

We first have to define a number of concepts, which play a foundational role in both classical and quantum mechanics. Apart from transition probabilities, Poisson brackets play a central role in dynamical theories. Recall that a **Poisson manifold** [55, 39] is a manifold P with a Lie bracket $\{\,,\,\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$, such that $C^\infty(P)$ equipped with this Lie bracket, and pointwise multiplication as the Jordan product $\circ$, is a Poisson algebra. Symplectic manifolds are special instances of Poisson manifolds; in the symplectic case the Hamiltonian vector fields span the

tangent space at every point of P. Recall from classical mechanics [39] that any $H \in C^\infty(P)$ defines a so-called **Hamiltonian vector field** X_H by $X_H f = \{H, f\}$; the flow of X_H is called a **Hamiltonian flow**; similarly, one speaks of a **Hamiltonian curve**.

The most important result in the theory of Poisson manifolds states that a Poisson manifold P admits a decomposition into symplectic leaves [55, 39]. This means that there exists a family S_α of symplectic manifolds, as well as continuous injections $\iota_\alpha : S_\alpha \to P$, such that $P = \cup_\alpha \iota_\alpha(S_\alpha)$ (disjoint union), and

$$\{f, g\}(\iota_\alpha(\sigma)) = \{\iota_\alpha^* f, \iota_\alpha^* g\}_\alpha(\sigma), \tag{1.20}$$

for all α and all $\sigma \in S_\alpha$. Here $\{\,,\,\}_\alpha$ is the Poisson bracket associated to the symplectic structure on S_α [39], and $(\iota_\alpha^* f)(\sigma) = f(\iota_\alpha(\sigma))$, etc.

We will need a generalization of the notion of a Poisson manifold, which is inspired by the above decomposition.

Definition 2. *A **Poisson space** P is a Hausdorff topological space together with a linear subspace $\mathfrak{A} \subset C(P)$ and a collection S_α of symplectic manifolds, as well as continuous injections $\iota_\alpha : S_\alpha \to P$, such that:*

- *$P = \cup_\alpha \iota_\alpha(S_\alpha)$ (disjoint union);*
- *$\mathfrak{A}$ separates points;*
- *$\mathfrak{A} \subseteq C_L^\infty(P)$, where $C_L^\infty(P)$ consists of all $f \in C(P)$ for which $\iota_\alpha^* f \in C^\infty(S_\alpha)$ for each α;*
- *$\mathfrak{A}$ is closed under Poisson brackets.*

The last requirement means, of course, that the Poisson bracket, computed from the symplectic structure on the S_α and the above decomposition of P through (1.20), maps $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{A}$. In the context of Poisson spaces, each subspace $\iota_\alpha(S_\alpha)$ of P is called a **symplectic leaf** of P. This terminology is sometimes applied to the S_α themselves as well.

In general, this decomposition falls under neither foliation theory nor (Whitney) stratification theory (cf. [51] for this theory in a symplectic context).

If the ambient space P carries additional structure, such as a uniformity, or a smooth structure, one can refine the above definition in the obvious way; such refinements will play an important role in what follows.

Definition 3. *A **uniform Poisson space is** a Poisson space P in which the topology is defined by a uniformity on P, and which satisfies Definition 2 with $C(P)$ replaced by $C_u(P)$.*

Here $C_u(P)$ is the space of uniformly continuous functions on P; it follows that elements of $C_L^\infty(P)$ are now required to be uniformly continuous.

Similarly, a **smooth Poisson space** is a Poisson space for which P is a manifold, and $C(P)$ in Definition 2 is replaced by $C^\infty(P)$. Hence $C_L^\infty(P) = C^\infty(P)$. By the symplectic decomposition theorem, a smooth Poisson space with $\mathfrak{A} = C^\infty(P)$ is nothing but a Poisson manifold.

In any case, $C_L^\infty(P)$ is the function space intrinsically related to a (general, uniform, or smooth) Poisson space P.

The pure state space $\mathcal{P}(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}_\mathbb{C}$ is a uniform Poisson space in the following way. We refer to (1.17) and subsequent text.

Firstly, it follows directly from the definition of the w^*-uniformity on $\mathcal{P}(\mathfrak{A})$ that each $\hat{A}$, $A \in \mathfrak{A}$, is in $C_u(\mathcal{P}(\mathfrak{A}))$; hence $\mathfrak{A} \subset C_u(\mathcal{P}(\mathfrak{A}))$, as required. As is well known, a C^*-algebra separates the points of its pure state space (cf. [30]).

Secondly, it is not difficult to show that the natural manifold topology on a projective Hilbert space $\mathbb{P}\mathcal{H}$ coincides with the w^*-topology it inherits from the canonical inclusion $\mathbb{P}\mathcal{H} \subset \mathcal{S}(\mathfrak{B}(\mathcal{H}))^*$. It follows that the inclusion map of any sector $\mathbb{P}\mathcal{H}_\alpha$ (equipped with the manifold topology) into $\mathcal{P}(\mathfrak{A})$ (with the w^*-topology) is continuous.

Thirdly, there is a unique Poisson structure $\{\,,\,\}$ on $\mathcal{P}(\mathfrak{A})$ such that

$$\{\hat{A}, \hat{B}\} = i(\widehat{AB - BA})\,. \tag{1.21}$$

This Poisson bracket is defined by letting the sectors $\mathbb{P}\mathcal{H}_\alpha$ of $\mathcal{P}(\mathfrak{A})$ coincide with its symplectic leaves, and making each $\mathbb{P}\mathcal{H}_\alpha$ into a symplectic manifold by endowing it with the (suitably normalized) Fubini–Study symplectic form [53, 38, 18, 19, 20, 39]. The reason that this structure is uniquely determined by (1.21) is that in an irreducible representation $\pi(\mathfrak{A}_\mathbb{C})$ on a Hilbert space $\mathcal{H}$ the collection of differentials $\{\widehat{d\pi(A)}, A \in \mathfrak{A}\}$ is dense in the cotangent space at each point of $\mathbb{P}\mathcal{H}$. Note that the precise choice of $\mathcal{H}_\alpha$ in its unitary equivalence class does not affect the definition of this Poisson structure, since it is invariant under unitary transformations. Since $\mathfrak{A}_\mathbb{C}$ is a C^*-algebra, $\mathfrak{A}$ is closed under the right-hand side of (1.21), and therefore under the Poisson bracket on the left-hand side as well.

We now return to general Poisson spaces.

If $\mathcal{P}$ is simultaneously a (general, uniform, or smooth) Poisson space and a transition probability space, two function spaces are intrinsically associated with it: $C_L^\infty(\mathcal{P})$ and $\mathfrak{A}(\mathcal{P})$, respectively. The space naturally tied with both structures in concert is therefore

$$\mathfrak{A}_L(\mathcal{P}) = \mathfrak{A}(\mathcal{P}) \cap C_L^\infty(\mathcal{P})\,. \tag{1.22}$$

Since elements of $\mathfrak{A}_L(\mathcal{P})$ are smooth on each symplectic leaf of $\mathcal{P}$, they generate a well-defined Hamiltonian flow, which, of course, stays inside a given leaf.

Definition 4. *A (general, uniform, or smooth) Poisson space which is simultaneously a transition probability space is called **unitary** if the Hamiltonian flow on $\mathcal{P}$ defined by each element of $\mathfrak{A}_L(\mathcal{P})$ preserves the transition probabilities. That is, if $\rho(t)$ and $\sigma(t)$ are Hamiltonian curves (with respect to a given $H \in \mathfrak{A}_L(\mathcal{P})$) through $\rho(0) = \rho$ and $\sigma(0) = \sigma$, respectively, then*

$$p(\rho(t), \sigma(t)) = p(\rho, \sigma) \tag{1.23}$$

for each t for which both flows are defined.

We now come to the central concept of this work.

Definition 5. *A* (**general**, **uniform**, *or* **smooth**) **Poisson space with a transition probability** *is a set $\mathcal{P}$ which is a well-behaved transition probability space and a unitary (general, uniform, or smooth) Poisson space, for which $\mathfrak{A} = \mathfrak{A}_L(\mathcal{P})$.*

This definition imposes two closely related compatibility conditions between the Poisson structure and the transition probabilities: firstly, it makes a definite choice for the space $\mathfrak{A}$ appearing in the definition of a Poisson space, and secondly it imposes the unitarity requirement.

If $(\mathcal{P}, p)$ is a classical transition probability space (that is, p is given by (1.12)), then any Poisson structure is unitary. This is, indeed, the situation in classical mechanics, where $\mathcal{P}$ is the phase space of a physical system. The best-known example is, of course, $\mathcal{P} = \mathbb{R}^{2n}$ with canonical symplectic structure.

The pure state space $\mathcal{P}(\mathfrak{A})$ of a C^*-algebra is a uniform Poisson space with a transition probability. Indeed, we infer from (1.18) that $\mathfrak{A}(\mathcal{P}) \subset C_L^\infty(\mathcal{P}(\mathfrak{A}))$, so that $\mathfrak{A}_L(\mathcal{P}(\mathfrak{A}))$ as defined in (1.22) coincides with $\mathfrak{A}$ as given in (1.19). Moreover, the flow of each $A \in \mathfrak{A}$ on a given symplectic leaf (= sector) $\mathbb{P}\mathcal{H}_\alpha$ of $\mathcal{P}(\mathfrak{A})$ is the projection of the flow $\Psi(t) = \exp(-itA)\Psi$ on $\mathcal{H}_\alpha$. Since A is self-adjoint, $\exp(-itA)$ is a unitary operator, and the transition probabilities (1.16) are clearly invariant under such flows.

2. Axioms for Pure State Spaces

As remarked above, a direct translation of the Axioms AHS1–AHS5 for compact convex sets to axioms on their extreme boundaries is difficult. Nevertheless, we can work with a set of axioms on a set $\mathcal{P}$, some of which are similar to AHS1–AHS5. In particular, AHS2 can be directly translated:

Definition 6. *A well-behaved transition probability space $\mathcal{P}$ is said to have the* **two-sphere property** *if for any two points ρ, σ (with $\rho \neq \sigma$) lying in the same sector of $\mathcal{P}$, the space $\{\rho, \sigma\}^{\perp\perp}$ is isomorphic as a transition probability space to the two-sphere S^2, with transition probabilities given by $p(z, w) = \frac{1}{2}(1 + \cos\theta(z, w))$ (where $\theta(z, w)$ is the angular distance between z and w, measured along a great circle).*

Here the orthoclosed space $\{\rho, \sigma\}^{\perp\perp} = \rho \vee \sigma$ may be regarded as an element of the lattice $\mathfrak{L}(\mathcal{P})$. If ρ and σ lie in different sectors of $\mathcal{P}$, then $\rho \vee \sigma = \{\rho, \sigma\}$; this follows from repeated application of De Morgan's laws [31] and $\rho^{\perp\perp} = \rho$ (etc.).

To understand the nature of the two-sphere property, note that a two-sphere S^2 with radius 1 may be regarded as the extreme boundary of the unit ball $B^3 \subset \mathbb{R}^3$, seen as a compact convex set. As we saw in Sec. 1.2, $B^3 \simeq \mathcal{S}(\mathfrak{M}_2(\mathbb{C}))$. Restricted to the extreme boundary, the parametrization (1.3) leads to a bijection between $\mathcal{P}(\mathfrak{M}_2(\mathbb{C})) \simeq \mathbb{P}\mathbb{C}^2$ and S^2. Under this bijection the transition probabilities (1.16) on $\mathbb{P}\mathbb{C}^2$ are mapped into the ones stated in Definition 6.

In other words, the two-sphere property states that there exists a fixed reference two-sphere $S_{\text{ref}}^2 \simeq \mathbb{P}\mathbb{C}^2$, equipped with the standard Hilbert space transition pro-

babilities $p = p_{\mathbb{C}^2}$ given by (1.16), and a collection of bijections $T_{\rho \vee \sigma} : \rho \vee \sigma \to S^2_{\mathrm{ref}}$, defined for each orthoclosed subspace of the type $\rho \vee \sigma \subset \mathcal{P}$ (where ρ and $\sigma \neq \rho$ lie in the same sector of $\mathcal{P}$), such that

$$p_{\mathbb{C}^2}(T_{\rho \vee \sigma}(\rho'), T_{\rho \vee \sigma}(\sigma')) = p(\rho', \sigma') \tag{2.1}$$

for all $\rho', \sigma' \in \rho \vee \sigma$.

Now consider the following axioms on a set $\mathcal{P}$:

Axiom 1. $\mathcal{P}$ *is a uniform Poisson space with a transition probability*;

Axiom 2. $\mathcal{P}$ *has the two-sphere property*;

Axiom 3. *The sectors of* $\mathcal{P}$ *as a transition probability space coincide with the symplectic leaves of* $\mathcal{P}$ *as a Poisson space*;

Axiom 4. *The space* $\mathfrak{A}$ (*defined through Axiom* 1 *by* (1.22)) *is closed under the Jordan product constructed from the transition probabilities*;

Axiom 5. *The pure state space* $\mathcal{P}(\mathfrak{A})$ *of* $\mathfrak{A}$ *coincides with* $\mathcal{P}$.

The meaning of Axiom 4 will become clear as soon as we have explained how to construct a Jordan product on $\mathfrak{A}(\mathcal{P})$, for certain transition probability spaces $\mathcal{P}$. This axiom turns $\mathfrak{A}$ into a JB-algebra, which is contained in $C(\mathcal{P})$. Hence each element of $\mathcal{P}$ defines a pure state on $\mathfrak{A}$ by evaluation; Axiom 5 requires that all pure states of $\mathfrak{A}$ be of this form (note that, by Axiom 1, $\mathfrak{A}$ already separates points).

Axioms 2 and 4 are direct analogues of Axioms AHS2 and AHS4, respectively (also cf. the end of Sec. 4.2). The 'bootstrap' Axiom 5 restricts the possible uniformities on $\mathcal{P}$; it is somewhat analogous to Axiom AHS3.

In the previous section we have seen that the pure state space of a unital C^*-algebra satisfies Axioms 1–5.

The remainder of this paper is devoted to the proof of the following

Theorem. *If a set* $\mathcal{P}$ *satisfies Axioms* 1–5 (*with* $\mathcal{P}$ *as a transition probability space containing no sector of dimension* 3), *then there exists a unital* C^*-*algebra* $\mathfrak{A}_{\mathbb{C}}$, *whose self-adjoint part is* $\mathfrak{A}$ (*defined through Axiom* 1). *This* $\mathfrak{A}_{\mathbb{C}}$ *is unique up to isomorphism, and can be explicitly reconstructed from* $\mathcal{P}$, *such that*

(1) $\mathcal{P} = \mathcal{P}(\mathfrak{A})$ (*i.e.,* $\mathcal{P}$ *is the pure state space of* $\mathfrak{A}$);
(2) *the transition probabilities* (1.4) *coincide with those initially given on* $\mathcal{P}$;
(3) *the Poisson structure on each symplectic leaf of* $\mathcal{P}$ *is proportional to the Poisson structure imposed on the given leaf by* (1.21);
(4) *the* w^*-*uniformity on* $\mathcal{P}(\mathfrak{A})$ *defined by* $\mathfrak{A}$ *is contained in the initial uniformity on* $\mathcal{P}$;
(5) *the* C^*-*norm on* $\mathfrak{A} \subset \mathfrak{A}_{\mathbb{C}}$ *is equal to the sup-norm inherited from the inclusion* $\mathfrak{A} \subset \ell^{\infty}(\mathcal{P})$.

The unfortunate restriction to transition probability spaces without 3-dimensional sectors (where the notion of dimension is as defined after (1.10), i.e., as the cardinality of a basis of $\mathcal{P}$ as a transition probability space) follows from our method of proof, which uses the von Neumann co-ordinatization theorem for Hilbert lattices

[23, 54, 32]. In view of the parallel between our axioms and those in [5], however, we are confident that the theorem holds without this restriction. To make progress in this direction one has to either follow our line of proof and exclude the possibility of non-Desarguesian projective geometries (cf. [23, 24] in the present context), or abandon the use of Hilbert lattices and develop a spectral theory of well-behaved transition probability spaces, analogous to the spectral theory of compact convex sets of Alfsen and Shultz [6, 7]. Despite considerable efforts in both directions the author has failed to remove the restriction.

The theorem lays out a possible mathematical structure of quantum mechanics with superselection rules. Like all other attempts to do so (cf. [43, 44, 49, 37, 14]), the axioms appear to be contingent. This is particularly true of Axiom AHS2 and of our Axiom 2, which lie at the heart of quantum mechanics. One advantage of the axiom schemes in [5] and the present paper is that they identify the incidental nature of quantum mechanics so clearly.

If $\mathcal{P}$ is merely assumed to be a Poisson space with a transition probability (i.e., no uniformity is present), then the above still holds, with the obvious modifications. In that way, however, only perfect C^*-algebras [50, 2] can be reconstructed (cf. Sec. 1.4).

3. From Transition Probabilities to C^*-algebras

The proof of the theorem above essentially consists of the construction of a C^*-algebra $\mathfrak{A}_{\mathbb{C}}$ from the given set $\mathcal{P}$. In summary, we can say that in passing from pure states to algebras of observables one has the following correspondences.

Pure state space	*Algebra of observables*
transition probabilities	Jordan product
Poisson structure	Poisson bracket
unitarity	Leibniz rule

To avoid unnecessary interruptions of the argument, some of the more technical arguments are delayed to Chapter 4.

3.1. *Identification of $\mathcal{P}$ as a transition probability space*

This identification follows from Axiom 1 (of which only the part stating that $\mathcal{P}$ be a well-behaved transition probability space is needed) and Axiom 2, as a consequence of the following result.

Proposition 1. *Let a well-behaved transition probability space $\mathcal{P}$ (with associated lattice $\mathfrak{L}(\mathcal{P})$) have the two-sphere property. If $\mathcal{P}$ has no sector of dimension 3, then $\mathcal{P} \simeq \cup_\alpha \mathbb{P}\mathcal{H}_\alpha$ as a transition probability space (for some family $\{\mathcal{H}_\alpha\}$ of complex Hilbert spaces), where each sector $\mathbb{P}\mathcal{H}_\alpha$ is equipped with the transition probabilities (1.16).*

This statement is not necessarily false when $\mathcal{P}$ does have sectors of dimension 3 (in fact, we believe it to be always true in that case as well); unfortunately our proof does not work in that special dimension.

In any case, it is sufficient to prove the theorem for each sector separately, so we may assume that $\mathcal{P}$ is irreducible (as a transition probability space). Even so, the proof is quite involved, and will be given in Sec. 4.1.

3.2. *Spectral theorem*

For each orthoclosed subset Q of a well-behaved transition probability space $\mathcal{P}$, define a function p_Q on $\mathcal{P}$ by

$$p_Q = \sum_{i=1}^{\dim(Q)} p_{e_i} \, ; \tag{3.1}$$

here $\{e_i\}$ is a basis of Q; it is easily seen that p_Q is independent of the choice of this basis (cf. [59]).

Definition 7. *Let $\mathcal{P}$ be a well-behaved transition probability space. A* **spectral resolution** *of an element $f \in \ell^\infty(\mathcal{P})$ is an expansion (in the topology of pointwise convergence)*

$$f = \sum_j \lambda_j p_{Q_j} \, , \tag{3.2}$$

where $\lambda_j \in \mathbb{R}$, and $\{Q_j\}$ is an orthogonal family of orthoclosed subsets of $\mathcal{P}$ (cf. (3.1)) for which $\sum_j p_{Q_j}$ equals the unit function on $\mathcal{P}$.

Proposition 2. *If $\mathcal{P} = \cup_\alpha \mathbb{P}\mathcal{H}_\alpha$ (with transition probabilities (1.16)) then any $f \in \mathfrak{A}_{00}(\mathcal{P})$ has a unique spectral resolution.*

By the previous section this applies, in particular, to a transition probability space $\mathcal{P}$ satisfying Axioms 1 and 2.

Proof. Firstly, the case of reducible $\mathcal{P}$ may be reduced to the irreducible one by grouping the ρ_i in $f = \sum_{i=1}^N c_i p_{\rho_i}$ into mutually orthogonal groups, with the property that $(\cup \rho)^{\perp\perp}$ is irreducible if the union is over all ρ_i in a given group. Thus we henceforth assume that $\mathcal{P}$ is irreducible, hence of the form $\mathcal{P} = \mathbb{P}\mathcal{H}$ with the transition probabilities (1.16).

If $\mathcal{P}$ is finite-dimensional the proposition is simply a restatement of the spectral theorem for Hermitian matrices. In the general case, let f be as above, and $Q := \{\rho_1, \ldots, \rho_N\}^{\perp\perp}$. If $\sigma \in Q$ then $f(\sigma) = \sum_j \lambda_j p_{Q_j}(\sigma)$ for some λ_j and mutually orthogonal $Q_j \subset Q$, as in the previous paragraph. If $\sigma \in Q^\perp$ this equation trivially holds, as both sides vanish.

Let us assume, therefore, that σ lies neither in Q nor in $Q^\perp$. Define $\varphi_Q(\sigma)$ by the following procedure: lift σ to a unit vector Σ in $\mathcal{H}$, project Σ onto the subspace defined by Q, normalize the resulting vector to unity, and project back to $\mathbb{P}\mathcal{H}$ (this is a Sasaki projection in the sense of lattice theory [14,31]). In the Hilbert space case relevant to us, the transition probabilities satisfy

$$p(\sigma, \rho) = p(\sigma, \varphi_Q(\sigma)) p(\varphi_Q(\sigma), \rho) \tag{3.3}$$

for $\rho \in Q$ and $\sigma \notin Q^\perp$. We now compute $f(\sigma)$ by using this equation, followed by the use of the spectral theorem in Q, and subsequently we recycle the same equation in the opposite direction. This calculation establishes the proposition for $\sigma \notin Q^\perp$.
□

If $\mathcal{P}$ is a classical transition probability space (see (1.12)) then a spectral theorem obviously holds as well; it simply states that a function f with finite support $\{\sigma_i\}$ is given by $f = \sum_i f(\sigma_i) p_{\sigma_i}$.

3.3. Jordan structure

Proposition 3. *If* $\mathcal{P} = \cup_\alpha \mathbb{P}\mathcal{H}_\alpha$ (*with   transition   probabilities   (1.16)),* $f = \sum_j \lambda_j p_{Q_j}$ *is the spectral resolution of* $f \in \mathfrak{A}_{00}(\mathcal{P})$, *and* f^2 *is defined by* $f^2 = \sum_j \lambda_j^2 p_{Q_j}$, *then the product* $\circ$ *defined by*

$$f \circ g = \tfrac{1}{4}((f+g)^2 - (f-g)^2) \tag{3.4}$$

turns $\mathfrak{A}_{00}(\mathcal{P})$ *into a Jordan algebra. Moreover, this Jordan product* $\circ$ *can be extended to* $\mathfrak{A}_0(\mathcal{P})$ *by* (*norm-*) *continuity, which thereby becomes a JB-algebra* (*with the sup-norm inherited from* $\ell^\infty(\mathcal{P})$). *Finally, the bidual* $\mathfrak{A}(\mathcal{P})$ *is turned into a JB-algebra by extending* $\circ$ *by* w^*-*continuity.*

The bilinearity of (3.4) is not obvious, and would not necessarily hold for arbitrary well-behaved transition probability spaces in which a spectral theorem (in the sense of Proposition 2) is valid. In the present case, it follows, as a point of principle, from the explicit form of the transition probabilities in $\mathbb{P}\mathcal{H}$. The quickest way to establish bilinearity, of course, is to look at a function p_Q (where Q lies in a sector $\mathbb{P}\mathcal{H}$ of $\mathcal{P}$) as the Gel'fand transform of a projection operator on $\mathcal{H}$.

Given bilinearity, the claims of the proposition follow from the literature. The extension to $\mathfrak{A}_0(\mathcal{P})$ by continuity, turning it into a JB-algebra, is in [6, Thm. 12.12] or [8, Prop. 6.11]. For the the extension to $\mathfrak{A}(\mathcal{P})$ see Sec. 3 of [9] and Sec. 2 and Prop. 6.13 of [8]. (There is a spectral theorem in $\mathfrak{A}(\mathcal{P})$, which is a so-called JBW-algebra, as well, cf. [6, 7, 9], but we will not need this.)

The norm in $\mathfrak{A}(\mathcal{P})$ is the sup-norm inherited from $\ell^\infty(\mathcal{P})$ as well; this establishes item 5 of the Theorem.   □

If $\mathcal{P}$ is classical, $\mathfrak{A}(\mathcal{P}) = \ell^\infty(\mathcal{P})$, and the Jordan product constructed above is given by pointwise multiplication. This explains why the latter is used in classical mechanics.

3.4. Explicit description of $\mathfrak{A}(\mathcal{P})$

Proposition 4. *Let* $\mathcal{P} = \cup_\alpha \mathbb{P}\mathcal{H}_\alpha$ (*with transition probabilities* (1.16)), *and regard self-adjoint elements* $A = \oplus_\alpha A_\alpha$ *of the von Neumann algebra* $\mathfrak{M}_\mathbb{C} = \oplus_\alpha \mathfrak{B}(\mathcal{H}_\alpha)$ *as functions* $\hat{A}$ *on* $\mathcal{P}$ *in the obvious way: if* $\rho \in \mathbb{P}\mathcal{H}_\alpha$ *then* $\hat{A}(\rho) = \rho(A_\alpha)$. *Denote the subspace of* $\ell^\infty(\mathcal{P})$ *consisting of all such* $\hat{A}$, $A \in \mathfrak{M}$, *by* $\hat{\mathfrak{M}}$. *Then*

$$\mathfrak{A}(\mathcal{P}) = \hat{\mathfrak{M}}. \tag{3.5}$$

Note that the identification of $A \in \mathfrak{M}$ with $\hat{A} \in \ell^\infty(\mathcal{P})$ is norm-preserving relative to the operator norm and the sup-norm, respectively. Also, it is clear that this proposition proves (1.18).

Proof. Inspired by [1, 19], we define a (locally non-trivial) fiber bundle $\mathcal{B}(\mathcal{P})$, whose base space B is the space of sectors, equipped with the discrete topology, and whose fiber above a given base point α is $\mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$; here $\mathcal{H}_\alpha$ is such that the sector α is $\mathbb{P}\mathcal{H}_\alpha$. Moreover, $\mathcal{P}$ itself may be seen as a fiber bundle over the same base space; now the fiber above α is $\mathbb{P}\mathcal{H}_\alpha$. We will denote the projection of the latter bundle by pr. A cross-section s of $\mathcal{B}(\mathcal{P})$ then defines a function $\hat{s}$ on $\mathcal{P}$ by $\hat{s}(\rho) = [s(pr(\rho))](\rho)$. The correspondence $s \leftrightarrow \hat{s}$ is isometric if we define the norm of a cross-section of $\mathcal{B}(\mathcal{P})$ by $\|s\| = \sup_{\alpha \in B} \|s(\alpha)\|$ (where the right-hand side of course contains the operator norm in $\mathfrak{B}(\mathcal{H}_\alpha)$), and the norm of $\hat{s}$ as the sup-norm in $\ell^\infty(\mathcal{P})$.

It follows directly from its definition that the space $\mathfrak{A}_{00}(\mathcal{P})$ consists of section s of $\mathcal{B}(\mathcal{P})$ with finite support, and such that $s(\alpha)$ has finite rank for each α. Its closure $\mathfrak{A}_0(\mathcal{P})$ contains all sections such that the function $\alpha \to \|s(\alpha)\|$ vanishes at infinity, and $s(\alpha)$ is a compact operator. It follows from elementary operator algebra theory that the dual $\mathfrak{A}_0(\mathcal{P})^*$ may be realized as the space of sections for which $s(\alpha)$ is of trace-class and $\alpha \to \|s(\alpha)\|_1$ (the norm here being the trace-norm) is in $\ell^1(B)$. The bidual $\mathfrak{A}(\mathcal{P})$ then consists of all sections of $\mathcal{B}(\mathcal{P})$ for which $\alpha \to \|s(\alpha)\|$ is in $\ell^\infty(B)$ (here the crucial point is that $\mathfrak{K}(\mathcal{H})^{**} = \mathfrak{B}(\mathcal{H})$). Eq. (3.5) is then obvious. $\qquad\square$

For later use, we note that $\mathfrak{A}_0(\mathcal{P})$ and even $\mathfrak{A}_{00}(\mathcal{P})$ are dense in $\mathfrak{A}(\mathcal{P})$ in the topology of pointwise convergence. This is because firstly $\mathfrak{K}(\mathcal{H})$ is dense in $\mathfrak{B}(\mathcal{H})$ in the weak operator topology [30] (as is the set of operators of finite rank), hence certainly in the coarser topology of pointwise convergence on $\mathcal{P}$, and secondly the topology of pointwise convergence on $\ell^\infty(B)$ is contained in the w^*-topology ($\ell^\infty(B)$ being the dual of $\ell^1(B)$, which in turn is the dual of $\ell_0(B)$); recall that any (pre-) Banach space is w^*-dense in its double dual (e.g., [30]).

Under the correspondence $\mathfrak{A}(\mathcal{P}) = \hat{\mathfrak{M}} \leftrightarrow \mathfrak{M}$ the Jordan product constructed in the previous section is then simply given by the anti-commutator of operators in $\mathfrak{M}$.

3.5. *Algebra of observables*

By Axiom 1, the space of observables $\mathfrak{A}$ is defined by (1.22). We now use Axiom 3, which implies that each symplectic leaf of $\mathcal{P}$ is a projective Hilbert space $\mathbb{P}\mathcal{H}_\alpha$. For the moment we assume that each leaf $\mathbb{P}\mathcal{H}_\alpha$ has a manifold structure relative to which all functions $\hat{A}$, where $A \in \mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$, are smooth (such as its usual manifold structure). Then $\mathfrak{A}(\mathcal{P}) \cap C_u(\mathcal{P}) \subset C_L^\infty(\mathcal{P})$ by the explicit description of $\mathfrak{A}(\mathcal{P})$ just obtained. It then follows from (1.22) that

$$\mathfrak{A} = \mathfrak{A}(\mathcal{P}) \cap C_u(\mathcal{P}). \tag{3.6}$$

It is easily shown that $\mathfrak{A}$ is closed (in the sup-norm). This follows from the fact that $\mathfrak{A}(\mathcal{P})$ is closed, plus the observation that the subspace of functions in $\ell^\infty(\mathcal{P})$

which are uniformly continuous with respect to any uniformity on $\mathcal{P}$, is closed; this generalizes the well-known fact that the subspace of continuous functions relative to any topology on $\mathcal{P}$ is sup-norm closed (the proof of this observation proceeds by the same $\varepsilon/3$-argument).

Note that $\mathfrak{A}_0(\mathcal{P})$ is not necessarily a subspace of $\mathfrak{A}$; it never is if the C^*-algebra $\mathfrak{A}_\mathbb{C}$ to be constructed in what follows is antiliminal [21].

We can construct a Jordan product in $\mathfrak{A}$ by the procedure in Sec. 3.3. By Proposition 3 and Axiom 4, this turns $\mathfrak{A}$ into a JB-algebra. At this stage we can already construct the pure state space $\mathcal{P}(\mathfrak{A})$; the first claim of the Theorem then holds by Axiom 5.

We may regard the restriction of $\mathfrak{A}$ to a given sector $\mathbb{P}\mathcal{H}_\alpha$ as the Gel'fand transform of a Jordan subalgebra of $\mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$. This subalgebra must be weakly dense in $\mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$, for otherwise Axiom 5 cannot hold.

Let us now assume that some $\mathbb{P}\mathcal{H}_\alpha$ have an exotic manifold structure such that $\mathfrak{A}(\mathcal{P}) \cap C_u(\mathcal{P})$ is not contained in $C_L^\infty(\mathcal{P})$, so that $\mathfrak{A} \subset \mathfrak{A}(\mathcal{P}) \cap C_u(\mathcal{P})$ is a proper inclusion (rather than the equality (3.6)). It follows from Axiom 5 that the statement in the previous paragraph must still hold. This weak density suffices for the results in Secs. 3.7 and 3.8 to hold, and we can construct a C^*-algebra $\mathfrak{A}_\mathbb{C}$ with pure state space $\mathcal{P}$. The proper inclusion above would then contradict (1.19). Hence such exotic manifold structures are excluded by the axioms.

3.6. *Unitarity*, *Leibniz rule*, *and Jordan homomorphisms*

It is instructive to discuss a slightly more general context than is strictly necessary for our purposes.

Proposition 5. *Let $\mathcal{P}$ be a Poisson space with a transition probability in which every $f \in \mathfrak{A}_{00}(\mathcal{P})$ has a unique spectral resolution (in the sense of Definition 7). Assume that for each $H \in \mathfrak{A}_L(\mathcal{P})$ (cf. (1.22)) the map $f \to \{H, f\}$ is bounded on $\mathfrak{A}_L(\mathcal{P}) \subset \ell^\infty(\mathcal{P})$ (with sup-norm). If a Jordan product $\circ$ is defined on $\mathfrak{A}_L(\mathcal{P})$ through the transition probabilities, in the manner of Proposition 3, then $\circ$ and the Poisson bracket satisfy the Leibniz rule (1.1).*

The boundedness assumption holds in the case at hand (cf. the next section); it is mainly made to simplify the proof. The proposition evidently holds when $\mathfrak{A}_L(\mathcal{P})$ is a Poisson algebra, for which the assumption is violated.

Proof. Writing $\delta_H(f)$ for $\{H, f\}$, the boundedness of δ_H implies that the series $\alpha_t(f) = \sum_{n=0}^\infty t^n \delta_H^n(f)/n!$ converges uniformly, and defines a uniformly continuous one-parameter group of maps on $\mathfrak{A}_L(\mathcal{P})$ (cf. [15]). On the other hand, if $\sigma(t)$ is the Hamiltonian flow of H on $\mathcal{P}$ (with $\sigma(0) = \sigma$), then α_t as defined by $\alpha_t(f) : \sigma \to f(\sigma(t))$ must coincide with the definition above, for they each satisfy the same differential equation with the same initial condition. In particular, the flow in question must be complete. Moreover, it follows that the Leibniz rule (yet to be established) is equivalent to the property that α_t is a Jordan morphism for each t; this, in turn, can be rephrased by saying that $\alpha_t(f^2) = \alpha_t(f)^2$ for all $f \in \mathfrak{A}_L(\mathcal{P})$.

Let $f \in \mathfrak{A}_{00}(\mathcal{P}) \cap \mathfrak{A}_L(\mathcal{P})$, so that $f = \sum_k \lambda_k p_{e_k}$, where all e_k are orthogonal (cf. Sec. 3.2). Unitarity implies firstly that $\alpha_t(f) = \sum_k \lambda_k p_{e_k(-t)}$, and secondly that the $e_k(-t)$ are orthogonal. Hence $\alpha_t(f)$ is given in its spectral resolution, so that $\alpha_t(f)^2 = \sum_k \lambda_k^2 p_{e_k(-t)}$. Repeating the first use of unitarity, we find that this equals $\alpha_t(f^2)$. Hence the property holds on $\mathfrak{A}_{00}(\mathcal{P})$.

Now $\mathfrak{A}_{00}(\mathcal{P})$ is dense in $\mathfrak{A}(\mathcal{P})$ in the topology of pointwise convergence in $\ell^\infty(\mathcal{P})$. But $f_\lambda \to f$ pointwise clearly implies $\alpha_t(f_\lambda) \to \alpha_t(f)$ pointwise. This, plus the w^*-continuity of the Jordan product [9] proves the desired result. $\qquad\square$

3.7. *Poisson structure*

Item 3 of the Theorem follows from Axiom 3, the penultimate paragraph of Sec. 3.5, and the following

Proposition 6. *Let $\mathbb{P}\mathcal{H}$, equipped with the transition probabilities* (1.16), *be a unitary Poisson space for which the Poisson structure is symplectic, and for which $\mathfrak{A}$ is the Gel'fand transform of a weakly dense subspace of $\mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$.*

Then the Poisson structure is determined up to a multiplicative constant, and given by (1.21) *times some $\hbar^{-1} \in \mathbb{R}$.*

Proof. Axiom 3 implies that each sector $\mathbb{P}\mathcal{H}$ (for some $\mathcal{H}$) is a symplectic space. Unitarity (in our sense) and Wigner's theorem (cf. [54, 14, 50] for the latter) imply that each $\hat{A} \in \mathfrak{A}$ generates a Hamiltonian flow on $\mathbb{P}\mathcal{H}$ which is the projection of a unitary flow on $\mathcal{H}$. Therefore, $\{\hat{A}, \hat{B}\}(\psi) = \frac{d}{dt}\hat{B}(\exp(it\hat{C}(A))\psi)_{t=0}$ for some self-adjoint operator C, depending on A (here $\exp(it\hat{C}(A))\psi$ is by definition the projection of $\exp(itC(A))\psi$ to $\mathbb{P}\mathcal{H}$, where ψ is some unit vector in $\mathcal{H}$ which projects to $\psi \in \mathbb{P}\mathcal{H}$). The right-hand side equals $i(\widehat{CB - BC})(\psi)$. Anti-symmetry of the left-hand side implies that $C = \hbar^{-1}A$ for some $\hbar^{-1} \in \mathbb{R}$. By the weak density assumption, the collection of all differentials $d\hat{A}$ spans the fiber of the cotangent bundle at each point of $\mathbb{P}\mathcal{H}$. Thus the Poisson structure is completely determined. $\qquad\square$

This shows that the symplectic structure on each leaf is $\hbar\omega_{FS}$, where ω_{FS} is the Fubini–Study structure [53, 38, 18, 19, 20, 39]. (A closely related fact is that the Kähler metric associated to ω_{FS} is determined, up to a multiplicative constant, by its invariance under the induced action of all unitary operators on $\mathcal{H}$, cf. [1, 39].) The multiplicative constant is Planck's constant $\hbar$, which, as we see, may depend on the sector. To satisfy Axiom 4, $\hbar^{-1}$ must be nonzero in every sector whose dimension is greater than 1. In one-dimensional sectors the Poisson bracket identically vanishes, so that the value of $\hbar$ is irrelevant.

The Poisson structure on $\mathcal{P}$ is determined by the collection of symplectic structures on the sectors of $\mathcal{P}$, for the Poisson bracket $\{f, g\}(\rho)$ is determined by the restrictions of f and g to the leaf through ρ; cf. (1.20).

The choice (1.21) for the Poisson bracket on $\mathfrak{A}$ corresponds to taking $\hbar$ a sector-independent constant (put equal to 1). In general, we may regard $\hbar$ as a function on $\mathcal{P}(\mathfrak{A})$, which is constant on each sector. If $\hat{A}$ denotes an element of $\mathfrak{A}$, the restriction

of $\hat{A}$ to a sector $\mathbb{P}\mathcal{H}_\alpha$ corresponds to an operator $A_\alpha \in \mathfrak{B}(\mathcal{H}_\alpha)_{\mathrm{sa}}$ (cf. Sec. 3.5). The sector in which $\rho \in \mathcal{P}(\mathfrak{A})$ lies is called $\alpha(\rho)$. With this notation, and denoting $AB - BA$ by $[[A, B]]$ (recall that $[A, B]$ denotes the Lie bracket in a Jordan–Lie algebra) the Poisson bracket on $\mathfrak{A}$ is then given by

$$\{\hat{A}, \hat{B}\}(\rho) = \frac{i}{\hbar(\rho)}[[\widehat{A_{\alpha(\rho)}, B_{\alpha(\rho)}}]](\rho). \tag{3.7}$$

The sector-dependence of $\hbar$ cannot be completely arbitrary, however; Axiom 1 implies that $\hbar$ must be a uniformly continuous function on $\mathcal{P}$. For suppose $\hbar$ is not uniformly continuous. We then take $\hat{A}, \hat{B} \in \mathfrak{A}$ in such a way that A_α and B_α are independent of α in a neighbourhood of a point σ of discontinuity of $\hbar$, with $[[A_{\alpha(\sigma)}, B_{\alpha(\sigma)}]] \neq 0$. Then the real-valued function on $\mathcal{P}(\mathfrak{A})$ defined by $\rho \to \hbar(\rho)\{\hat{A}, \hat{B}\}(\rho)$ is certainly uniformly continuous near σ, since its value at ρ is equal to $i[[\widehat{A_{\alpha(\rho)}, B_{\alpha(\rho)}}]](\rho)$. But, by assumption, $\{\hat{A}, \hat{B}\}$ is uniformly continuous as well. Because of the factor $\hbar$, the product $\hbar\{\hat{A}, \hat{B}\}$ cannot be uniformly continuous. This leads to a contradiction.

3.8. C^*-structure

We now turn $\mathfrak{A}$ into a Jordan–Lie algebra, and thence into the self-adjoint part of a C^*-algebra $\mathfrak{A}_\mathbb{C}$ (cf. Sec. 1.1).

On each leaf, the associator equation (1.2) is identically satisfied by the Poisson bracket (3.7). However, the 'constant' $k \equiv \hbar^2/4$ may depend on the leaf. Therefore, we have to rescale the Poisson bracket so as to undo its $\hbar$-dependence. From (3.7) this is obviously accomplished by putting $[f, g](\rho) = \hbar(\rho)\{f, g\}(\rho)$. With the Jordan product $\circ$ defined in Sec. 3.3, Eq. (1.2) is now satisfied. Hence we define a product $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}_\mathbb{C}$ by

$$f \cdot g = f \circ g - \frac{1}{2}i[f, g], \tag{3.8}$$

and extend this to $\mathfrak{A}_\mathbb{C} \times \mathfrak{A}_\mathbb{C}$ by complex linearity.

As explained in Sec. 1.1, this product is associative. Indeed, in the notation introduced in the previous section one simply has

$$\hat{A} \cdot \hat{B}(\rho) = \widehat{A_{\alpha(\rho)}B_{\alpha(\rho)}}(\rho), \tag{3.9}$$

where the multiplication on the right-hand side is in $\mathfrak{B}(\mathcal{H}_{\alpha(\rho)})$.

By Axiom 1 (in particular, closure of $\mathfrak{A}$ under the Poisson bracket), Axiom 4, and the uniform continuity of $\hbar(\cdot)$, $\mathfrak{A}_\mathbb{C}$ is closed under this associative product.

Let $\mathfrak{A}$ be a JB-algebra, and $\mathfrak{A}_\mathbb{C} = \mathfrak{A} \oplus i\mathfrak{A}$ its complexification. As shown in [58], one may construct a norm on $\mathfrak{A}_\mathbb{C}$, which turns it into a so-called JB^*-algebra [28]; the involution is the natural one, i.e., $(f + ig)^* = f - ig$ for $f, g \in \mathfrak{A}$. Now given a JB^*-algebra $\mathfrak{A}_\mathbb{C}$ whose Jordan product $\circ$ is the anti-commutator of some associative product $\cdot$, it is shown in [47] that $(\mathfrak{A}_\mathbb{C}, \cdot)$ is a C^*-algebra iff $(\mathfrak{A}_\mathbb{C}, \circ)$ is JB^*-algebra.

Hence one can find a norm on $\mathfrak{A}_\mathbb{C}$ (whose restriction to its self-adjoint part $\mathfrak{A}$, realized as in (1.19), is the sup-norm) such that it becomes a C^*-algebra equipped with the associative product (3.8). Since the unit function evidently lies in $\mathfrak{A}(\mathcal{P})$ (cf. (3.5)) as well as in $C_u(\mathcal{P})$, it lies in $\mathfrak{A}$ (cf. (3.6)). In conclusion, the unital C^*-algebra mentioned in the theorem has been constructed.

An alternative argument showing that $\mathfrak{A}$ is closed under the commutator (Poisson bracket) is to combine the results of section 4.3 below and [5, §7]. This avoids the rescaling of the Poisson bracket by $\hbar(\cdot)$, but relies on the deep analysis of [5].

It is also possible to have $+$ instead of $-$ in (3.8). This choice produces a C^*-algebra $\mathfrak{A}_\mathbb{C}^{(+)}$ which is canonically anti-isomorphic to $\mathfrak{A}_\mathbb{C} \equiv \mathfrak{A}_\mathbb{C}^{(-)}$. Moreover, in some cases $\mathfrak{A}_\mathbb{C}^{(+)}$ is isomorphic to $\mathfrak{A}_\mathbb{C}^{(-)}$ in a non-canonical fashion. Choose a faithful representation $\pi(\mathfrak{A}_\mathbb{C})$ on some Hilbert space $\mathcal{H}$, and choose a basis $\{\mathbf{e}_i\}$ in $\mathcal{H}$. Then define an anti-linear map $J : \mathcal{H} \to \mathcal{H}$ by $J \sum_i c_i \mathbf{e}_i = \sum_i \overline{c}_i \mathbf{e}_i$, and subsequently a linear map j on $\pi(\mathfrak{A}_\mathbb{C})$ by $j(A) = J\pi(A)^* J$. If j maps $\pi(\mathfrak{A}_\mathbb{C})$ into itself, it defines an isomorphism between $\mathfrak{A}_\mathbb{C}^{(-)}$ and $\mathfrak{A}_\mathbb{C}^{(+)}$.

In [5] (or [50]) this sign change would correspond to reversing the orientation of K (or $\mathcal{P}$).

3.9. *Transition probabilities and uniform structure*

Recall Mielnik's definition (1.4) of the transition probability in the extreme boundary $\partial_e K$ of a compact convex set [41].

By Axiom 5, the extreme boundary of the state space $K = \mathcal{S}(\mathfrak{A})$ of $\mathfrak{A}$ is $\mathcal{P}$. Hence $\mathcal{P}$ acquires transition probabilities by (1.4), which are to be compared with those originally defined on it. In Sec. 4.2 we show that these transition probabilities coincide, and this proves item 2 of the Theorem.

It is immediate from the previous paragraph that $\mathfrak{A}(\mathcal{P}(\mathfrak{A})) = \mathfrak{A}(\mathcal{P})$. The w^*-uniformity appearing in (1.19) is the weakest uniformity relative to which all elements of $\mathfrak{A}$ are uniformly continuous. It then follows from (1.19) and (3.6) (in which the uniformity is the initially given one) that the initial uniformity on $\mathcal{P}$ must contain the w^*-uniformity it acquires as the space of pure states of $\mathfrak{A}_\mathbb{C}$. This proves item 4.

This completes our construction, as well as the proof of the theorem. $\qquad\square$

4. Proofs
4.1. *Proof of Proposition 1*

The strategy of the proof is to characterize the lattice $\mathfrak{L}(\mathcal{P})$ (cf. Sec. 1.4), and then use the so-called co-ordinatization theorem in lattice theory [14, 32] to show that $\mathfrak{L}(\mathcal{P})$ is isomorphic to the lattice $\mathcal{L}(\mathcal{H})$ of closed subspaces of some complex Hilbert space $\mathcal{H}$ (see [54, 14, 31, 32] for extensive information on this lattice; an equivalent description is in terms of the projections in the von Neumann algebra $\mathfrak{B}(\mathcal{H})$).

It is known that $\mathfrak{L}(\mathcal{P})$ is complete, atomic, and orthomodular [59, 13, 14] if $\mathcal{P}$ is a well-behaved transition probability space; hence it is also atomistic [14, 31]. Using

the connection between the center of an orthomodular lattice and its reducibility [31], it is routine to show that the irreducibility of $\mathcal{P}$ as a transition probability space (which we assume for the purpose of this proof) is equivalent to the irreducibility of $\mathfrak{L}(\mathcal{P})$ as a lattice. Hence $\mathfrak{L}(\mathcal{P})$ is also irreducible.

Lemma 1. $\mathfrak{L}(\mathcal{P})$ *has the covering property (i.e., satisfies the exchange axiom).*

See [14, 31, 32] for the relevant definitions and context.

Proof. Consistent with previous notation, we denote atoms of $\mathfrak{L}(\mathcal{P})$ (hence points of $\mathcal{P}$) by ρ, σ, and arbitrary elements by Q, Q_i, R, S.

Let $n = \dim(Q)$ (as a transition probability space); for the moment we assume $n < \infty$. We will first use induction to prove that if $\rho \notin Q$, the element $(\rho \vee Q) \wedge Q^\perp$ is an atom.

To start, note that if $Q_1 \le Q_2$ for orthoclosed Q_1, Q_2 of the same finite dimension, then $Q_1 = Q_2$, for an orthoclosed set in $\mathcal{P}$ is determined by a basis of it, which in turn determines its dimension. This implies that $\dim(\rho \vee Q) > \dim(Q)$ if $\rho \notin Q$ (take $Q_1 = Q$ and $Q_2 = \rho \vee Q$). Accordingly, it must be that $(\rho \vee Q) \wedge Q^\perp > \emptyset$, for equality would imply that $\dim(\rho \vee Q) = \dim(Q)$.

For $n = 1$, Q is an atom. By assumption, $\rho \vee Q$ is S^2, hence $(\rho \vee Q) \wedge Q^\perp$ is the anti-podal point to Q in $\rho \vee Q$, which is an atom, as desired. Now assume $n > 1$. Choose a basis $\{e_i\}_{i=1,\ldots,\dim(Q)}$ of Q; then $Q = \vee_{i=1}^n e_i$. Put $R = \vee_{i=1}^{n-1} e_i$; then $R < Q$ hence $Q^\perp < R^\perp$, so that $(\rho \vee Q) \wedge Q^\perp \le (\rho \vee Q) \wedge R^\perp$. The assumption $(\rho \vee Q) \wedge Q^\perp = (\rho \vee Q) \wedge R^\perp$ is equivalent, on use of $Q = R \vee e_n$, De Morgan's laws [31], and the associativity of $\wedge$, to $((\rho \vee Q) \wedge R^\perp) \wedge e_n^\perp = (\rho \vee Q) \wedge R^\perp$, which implies that $(\rho \vee Q) \wedge R^\perp \le e_n^\perp$. This is not possible, since the left-hand side contains e_n. Hence

$$\emptyset < (\rho \vee Q) \wedge Q^\perp < (\rho \vee Q) \wedge R^\perp. \tag{4.1}$$

It follows from the orthomodularity of $\mathfrak{L}(\mathcal{P})$ that if $R \le S$ and $R \le Q$, then

$$(S \vee Q) \wedge R^\perp = (S \wedge R^\perp) \vee (Q \wedge R^\perp). \tag{4.2}$$

Since $R < Q$ and $R \le \rho \vee R$, one has $\rho \vee Q = (\rho \vee R) \vee Q$. Now use (4.2) with $S = \rho \vee R$ to find

$$(\rho \vee Q) \wedge R^\perp = ((\rho \vee R) \vee Q) \wedge R^\perp = ((\rho \vee R) \wedge R^\perp) \vee (Q \wedge R^\perp).$$

By the induction hypothesis $(\rho \vee R) \wedge R^\perp$ is an atom (call it σ), so the right-hand side equals $\sigma \vee e_n$. The equality $\sigma = e_n$ would imply that $\rho \in Q$, hence $\sigma \ne e_n$. But then (4.1) and the S^2-assumption imply $0 < \dim((\rho \vee Q) \wedge Q^\perp) < 2$, so that $(\rho \vee Q) \wedge Q^\perp$ must indeed be an atom.

It follows that $\dim(\rho \vee Q) = \dim(Q) + 1$. Hence any $S \subset \mathcal{P}$ satisfying $Q \le S \le \rho \vee Q$ must have $\dim(S)$ equal to $\dim(Q)$ or to $\dim(Q) + 1$. In the former case, it must be that $S = Q$ by the dimension argument earlier. Similarly, in the latter case

the only possibility is $S = \rho \vee Q$. All in all, we have proved the covering property for finite-dimensional sublattices.

A complicated technical argument involving the dimension theory of lattices then shows that the covering property holds for all $x \in \mathfrak{L}(\mathcal{P})$; see Sec. 13 in [31] and Sec. 8 in [32]. $\qquad\square$

We have, therefore, shown that $\mathfrak{L}(\mathcal{P})$ is a complete atomistic irreducible orthomodular lattice with the covering property. If $\mathfrak{L}(\mathcal{P})$ is in addition infinite-dimensional, one speaks of a Hilbert lattice (recently, there has been a major breakthrough in the theory of such lattices [52, 29], but since the infinite-dimensionality is used explicitly in this work we derive no direct benefit from this). In any case, we are in a position to apply the standard co-ordinatization theorem of lattice theory; see [23, 54, 14, 32 29]. For this to apply, the dimension of $\mathfrak{L}(\mathcal{P})$ as a lattice [31] (which is easily seen to coincide with the dimension of $\mathcal{P}$ as a transition probability space) must be ≥ 4, so that we must now assume that $\dim(\mathcal{P}) \neq 3$; the case $\dim(\mathcal{P}) = 2$ is covered directly by Axiom 2. (The fact that dimension 3 is excluded is caused by the existence of so-called non-Desarguesian projective geometries; see [24] for a certain analogue of the co-ordinatization procedure in that case.)

Accordingly, for $\dim(\mathcal{P}) \neq 3$ there exists a vector space V over a division ring $\mathbb{D}$ (both unique up to isomorphism), equipped with an anisotropic Hermitian form θ (defined relative to an involution of $\mathbb{D}$, and unique up to scaling), such that $\mathfrak{L}(\mathcal{P}) \simeq \mathcal{L}(V)$ as orthocomplemented lattices. Here $\mathcal{L}(V)$ is the lattice of orthoclosed subspaces of V (where the orthoclosure is meant with respect to the orthogonality relation defined by θ).

We shall now show that we can use Axiom 2 once again to prove that $\mathbb{D} = \mathbb{C}$ as division rings. While this may seem obvious from the fact [23, 54] that for any irreducible projection lattice one has $\mathbb{D} \simeq (\rho \vee \sigma)\backslash \sigma$ (for arbitrary atoms $\rho \neq \sigma$), which is $\mathbb{C}$ by Axiom 2, this argument does not prove that $\mathbb{D} = \mathbb{C}$ as division rings.

The following insight (due to [34], and used in exactly the same way in [60] and [17]) is clear from the explicit construction of addition and mutliplication in $\mathbb{D}$ [54, 23]. Let V be 3-dimensional, and let $\mathcal{L}(V)$ carry a topology for which the lattice operations $\vee$ and $\wedge$ are jointly continuous. Then $\mathbb{D}$ (regarded as a subset of the collection of atoms in $\mathcal{L}(V)$), equipped with the topology inherited from $\mathcal{L}(V)$, is a topological division ring (i.e., addition and multiplication are jointly continuous).

Let $F \in \mathfrak{L}(\mathcal{P})$ be finite-dimensional. We can define a topology on $[\emptyset, F]$ (i.e., the set of all $Q \in \mathfrak{L}(\mathcal{P})$ for which $Q \subseteq F$) through a specification of convergence.

Given a net $\{Q_\lambda\}$ in F, we say that $Q_\lambda \to Q$ when eventually $\dim(Q_\lambda) = \dim(Q)$, and if there exists a family of bases $\{e_i^\lambda\}$ for $\{Q_\lambda\}$, and a basis $\{e_j\}$ of Q, such that $\sum_{i,j} p(e_i^\lambda, e_j) \to \dim(Q)$. This notion is actually independent of the choice of all bases involved, since $\sum_j p(\rho, e_j)$ is independent of the choice of the basis in Q for any $\rho \in \mathcal{P}$, and similarly for the bases of Q_λ (to see this, extend $\{e_j\}_{j=1}^{\dim(Q)}$ to a basis $\{e_j\}_{j=1}^{\dim(\mathcal{P})}$, and use the property $\sum_{j=1}^{\dim(\mathcal{P})} p(e_j, \rho) = 1$ for all $\rho \in \mathcal{P}$).

An equivalent definition of this convergence is that $Q_\lambda \to Q$ if $p(\rho_\lambda, \sigma) \to 0$ for all $\sigma \in F \wedge Q^\perp$ and all $\{\rho_\lambda\}$ such that $\rho_\lambda \in Q_\lambda$.

Using the criteria in [33], it is easily verified that this defines a topology on F. Moreover, this topology is Hausdorff. For let $Q_\lambda \to Q$ and $Q_\lambda \to R$. Then $p(\rho_\lambda, \sigma) \to 0$ for all $\sigma \in Q^\perp \vee R^\perp = (Q \wedge R)^\perp$, and $\{\rho_\lambda\}$ as specified above. Choose a basis $\{e_j\}$ of Q which extends a basis of $Q \wedge R$. Then $\sum_{j=1}^{\dim(Q \wedge R)} p(\rho_\lambda, e_j) = 1$, but also $\sum_{j=1}^{\dim(Q)} p(\rho_\lambda, e_j) = 1$ since $Q_\lambda \to Q$. Hence $p(\rho_\lambda, \sigma) \to 0$ for all $\sigma \in Q \wedge (Q \wedge R)^\perp$. This leads to a contradiction unless $Q = R$.

Lemma 2. *The restriction of this topology to any two-sphere $\rho \vee \sigma \simeq S^2$ in F induces the usual topology on S^2. Moreover, $\vee$ and $\wedge$ are jointly continuous on any $[\emptyset, F]$, where F is a 3-dimensional subspace of $\mathfrak{L}(\mathcal{P})$.*

Proof. If we restrict this topology to the atoms in F, then $\rho_\lambda \to \rho$ if $p(\rho_\lambda, \rho) \to 1$. This induces the usual topology on $F = \sigma \vee \tau \simeq S^2$, since one can easily show that, in $F = \sigma \vee \tau$, $p(\rho_\lambda, \rho) \to 1$ is equivalent to $p(\rho_\lambda, \nu) \to p(\rho, \nu)$ for all $\nu \in \sigma \vee \tau$ (cf. [17]).

We now take F to be a 3-dimensional subspace. We firstly show that $\rho_\lambda \to \rho$ and $\sigma_\lambda \to \sigma$, where ρ and σ are atoms, imply $\rho_\lambda \vee \sigma_\lambda \to \rho \vee \sigma$. Let $\tau_\lambda = (\rho_\lambda \vee \sigma_\lambda)^\perp \wedge F$, and $\tau = (\rho \vee \sigma)^\perp \wedge F$; these are atoms. Let ρ'_λ be the anti-podal point to ρ_λ in $\rho_\lambda \vee \sigma_\lambda$ (i.e., $\rho'_\lambda = \rho_\lambda^\perp \wedge (\rho_\lambda \vee \sigma_\lambda)$), and let σ'_λ be the anti-podal to σ_λ in $\rho_\lambda \vee \sigma_\lambda$. Then $\{\rho_\lambda, \rho'_\lambda, \tau_\lambda\}$ is a basis of F, and so is $\{\sigma_\lambda, \sigma'_\lambda, \tau_\lambda\}$. The definition of a basis and of $\rho_\lambda \to \rho$, $\sigma_\lambda \to \sigma$ imply that $p(\rho, \tau_\lambda) \to 0$ and $p(\sigma, \tau_\lambda) \to 0$. Hence $p(\tau, \tau_\lambda) \to 1$.

Now take an arbitrary atom $\alpha_\lambda \in \tau_\lambda^\perp \wedge F$, and complete to a basis $\{\alpha_\lambda, \alpha'_\lambda, \tau_\lambda\}$, where $\alpha'_\lambda \in \rho_\lambda \vee \sigma_\lambda$. Again, the definition of a basis implies that $p(\alpha_\lambda, \tau) \to 0$. Hence by our second definition of convergence $\rho_\lambda \vee \sigma_\lambda \to \rho \vee \sigma$.

Secondly, we show that $Q_\lambda \to Q$ and $R_\lambda \to R$, where Q and R are two-dimensional subspaces of F, implies $Q_\lambda \wedge R_\lambda \to Q \wedge R$ (we assume $Q \neq R$, so eventually $Q_\lambda \neq R_\lambda$). Let $\alpha = Q^\perp \wedge F$, $\beta = R^\perp \wedge F$, $\gamma = Q \wedge R$, and $\gamma_\lambda = Q_\lambda \wedge R_\lambda$; as a simple dimension count shows, these are all atoms. By assumption, $p(\gamma_\lambda, \alpha) \to 0$ and $p(\gamma_\lambda, \beta) \to 0$. Since $(\alpha \cup \beta)^\perp = (\alpha \vee \beta)^\perp$ by definition of $\vee$, and $(\alpha \vee \beta)$ is two-dimensional, γ is the only point in F which is orthogonal to α and β. Hence $p(\gamma_\lambda, \gamma) \to 1$; if not, the assumption would be contradicted. But this is precisely the definition of $Q_\lambda \wedge R_\lambda \to Q \wedge R$. $\qquad\square$

From the classification of locally compact connected division rings [56] we conclude that $\mathbb{D} = \mathbb{C}$ as division rings; the ring structure is entirely determined by the topology. Moreover, Lemma 2 implies that the orthocomplementation is continuous on 3-dimensional subspaces. If one inspects the way the involution of $\mathbb{D}$ is constructed in the proof of the lattice co-ordinatization theorem, one immediately infers that this involution (of $\mathbb{C}$ in our case) must then be continuous as well. It can be shown that $\mathbb{C}$ only possesses two continuous involutions: complex conjugation and the identity map [54]. The latter cannot define a non-degenerate sesquilinear form (so that, in particular, the lattice $\mathcal{L}(V)$ could not be orthomodular). Hence one is left with complex conjugation, and V must be a complex pre-Hilbert space.

The fact that V is actually complete follows from the orthomodularity of $\mathcal{L}(\mathcal{P})$ (hence of $\mathcal{L}(V)$). The proof of this statement is due to [10]; see (also cf. [32, Thm.

11.9], or [14, Thm. 21.4.1]). We will therefore write $V = \mathcal{H}$.

We conclude that $\mathcal{L}(\mathcal{P})$ is isomorphic to the lattice $\mathcal{L}(\mathcal{H})$ of closed subspaces of some complex Hilbert space $\mathcal{H}$. Therefore, their respective collections of atoms $\mathcal{P}$ and $\mathbb{P}\mathcal{H}$ must be isomorphic. Accordingly, we may identify $\mathcal{P}$ and $\mathbb{P}\mathcal{H}$ as sets. Denote the standard transition probabilities (1.16) on $\mathbb{P}\mathcal{H}$ by $p_{\mathcal{H}}$. With p the transition probabilities in $\mathcal{P}$, we will show that $p = p_{\mathcal{H}}$.

Refer to the text following Definition 6. We may embed S_{ref}^2 isometrically in $\mathbb{P}\mathcal{H}$; one then simply has $p = p_{\mathcal{H}}$ on S_{ref}^2. Equation (2.1) now reads

$$p_{\mathcal{H}}(T_{\rho \vee \sigma}(\rho'), T_{\rho \vee \sigma}(\sigma')) = p(\rho', \sigma'); \tag{4.3}$$

in particular, $p_{\mathcal{H}}(T_{\rho \vee \sigma}(\rho'), T_{\rho \vee \sigma}(\sigma')) = 0$ iff $p(\rho', \sigma') = 0$. On the other hand, we know that p and $p_{\mathcal{H}}$ generate isomorphic lattices, which implies that $p_{\mathcal{H}}(\rho', \sigma') = 0$ iff $p(\rho', \sigma') = 0$. Putting this together, we see that $p_{\mathcal{H}}(T_{\rho \vee \sigma}(\rho'), T_{\rho \vee \sigma}(\sigma')) = 0$ iff $p_{\mathcal{H}}(\rho', \sigma') = 0$. A fairly deep generalization of Wigner's theorem (see [54, Thm. 4.29]; here the theorem is stated for infinite-dimensional $\mathcal{H}$, but it is valid in finite dimensions as well, for one can isometrically embed any finite-dimensional Hilbert space in an infinite-dimensional separable Hilbert space) states that a bijection $T : \mathbb{P}\mathcal{H}_1 \to \mathbb{P}\mathcal{H}_2$ (where the $\mathcal{H}_i$ are separable) which merely preserves orthogonality (i.e., $p_{\mathcal{H}_2}(T(\rho'), T(\sigma')) = 0$ iff $p_{\mathcal{H}_1}(\rho', \sigma') = 0$) is induced by a unitary or an antiunitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$. We use this with $\mathcal{H}_1 = \rho \vee \sigma$, $\mathcal{H}_2 = S_{\text{ref}}^2$, and $T = T_{\rho \vee \sigma}$. Since $T_{\rho \vee \sigma}$ is induced by a(n) (anti-) unitary map, which preserves $p_{\mathcal{H}}$, we conclude from (4.3) that $p_{\mathcal{H}}(\rho', \sigma') = p(\rho', \sigma')$. Since ρ and σ (and $\rho', \sigma' \in \rho \vee \sigma$) were arbitrary, the proof of Theorem 1 is finished. $\qquad \square$

4.2. Transition probabilities

Our aim is to show that the transition probabilities defined by (1.4) on the pure state space $\mathcal{P}(\mathfrak{A})$ of the C^*-algebra $\mathfrak{A}_{\mathbb{C}}$ (i.e., $K = \mathcal{S}(\mathfrak{A})$; recall that $\mathfrak{A}_{\mathbb{C}}$ is unital) coincide with those originally defined on $\mathcal{P} = \mathcal{P}(\mathfrak{A}) = \partial_e K$ (cf. Axiom 5); from Proposition 1 we know that these are given by (1.16).

Firstly, $\mathfrak{A}$ as a Banach space (and as an order-unit space) is isomorphic to the space $A(K)$ of continuous affine functions on K, equipped with the sup-norm. The double dual $\mathfrak{A}^{**}$ is isomorphic to $A_b(K)$ (with sup-norm), and the w^*-topology on $A_b(K)$ as the dual of $A(K)^*$ is the topology of pointwise convergence, cf. [12, 6]. Since $A(K)$ is w^*-dense in $A_b(K)$, one may take the infimum in (1.4) over all relevant f in $A(K)$. Since $\mathfrak{A} \subseteq \mathfrak{M} \subseteq \mathfrak{A}^{**}$ (where $\mathfrak{M}$ was defined in Proposition 4), by (3.5) one may certainly take the infimum over $\mathfrak{A}(\mathcal{P})$. But, as we saw in Sec. 3.4, $\mathfrak{A}_{00}(\mathcal{P})$ is dense in $\mathfrak{A}(\mathcal{P})$ when both are seen as subspaces of $\ell^\infty(\mathcal{P})$ with the topology of pointwise convergence. Hence we may take the infimum in (1.4) over all relevant f in $\mathfrak{A}_{00}(\mathcal{P})$.

Let Q be an orthoclosed subspace of $\mathcal{P}$, and recall that p_Q was defined in (3.1). We now show that an equation similar to Eq. (2.19) in [6] holds, viz.

$$p_Q = \inf \left\{ g \in \mathfrak{A}_{00}(\mathcal{P}) \mid 0 \leq g \leq 1, \, g \restriction Q = 1 \right\}. \tag{4.4}$$

For suppose there exists a $0 \leq h < p_Q$ for which the infimum is reached. We must

have $h = 1$ on Q and $h = 0$ on $Q^\perp$, since $p_Q = 0$ on $Q^\perp$. Then the function $p_Q - h$ is ≥ 0, and vanishes on Q and $Q^\perp$. But such functions must vanish identically: let $p_Q - h = \sum_i \lambda_i \, p_{\rho_i}$. Choose a basis $\{e_j\}$ in $Q \cup Q^\perp$. For every point $\rho \in \mathcal{P}$, one must have $\sum_j p(\rho, e_i) = 1$. Hence $\sum_j (p_Q - h)(e_j) = \sum_i \lambda_i = 0$. Suppose that $p_Q - h > 0$. Then there will exist another basis $\{u_j\}$ such that $f(u_j) > 0$ for at least one j. This implies $\sum_i \lambda_i > 0$, which contradicts the previous condition. We conclude that $p_Q = h$, and (4.4) has been proved.

The desired result now follows immediately from (4.4) and the observation that by definition $p_\rho(\sigma) = p(\rho, \sigma)$ for atoms $Q = \rho$. $\qquad\qquad\qquad\square$

We close this section with a technical comment. If $F \subset K \subset \mathfrak{A}^*$ (again with $K = \mathcal{S}\mathfrak{A}$) is a w^*-closed face, then $\partial_e F \subseteq \partial_e K$ may be equipped with transition probabilities defined by (1.4), in which $A_b(K)$ is replaced by $A_b(F)$. These coincide with the transition probabilities inherited from $\partial_e K$. For $F = K \cap H$ for some w^*-closed hyperplane $H \subset \mathfrak{A}^*$ (see, e.g., [3, II.5], [6, Sec. 1]), so that $A_b(F) \simeq H^*$. By Hahn–Banach, each element of H^* can be extended to an element of $\mathfrak{A}^*$, so that any element of $A_b(F)$ extends to some element of $A_b(K)$. The converse is obvious. The claim then follows from the definition (1.4). This shows, in particular, that Axiom AHS2 is equivalent to our Axiom 2.

4.3. *Poisson structure and orientability*

While not necessary for the main argument in this paper, it is enlightening to see that (given the other axioms) the existence of a Poisson structure on $\mathcal{P}$ implies Axiom AHS5, i.e., orientability in the sense of Alfsen et al. [5] (also cf. [50]). We still write K for $\mathcal{S}(\mathfrak{A})$.

These authors define the object $\mathcal{B}(K)$ as the space of all affine isomorphisms from B^3 onto a face of K (which in our setting is the state space of $\mathfrak{A}(\mathcal{P})$ as a JB-algebra), equipped with the topology of pointwise convergence. It follows from Axiom 5 and the argument in [50, p. 499] (or section 3 of [18]) that one can work equally well with the space $\mathcal{B}(\mathcal{P})$ of all injective maps from $S^2 = \mathbb{P}\mathbb{C}^2$ into $\mathcal{P}$ which preserve transition probabilities, topologized by pointwise convergence. If $\varphi, \psi \in \mathcal{B}(\mathcal{P})$ have the same image, then by Axiom 2 and Wigner's theorem the map $\psi^{-1} \circ \varphi : S^2 \to S^2$ lies in $O(3)$ (acting on $S^2 \in \mathbb{R}^3$ in the obvious way). The maps ψ and φ are said to be equivalent if $\psi^{-1} \circ \varphi \in SO(3)$; the space of such equivalence classes is $\mathcal{B}(\mathcal{P})/SO(3)$. The space $\mathcal{P}$ is said to be orientable if the $\mathbb{Z}_2$-bundle $\mathcal{B}(\mathcal{P})/SO(3) \to \mathcal{B}(\mathcal{P})/O(3)$ is globally trivial (cf. [5 Sec. 7]). This notion of orientability is equivalent to the one used in [5], cf. [50].

Given $\varphi \in \mathcal{B}(\mathcal{P})$ and $f \in \mathfrak{A}$, we form $f \circ \varphi : S^2 \to \mathbb{R}$. We infer from the explicit description of $\mathfrak{A}$ in Chapter 3 that $f \circ \varphi$ is smooth. If $f, g \in \mathfrak{A}$ then by (3.7)

$$\{f, g\} \circ \varphi(z) = \mathrm{sgn}(\varphi)\hbar^{-1}(\varphi(z))\{f \circ \varphi, g \circ \varphi\}_{S^2}(z), \qquad\qquad (4.5)$$

where $\{\,,\,\}_{S^2}$ is the Fubini–Study Poisson bracket on S^2, and $\mathrm{sgn}(\varphi)$ is ± 1, depending on the orientation of φ.

Now suppose that K (hence $\mathcal{P}$) were not orientable. Then there exists a continuous family $\{\varphi_t\}_{t\in[0,1]}$ in $\mathcal{B}(\mathcal{P})$, for which φ_0 and φ_1 have the same image, but

opposite orientations (cf. the proof of Lemma 7.1 in [5], also for the idea of the present proof). We replace φ by φ_t in (4.5). Since $\{f, g\}$ is continuous, the left-hand side is continuous in t (pointwise in z). On the right-hand side, $\{f \circ \varphi_t, g \circ \varphi_t\}_{S^2}$ is continuous in t, and so is $\hbar^{-1} \circ \varphi_t$. But $\text{sgn}(\varphi_t)$ must jump from ± 1 to ∓ 1 between 0 and 1, and we arrive at a contradiction.

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