

Poisson structures on moduli spaces of sheaves over Poisson surfaces

Francesco Bottacin

Dip. di Mat. Pura e Appl., Univ. degli Studi di Padova, Via Belzoni 7, I-35131 Padova, Italy

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Summary. We introduce and study the notion of Poisson surface. We prove that the choice of a Poisson structure on a surface S canonically determines a Poisson structure on the moduli space \mathcal{M} of stable sheaves on S . This result generalizes previous results obtained by Mukai [14], for abelian or $K3$ surfaces, and by Tyurin [16].

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Introduction

In [14], Mukai proved that the moduli space \mathcal{M} of sheaves on an abelian or $K3$ surface S has a natural symplectic structure ω . However, in Mukai's paper, a symplectic structure is defined as a nowhere degenerate holomorphic 2-form, hence a natural question arises: can we prove directly that ω is closed?

We may remark that the reason for considering only abelian or $K3$ surfaces is that their canonical bundle is trivial, i.e., they are symplectic surfaces, and it is actually the choice of a symplectic structure on the surface S that induces a symplectic structure on the moduli space \mathcal{M} .

In a later paper [16], Tyurin generalized this result by showing that the choice of a 2-form $\omega \in H^0(S, \wedge^2 T^*S)$, which he calls a 'symplectic structure' on S (resp. of a bivector field $\theta \in H^0(S, \wedge^2 TS)$, which he calls a 'Poisson structure'), determines in a canonical way a 2-form $\tilde{\omega} \in H^0(\mathcal{M}, \wedge^2 T^*\mathcal{M})$ (resp. a bivector field $\tilde{\theta} \in H^0(\mathcal{M}, \wedge^2 T\mathcal{M})$), i.e., a symplectic structure (resp. a Poisson structure) on \mathcal{M} . In this case, again, no mention is made of the closure condition $d\tilde{\omega} = 0$ for the symplectic structure $\tilde{\omega}$, nor of the analogous condition that the Poisson bracket associated to a Poisson structure must satisfy the Jacobi identity.

In this paper we consider the general case of a Poisson surface S and show that, in correspondence to the choice of a Poisson structure on S , there is a canonically defined Poisson structure on the moduli space \mathcal{M} of stable sheaves on S , i.e., there is a bilinear antisymmetric bracket $\{\cdot, \cdot\}$, defined on the sheaf of regular functions on \mathcal{M} , that is a derivation in each entry and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

for any functions f, g, h on \mathcal{M} .

If S is an abelian or a $K3$ surface, i.e., in the symplectic case, our proof shows that the symplectic form defined by Mukai is actually closed.

This paper is organized as follows: in §1 we recall some basic definitions and results of symplectic geometry, then, in §2, we introduce and study Poisson surfaces. In §3 we collect some results on moduli spaces of sheaves on a Poisson surface S , and, in §4, we define the Poisson structure on the moduli space \mathcal{M} canonically associated to the choice of a Poisson structure on S . In §5, we shall prove that θ satisfies a certain closure condition, equivalent to the Jacobi identity for the Poisson bracket defined by θ . Finally, in §6, we conclude with some remarks on the rank of the Poisson structure θ , i.e., on the dimension of the symplectic leaves of the Poisson variety \mathcal{M} .

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1. Symplectic and Poisson structures

We recall here some definitions and results of symplectic geometry.

Let X be a smooth algebraic variety over the complex field \mathbb{C} . A (holomorphic) symplectic structure on X is a closed nondegenerate 2-form $\omega \in H^0(X, \Omega_X^2)$. Given a symplectic structure ω , we define the Hamiltonian vector field H_f of a regular function f by requiring that $\omega(H_f, v) = \langle df, v \rangle$, for every tangent field v . Then, for $f, g \in \Gamma(U, \mathcal{O}_X)$, we define the Poisson bracket $\{f, g\}$ of f and g by setting $\{f, g\} = \langle H_f, dg \rangle = \omega(H_g, H_f)$. The map $g \mapsto \{f, g\}$ is a derivation of $\Gamma(U, \mathcal{O}_X)$ whose corresponding vector field is precisely H_f . The pairing $\{\cdot, \cdot\}$ on \mathcal{O}_X is a bilinear antisymmetric map that is a derivation in each entry and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \quad (1.1)$$

for any $f, g, h \in \Gamma(U, \mathcal{O}_X)$. This implies that $[H_f, H_g] = H_{\{f, g\}}$, where $[u, v] = uv - vu$ is the commutator of the vector fields u and v .

A natural generalization of symplectic structures is given by the notion of Poisson structure.

A Poisson structure on X is a Lie algebra structure $\{\cdot, \cdot\}$ on \mathcal{O}_X satisfying the identity $\{f, gh\} = \{f, g\}h + g\{f, h\}$. Equivalently, this is given by an antisymmetric contravariant 2-tensor $\theta \in H^0(X, \wedge^2 TX)$, where we set $\{f, g\} = \langle \theta, df \wedge dg \rangle$. Then θ is a Poisson structure if the bracket it defines satisfies the Jacobi identity (1.1). For any $f \in \Gamma(U, \mathcal{O}_X)$, the map $g \mapsto \{f, g\}$ is a derivation of $\Gamma(U, \mathcal{O}_X)$,

hence corresponds to a vector field H_f on U , called the hamiltonian vector field associated to f .

Note that giving $\theta \in H^0(X, \wedge^2 TX)$ is equivalent to giving a homomorphism of vector bundles $B : T^*X \rightarrow TX$, with $\langle \theta, \alpha \wedge \beta \rangle = \langle B(\alpha), \beta \rangle$ (or $\langle \alpha, B(\beta) \rangle$, up to a sign), for 1-forms α, β .

Let us define an operator $\tilde{d} : H^0(X, \wedge^2 TX) \rightarrow H^0(X, \wedge^3 TX)$ as follows:

$$\begin{aligned} \tilde{d}\theta(\alpha, \beta, \gamma) &= B(\alpha)\theta(\beta, \gamma) - B(\beta)\theta(\alpha, \gamma) + B(\gamma)\theta(\alpha, \beta) \\ &\quad - \langle [B(\alpha), B(\beta)], \gamma \rangle + \langle [B(\alpha), B(\gamma)], \beta \rangle - \langle [B(\beta), B(\gamma)], \alpha \rangle, \end{aligned}$$

for 1-forms α, β, γ , where $[\cdot, \cdot]$ denotes the usual commutator of vector fields.

By a straightforward computation (using local coordinates), it is easy to prove the following

Proposition 1.1. *The bracket $\{\cdot, \cdot\}$, defined by an element $\theta \in H^0(X, \wedge^2 TX)$, satisfies the Jacobi identity if and only if $\tilde{d}\theta = 0$.*

Remark 1.2. A condition classically known to be equivalent to the Jacobi identity for the bracket $\{\cdot, \cdot\}$ is the vanishing of the so-called Schouten bracket $[\theta, \theta] = 0$ (see [15], for example). This is equivalent to the condition expressed by the preceding proposition.

When θ has maximal rank everywhere, to give θ is equivalent to giving its inverse 2-form $\omega \in H^0(X, \Omega_X^2)$, and the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$. In this case the Poisson structure induces a symplectic structure.

2. Poisson surfaces

In this section we consider Poisson structures on smooth algebraic surfaces. Let S be a smooth algebraic surface over the complex field \mathbb{C} . We shall denote by ω_S its canonical line bundle, by K_S its canonical divisor, and by $q = \dim H^1(S, \mathcal{O}_S)$ the irregularity of S . We have the following

Proposition 2.1. *A Poisson structure on S is given by a global section s of the anticanonical line bundle ω_S^{-1} .*

Proof. A Poisson structure on S is, by definition, an element $s \in H^0(S, \wedge^2 TS) = H^0(S, \omega_S^{-1})$ that satisfies the condition $\tilde{d}s = 0$. But S is a surface, hence the map \tilde{d} is identically zero. \square

Definition 2.2. *A Poisson surface S is a smooth algebraic surface which admits a non-zero Poisson structure, i.e., such that $H^0(S, \omega_S^{-1}) \neq 0$.*

We have the following

Proposition 2.3. *Let S be a connected Poisson surface. Then S is either a K3 surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 0$) or an abelian surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 2$) or a ruled surface (if ω_S is not trivial).*

Proof. Let s be a non-zero section of ω_S^{-1} ; we have an exact sequence

$$0 \rightarrow \omega_S \xrightarrow{s} \mathcal{O}_S \rightarrow \mathcal{O}_D \rightarrow 0,$$

where D is the divisor of s . It follows that either $H^0(S, \omega_S) = 0$ or $H^0(S, \omega_S) = \mathbb{C}$. In the second case ω_S has a non-vanishing global section, hence it is trivial; S is then either a $K3$ or an abelian surface, according to the value of q . If $H^0(S, \omega_S) = 0$, by considering the exact sequence

$$0 \rightarrow \omega_S^{n+1} \xrightarrow{s} \omega_S^n \rightarrow \omega_S^n|_D \rightarrow 0,$$

and using induction on n , it follows that $H^0(S, \omega_S^n) = 0$ for all $n \geq 1$. By recalling now a theorem of Enriques (cf., for example, [3, Chap. VI]), we conclude that S is a ruled surface. \square

Remark 2.4. Note that not every ruled surface is a Poisson surface. Let, for example, $S = C \times \mathbb{P}^1$; then we have, for every integer n ,

$$H^0(S, \omega_S^n) = H^0(C, \omega_C^n) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n)).$$

It follows that $H^0(S, \omega_S^{-1}) \neq 0$ if and only if C is a rational or an elliptic curve.

Remark 2.5. If S is a Poisson surface and ω_S^{-1} is ample, then it follows that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ or S is obtained from \mathbb{P}^2 by blowing-up n distinct points in general position, with $n \leq 8$. It follows, in particular, that S is rational ([3, Ch. V, Ex. 1]).

Remark 2.6. Let $s \in H^0(S, \omega_S^{-1})$ be a Poisson structure on S , and denote by D the divisor of s . Then the rank of the Poisson structure is 2 on the open subset $S \setminus D$, and is 0 on D . Hence the Poisson structure induces a symplectic structure on $S \setminus D$.

Example 2.7. Let $S = \mathbb{P}^2$ and take as the anticanonical divisor D a triple line. We may fix homogeneous coordinates (x_0, x_1, x_2) on \mathbb{P}^2 such that the section s defining the divisor D is given by $s = x_0^3$. The Poisson structure defined by s on \mathbb{P}^2 induces a symplectic structure on $\mathbb{C}^2 = \mathbb{P}^2 \setminus D$. If we consider the coordinates (X, Y) on \mathbb{C}^2 given by $X = x_1/x_0$ and $Y = x_2/x_0$, it is immediate to see that this symplectic structure is given by the 2-form $dX \wedge dY$. In other words, the Poisson structure determined by s on \mathbb{P}^2 induces on \mathbb{C}^2 the standard symplectic structure.

3. The moduli space of semistable sheaves on S

Let S be a connected Poisson surface, and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on S . Let us denote by D the divisor of s , $D \in |-K_S|$.

We recall now some basic definitions and results on moduli spaces of semistable sheaves on surfaces.

Let H be a very ample divisor on S . For every coherent torsion-free \mathcal{O}_S -module E , we set

$$p_E(n) = \frac{\chi(E(n))}{\text{rk}(E)},$$

where $\text{rk}(E)$ is the rank at the generic point of S , and $E(n) = E \otimes \mathcal{O}_S(nH)$.

From now on, by a sheaf on S we will always mean an \mathcal{O}_S -module.

Definition 3.1. *A sheaf E on S is said to be H -stable (resp. H -semistable) if it is a coherent torsion-free \mathcal{O}_S -module and, for every proper coherent subsheaf F of E , we have*

$$p_F(n) < p_E(n), \quad (\text{resp. } p_F(n) \leq p_E(n)),$$

for all sufficiently large integers n .

Let us denote by r , c_1 and c_2 , respectively the rank and the Chern classes of a coherent torsion-free sheaf on S . We have the following well-known result:

Theorem 3.2. *For fixed r, c_1, c_2 , there exists a coarse moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}(r, c_1, c_2)$ parametrizing S -equivalence classes of H -semistable sheaves of rank r and Chern classes c_1 and c_2 on S . $\overline{\mathcal{M}}$ is a projective variety and it contains an open subset $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ parametrizing isomorphism classes of H -stable sheaves.*

In the sequel we shall denote by E either an H -semistable (resp. H -stable) sheaf on S , or the point of $\overline{\mathcal{M}}$ (resp. \mathcal{M}) corresponding to the S -equivalence class (resp. isomorphism class) of E .

From infinitesimal deformation theory, it follows that there is a canonical isomorphism

$$T_E \overline{\mathcal{M}} \cong \text{Ext}^1(E, E), \quad (3.1)$$

where $T_E \overline{\mathcal{M}}$ denotes the tangent space to $\overline{\mathcal{M}}$ at E .

Then, from Grothendieck-Serre duality, it follows that the cotangent space to $\overline{\mathcal{M}}$ at E is given by

$$T_E^* \overline{\mathcal{M}} \cong \text{Ext}^1(E, E \otimes \omega_S). \quad (3.2)$$

We now turn to the problem of smoothness of moduli spaces. We have the following result:

Proposition 3.3. *The moduli space $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ is a smooth quasi-projective variety of dimension $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$.*

Proof. If S is a symplectic surface ($\omega_S \cong \mathcal{O}_S$), this is proved in [14, Theorem 0.1]. If S is a Poisson surface and ω_S is not trivial, then the divisor $D \in |-K_S|$ of a Poisson structure s is effective. It follows that $(D \cdot H) > 0$, hence $(K_S \cdot H) < 0$. Under this condition, the smoothness of \mathcal{M} is proved in [13, Corollary 6.7.3]. The computation of the dimension of \mathcal{M} is an easy application of the Riemann-Roch theorem, and is done in [13, Proposition 6.9]. \square

Since \mathcal{M} is a smooth variety, we may give global versions of (3.1) and (3.2).

First we need a definition:

Definition 3.4. *Let $f : X \rightarrow T$ be a T -scheme, and E, F two coherent \mathcal{O}_X -modules. The i -th relative Ext-sheaf $\mathcal{E}xt_{\mathcal{O}_T}^i(E, F)$ is the sheaf associated to the presheaf $U \mapsto \text{Ext}_{f^{-1}(U)}^i(E_U, F_U)$ for every open subset U of T .*

Now we note that, in general, a universal family \mathcal{E} on \mathcal{M} does not exist (not even on any Zariski open subset of \mathcal{M}); however it does exist locally in the étale

topology (or in the complex topology). As shown by Mukai, this is enough to ensure that the i -th relative Ext-sheaves $\mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^i(\mathcal{E}, \mathcal{E})$ on \mathcal{M} are well defined, for any integer i .

Then we have:

Proposition 3.5. *Let $p : \mathcal{M} \times S \rightarrow \mathcal{M}$ and $q : \mathcal{M} \times S \rightarrow S$ be the canonical projections. There are canonical isomorphisms*

$$T\mathcal{M} \cong \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^1(\mathcal{E}, \mathcal{E}), \quad (3.3)$$

and

$$T^*\mathcal{M} \cong \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^1(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)). \quad (3.4)$$

Let us denote by \mathcal{M}^0 the open subset of \mathcal{M} parametrizing isomorphism classes of H -stable locally free sheaves on S . Then, for $E \in \mathcal{M}^0$, we have

$$T_E\mathcal{M}^0 \cong H^1(S, \mathcal{E}nd(E)), \quad (3.5)$$

and

$$T_E^*\mathcal{M}^0 \cong H^1(S, \mathcal{E}nd(E) \otimes \omega_S). \quad (3.6)$$

Now, even if a universal family \mathcal{E} on \mathcal{M} does not exist, the sheaf $\mathcal{E}nd(\mathcal{E})$ on $\mathcal{M} \times S$ is well defined, and we have

$$\mathcal{E}nd(\mathcal{E})|_{\{E\} \times S} \cong \mathcal{E}nd(E), \quad \forall E \in \mathcal{M}.$$

Hence we may rewrite (3.3) and (3.4) for \mathcal{M}^0 as follows:

$$T\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E})), \quad (3.7)$$

and

$$T^*\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)). \quad (3.8)$$

This may also be proved directly by noting that, since \mathcal{M}^0 is smooth, the function $E \mapsto \dim H^1(S, \mathcal{E}nd(E))$ is constant on \mathcal{M}^0 . It follows that $R^1p_*(\mathcal{E}nd(\mathcal{E}))$ is locally free and we have isomorphisms $R^1p_*(\mathcal{E}nd(\mathcal{E})) \otimes \mathbb{C}(E) \cong H^1(S, \mathcal{E}nd(E))$, for every $E \in \mathcal{M}^0$. A similar argument works for the cotangent bundle.

4. Poisson structures on \mathcal{M}

Let S be a Poisson surface and choose a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on S . We shall define an element $\theta = \theta_s \in H^0(\mathcal{M}, \otimes^2 T\mathcal{M})$ as follows: for any $E \in \mathcal{M}$, $\theta(E) : T_E^*\mathcal{M} \times T_E^*\mathcal{M} \rightarrow \mathbb{C}$ is defined by

$$\begin{aligned} \theta(E) : \text{Ext}^1(E, E \otimes \omega_S) \times \text{Ext}^1(E, E \otimes \omega_S) &\xrightarrow{\circ} \\ \text{Ext}^2(E, E \otimes \omega_S^2) &\xrightarrow{s} \text{Ext}^2(E, E \otimes \omega_S) \xrightarrow{\text{Tr}} \mathbb{C}, \end{aligned} \quad (4.1)$$

where the first map is the composition map, the second is induced by the multiplication by the section s , and the third is the trace map.

Note that, by Grothendieck-Serre duality and the stability hypothesis on E , it follows that the trace map $\text{Tr} : \text{Ext}^2(E, E \otimes \omega_S) \rightarrow \mathbb{C}$ is an isomorphism.

A method analogous to the one used by Mukai in [14] may be used to prove the following

Proposition 4.1. *The sheaf $\mathcal{L} = \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^2(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S))$ is a trivial invertible sheaf on \mathcal{M} , and there is a bilinear map*

$\theta : \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^1(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \otimes \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^1(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \xrightarrow{\circ} \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^2(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S^2)) \xrightarrow{s} \mathcal{L}$
such that, for every $E \in \mathcal{M}$, $\theta \otimes \mathbb{C}(E)$ coincides with the map $\theta(E)$ defined in (4.1).

As we have previously stated, giving θ is equivalent to giving a homomorphism of vector bundles

$$B : T^*\mathcal{M} \rightarrow T\mathcal{M},$$

where we set $\theta(\alpha \otimes \beta) = \langle B(\alpha), \beta \rangle$.

It is easy to see that in this situation the homomorphism B is the map induced on Ext-sheaves by the multiplication by the section s . On the fibers over a point $E \in \mathcal{M}$, we have

$$B(E) : \text{Ext}^1(E, E \otimes \omega_S) \xrightarrow{s} \text{Ext}^1(E, E). \quad (4.2)$$

From now on we shall restrict ourselves to the open subset \mathcal{M}^0 of \mathcal{M} , and we shall use indifferently the expressions ‘locally-free sheaf’ or ‘vector bundle’.

If E is an H -stable locally-free sheaf, the map $\theta(E)$ may be written as

$$\begin{aligned} \theta(E) : H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \times H^1(S, \mathcal{E}nd(E) \otimes \omega_S) &\xrightarrow{\circ} \\ H^2(S, \mathcal{E}nd(E) \otimes \omega_S^2) &\xrightarrow{s} H^2(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\text{Tr}} \mathbb{C}. \end{aligned} \quad (4.3)$$

This is essentially the cup-product of two cohomology classes, followed by the multiplication by s . From the graded commutativity of the usual cup-product it follows that $\theta(E)$ is skew-symmetric (cf., for example, [10, p. 707]), hence to prove that θ defines a Poisson structure on \mathcal{M}^0 we have only to prove that it satisfies the closure condition $\tilde{d}\theta = 0$. Note that if we prove that θ defines a Poisson structure on the open subset \mathcal{M}^0 , then the same holds on the closure $\overline{\mathcal{M}^0}$.

As a final remark note that, for $E \in \mathcal{M}^0$, the global map θ of Proposition 4.1 may be written as

$$\begin{aligned} \theta : R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) \otimes R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) &\xrightarrow{\circ} \\ R^2p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S^2)) &\xrightarrow{s} \mathcal{L}, \end{aligned}$$

where the trivial invertible sheaf \mathcal{L} is given by $\mathcal{L} = R^2p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S))$, while the map B is the map induced on cohomology by the multiplication by the section s :

$$B(E) : H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\text{id} \otimes s} H^1(S, \mathcal{E}nd(E)). \quad (4.4)$$

5. The closure of θ

In this section we shall prove that $\tilde{d}\theta = 0$, thus proving that θ defines a Poisson structure on \mathcal{M}^0 . We note that this proof is original even in the symplectic case.

We start by recalling some preliminaries (see [5] for a detailed description of what follows).

Let $\mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers over \mathbb{C} . By convenience of notations, in the sequel it will be denoted simply by $\mathbb{C}[\epsilon]$. Let us denote by S_ϵ the fiber product $S \times \text{Spec}(\mathbb{C}[\epsilon])$. If $p_\epsilon : S_\epsilon \rightarrow S$ is the natural morphism and F is a vector bundle on S , we shall denote by $F[\epsilon]$ its trivial infinitesimal deformation, $F[\epsilon] = p_\epsilon^*(F)$.

Let $\pi : X \rightarrow Y$ be a morphism (locally of finite presentation) of schemes, and F, G two locally free sheaves on X . We denote by $\mathcal{D}iff_{X/Y}^1(F, G)$ the sheaf of relative differential operators from F to G of order ≤ 1 . There is an exact sequence ([11, Ch. IV, §16.8])

$$0 \rightarrow \mathcal{H}om_X(F, G) \rightarrow \mathcal{D}iff_{X/Y}^1(F, G) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \otimes \mathcal{H}om_X(F, G) \rightarrow 0,$$

where σ is the *symbol* morphism. Then, if $F = G$ and we restrict to differential operators with ‘scalar symbol’, written $\mathcal{D}_{X/Y}^1(F)$, we get the exact sequence

$$0 \rightarrow \mathcal{E}nd_X(F) \rightarrow \mathcal{D}_{X/Y}^1(F) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \rightarrow 0. \quad (5.1)$$

Let $p : \mathcal{M}^0 \times S \rightarrow \mathcal{M}^0$ and $q : \mathcal{M}^0 \times S \rightarrow S$ denote the canonical projections. To apply (5.1) to $q : \mathcal{M}^0 \times S \rightarrow S$, note that, as previously said, even if there is no Poincaré bundle \mathcal{E} on $\mathcal{M}^0 \times S$, the sheaf $\mathcal{E}nd(\mathcal{E})$ is well defined. By a similar argument it follows easily that the sheaf $\mathcal{D}_S^1(\mathcal{E}) = \mathcal{D}_{\mathcal{M}^0 \times S/S}^1(\mathcal{E})$ of first-order differential operators with scalar symbols that are $q^*(\mathcal{O}_S)$ -linear, is also well defined. Then we have an exact sequence

$$0 \rightarrow \mathcal{E}nd(\mathcal{E}) \rightarrow \mathcal{D}_S^1(\mathcal{E}) \rightarrow p^*T\mathcal{M}^0 \rightarrow 0.$$

By applying p_* , and noting that $p_*p^*T\mathcal{M}^0 \cong T\mathcal{M}^0$ since p is a proper morphism, we get a long exact cohomology sequence, a piece of which is

$$\dots \rightarrow T\mathcal{M}^0 \rightarrow R^1p_*(\mathcal{E}nd(\mathcal{E})) \rightarrow R^1p_*(\mathcal{D}_S^1(\mathcal{E})) \rightarrow \dots$$

It is easy to prove that the map $T\mathcal{M}^0 \rightarrow R^1p_*(\mathcal{E}nd(\mathcal{E}))$ is precisely the isomorphism (3.7), hence the image of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$ in $R^1p_*(\mathcal{D}_S^1(\mathcal{E}))$ is zero. It follows that, for every section $\{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$, there exist sections \dot{D}_i of $\mathcal{D}_S^1(\mathcal{E})$ over suitable open subsets V_i , such that

$$\eta_{ij} = \dot{D}_j - \dot{D}_i, \quad (5.2)$$

where, by simplicity of notations, we have not explicitly indicated the restrictions to the intersection $V_{ij} = V_i \cap V_j$.

If $E \in \mathcal{M}^0$, and we consider restrictions to $\{E\} \times S$, it follows that (5.2) is valid with $\{\eta_{ij}\}$ being a 1-cocycle with values in $\mathcal{E}nd(E)$, and \dot{D}_i being sections of $\mathcal{D}_S^1(E)$.

Let now X be a k -scheme. A tangent vector field on X is a k -linear map of sheaves $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that the induced map $D(U) : \Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$ is a k -derivation, for every open subset U of X . Equivalently, a vector field on X can be expressed by an automorphism over $\text{Spec } k[\epsilon]$

$$\begin{array}{ccc}
 X \times \mathrm{Spec}(k[\epsilon]) & \xrightarrow{\tilde{D}} & X \times \mathrm{Spec}(k[\epsilon]) \\
 & \searrow & \swarrow \\
 & \mathrm{Spec}(k[\epsilon]) &
 \end{array}$$

that restricts to the identity morphism of X when one looks at the fibers over $\mathrm{Spec} k$.

Over an open affine subset $U = \mathrm{Spec} A$ of X the tangent field $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is given equivalently by a k -derivation $D(U) : A \rightarrow A$. In this situation the automorphism \tilde{D} is determined by the k -algebra homomorphism $\tilde{D}(U) : A[\epsilon] \rightarrow A[\epsilon]$ given by $\tilde{D}(U) = 1 + \epsilon D(U)$.

Let now $D : \mathcal{O}_{\mathcal{M}^0} \rightarrow \mathcal{O}_{\mathcal{M}^0}$ be a tangent vector field on \mathcal{M}^0 and denote by \tilde{D} the corresponding automorphism of $\mathcal{M}^0 \times \mathrm{Spec} \mathbb{C}[\epsilon]$. Let \mathcal{E} be a local universal family for stable vector bundles (the local existence of \mathcal{E} in the étale topology is enough for our purposes), and $\mathcal{E}[\epsilon]$ be its pull-back to $\mathcal{M}^0 \times \mathrm{Spec} \mathbb{C}[\epsilon]$. The vector field D (or the automorphism \tilde{D}) may be described locally by giving the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathcal{E}[\epsilon]$ of the local universal family \mathcal{E} . At a point $\mathcal{E} \in \mathcal{M}^0$ the corresponding tangent vector is given by $E_\epsilon = \mathcal{E}_\epsilon|_{\{\mathcal{E}\} \times S_\epsilon}$, which is an infinitesimal deformation of the vector bundle E .

From what we have previously seen, the tangent field \mathcal{E}_ϵ corresponds to a global section $\eta = \{\eta_{ij}\}$ of $R^1 p_*(\mathcal{E}nd(\mathcal{E}))$, which can be described in terms of first-order differential operators. Let us give another useful interpretation of this fact.

Let $D : \mathcal{O}_{\mathcal{M}^0} \rightarrow \mathcal{O}_{\mathcal{M}^0}$ be the derivation corresponding to the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathcal{E}$ determined by the global section $\eta = \{\eta_{ij}\}$ of $R^1 p_*(\mathcal{E}nd(E))$. Let $(V_i)_{i \in I}$, $V_i = \mathrm{Spec}(A_i)$, be an open affine covering of $\mathcal{M}^0 \times S$. The vector field D is locally described by giving, for each $i \in I$, a $\mathbb{C}[\epsilon]$ -automorphism of $A_i[\epsilon]$ of the form $1 + \epsilon D_i$, where $D_i : A_i \rightarrow A_i$ is the \mathbb{C} -derivation determined by the restriction of D to V_i . Let $M_i = \Gamma(V_i, \mathcal{E})$ and $M_i[\epsilon] = \Gamma(V_i, \mathcal{E}[\epsilon])$. The infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathcal{E}[\epsilon]$ may be described as obtained by gluing the sheaves $\widetilde{M_i[\epsilon]}$ by means of suitable isomorphisms.

Let us denote by

$$1 + \epsilon \widetilde{\dot{D}_i} : \mathcal{E}_\epsilon|_{V_i \times \mathrm{Spec} \mathbb{C}[\epsilon]} \xrightarrow{\sim} \widetilde{M_i[\epsilon]}$$

the trivialization isomorphisms, where $\dot{D}_i : M_i \rightarrow M_i$ is a first-order differential operator with associated \mathbb{C} -derivation $D_i : A_i \rightarrow A_i$. By what we have previously seen, the gluing isomorphism on the intersection $V_i \cap V_j$ is given by $1 + \epsilon \eta_{ij} = (1 + \epsilon \dot{D}_j)(1 + \epsilon \dot{D}_i)^{-1} = 1 + \epsilon(\dot{D}_j - \dot{D}_i)$. It follows that $\eta_{ij} = \dot{D}_j - \dot{D}_i$, which is precisely (5.2).

Now let us consider two tangent vector fields $D^1, D^2 : \mathcal{O}_{\mathcal{M}^0} \rightarrow \mathcal{O}_{\mathcal{M}^0}$, corresponding to the global sections $\eta^1 = \{\eta_{ij}^1\}$ and $\eta^2 = \{\eta_{ij}^2\}$ of $R^1 p_*(\mathcal{E}nd(E))$. By what we have seen, there exist first-order differential operators \dot{D}_i^1 and \dot{D}_i^2 such that

$$\begin{aligned}
 \eta_{ij}^1 &= \dot{D}_j^1 - \dot{D}_i^1, \\
 \eta_{ij}^2 &= \dot{D}_j^2 - \dot{D}_i^2.
 \end{aligned}$$

Let us denote by

$$\begin{array}{ccc} \mathcal{M}^0 \times \text{Spec } \mathbb{C}[\epsilon, \epsilon'] \times S & \xrightarrow{\tilde{D}^h \times \text{id}_S} & \mathcal{M}^0 \times \text{Spec } \mathbb{C}[\epsilon, \epsilon'] \times S \\ & \searrow & \swarrow \\ & \text{Spec } \mathbb{C}[\epsilon, \epsilon'] & \end{array}$$

the automorphism corresponding to D^h , for $h = 1, 2$. The vector field D^h is given equivalently by the infinitesimal deformation $\mathcal{E}_\epsilon^h = (\tilde{D}^h \times \text{id}_S)^* \mathcal{E}[\epsilon]$ of \mathcal{E} , described by the global section $\{\eta_{ij}^h\}$ of $R^1 p_*(\mathcal{E}nd(E))$. Let us recall that if $f_i^h : \mathcal{E}_\epsilon^h|_{V_i} \rightarrow \widetilde{M_i^h[\epsilon]}$ are isomorphisms, then the sheaf \mathcal{E}_ϵ^h is constructed by gluing the sheaves $\widetilde{M_i^h[\epsilon]}$ and $\widetilde{M_j^h[\epsilon]}$ along the open sets V_{ij} by means of the isomorphisms $1 + \epsilon \eta_{ij}^h = f_j^h|_{V_{ij}} \circ f_i^h|_{V_{ij}}^{-1}$.

By applying the same reasoning, the second-order differential operator $D^1 D^2$ is equivalent to $(\tilde{D}^1 \times \text{id}_S)^*(\tilde{D}^2 \times \text{id}_S)^* \mathcal{E}$. We have isomorphisms

$$(\tilde{D}^1 \times \text{id}_S)^*(\tilde{D}^2 \times \text{id}_S)^* \mathcal{E}|_{V_i \times \text{Spec } \mathbb{C}[\epsilon, \epsilon']} \xrightarrow{(1 + \epsilon \dot{D}_i^1) \circ (1 + \epsilon' \dot{D}_i^2)} \widetilde{M_i[\epsilon, \epsilon']},$$

hence the gluing isomorphisms are given by $((1 + \epsilon \dot{D}_j^1) \circ (1 + \epsilon' \dot{D}_j^2)) \circ ((1 + \epsilon \dot{D}_i^1) \circ (1 + \epsilon' \dot{D}_i^2))^{-1} = 1 + \epsilon(\dot{D}_j^1 - \dot{D}_i^1) + \epsilon'(\dot{D}_j^2 - \dot{D}_i^2) + \epsilon\epsilon'(\dot{D}_j^1 \dot{D}_j^2 - \dot{D}_j^2 \dot{D}_i^1 - \dot{D}_j^1 \dot{D}_i^2 + \dot{D}_i^2 \dot{D}_i^1)$, that we can write in a simpler form as $1 + \epsilon \eta_{ij}^1 + \epsilon' \eta_{ij}^2 + \epsilon\epsilon'(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1)$.

In conclusion, we have proved that the second-order differential operator $D^1 D^2$, or, in other words the ‘infinitesimal deformation’ of $\eta^2 = \{\eta_{ij}^2\}$ in the direction given by $\eta^1 = \{\eta_{ij}^1\}$, is described by giving gluing isomorphisms of the form

$$1 + \epsilon \eta_{ij}^1 + \epsilon' \eta_{ij}^2 + \epsilon\epsilon'(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1). \quad (5.3)$$

Analogously, we find that $D^2 D^1$ is equivalent to the data of

$$1 + \epsilon \eta_{ij}^1 + \epsilon' \eta_{ij}^2 + \epsilon\epsilon'(\dot{D}_j^2 \eta_{ij}^1 - \eta_{ij}^1 \dot{D}_i^2). \quad (5.4)$$

Finally, it is easy to see that the vector field $[D^1, D^2]$ is determined by gluing isomorphisms of the form

$$1 + \epsilon\epsilon'([\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i).$$

Now we are able to prove the following

Theorem 5.1. *Let S be a Poisson surface and $s \in H^0(S, \omega_S^{-1})$ a Poisson structure on S . The antisymmetric contravariant 2-tensor $\theta = \theta_s \in H^0(\mathcal{M}^0, \wedge^2 T\mathcal{M}^0)$ defines a Poisson structure on the moduli space \mathcal{M}^0 of H -stable vector bundles on S .*

Proof. We have to prove that $\tilde{d}\theta = 0$. Let η^1, η^2, η^3 be three 1-forms on \mathcal{M}^0 , i.e., three global sections of $R^1 p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S))$. To compute $(B(\eta^1))(\theta(\eta^2, \eta^3)) = (B(\eta^1))(\langle s\eta^2, \eta^3 \rangle)$, i.e., the derivative of the function $\langle s\eta^2, \eta^3 \rangle$ along the vector field $B(\eta^1)$, we shall use first-order Taylor series expansions of η^2 and η^3 , i.e., we shall compute $\langle s\eta_\epsilon^2, \eta_\epsilon^3 \rangle$, where η_ϵ^2 and η_ϵ^3 are ‘infinitesimal deformations’ along the vector field $B(\eta^1)$ of η^2 and η^3 respectively.

Let us represent the cohomology classes η^h by 1-cocycles $\eta^h = \{\eta_{ij}^h\}$, for $h = 1, 2, 3$, so that $B(\eta^h)$ is represented by the 1-cocycle $\{s\eta_{ij}^h\}$. We know that there exist first-order differential operators \dot{D}_i^h such that

$$s\eta_{ij}^h = \dot{D}_j^h - \dot{D}_i^h, \quad \text{for } h = 1, 2, 3. \quad (5.5)$$

If we apply what we have previously seen to the second-order differential operator $B(\eta^h)B(\eta^k)$, we find that the infinitesimal deformation of $B(\eta^k) = s\eta^k$ along the vector field $B(\eta^h)$ is given by

$$s\eta_\epsilon^k = \{s\eta_{ij}^k + \epsilon(\dot{D}_j^h(s\eta_{ij}^k) - s\eta_{ij}^k \dot{D}_i^h)\} = \{s\eta_{ij}^k + s\epsilon(\dot{D}_j^h \eta_{ij}^k - \eta_{ij}^k \dot{D}_i^h)\},$$

because of the \mathcal{O}_S -linearity of the differential operators.

Since the multiplication by s is injective at the level of cocycles, it follows that

$$\eta_\epsilon^k = \{\eta_{ij}^k + \epsilon(\dot{D}_j^h \eta_{ij}^k - \eta_{ij}^k \dot{D}_i^h)\}, \quad (5.6)$$

for $h, k = 1, 2, 3$.

Then we have:

$$\begin{aligned} \langle s\eta_\epsilon^2, \eta_\epsilon^3 \rangle &= \langle \{s\eta_{ij}^2 + s\epsilon(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1)\}, \{\eta_{ij}^3 + \epsilon(\dot{D}_j^1 \eta_{ij}^3 - \eta_{ij}^3 \dot{D}_i^1)\} \rangle \\ &= \{s\eta_{ij}^2 \circ \eta_{jk}^3\} + \epsilon \{ (s\eta_{ij}^2 (\dot{D}_k^1 \eta_{jk}^3 - \eta_{jk}^3 \dot{D}_j^1) + s(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1) \eta_{jk}^3) \}, \end{aligned}$$

from which it follows that

$$(B(\eta^1))(\theta(\eta^2, \eta^3)) = \{s\eta_{ij}^2 (\dot{D}_k^1 \eta_{jk}^3 - \eta_{jk}^3 \dot{D}_j^1) + s(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1) \eta_{jk}^3\}. \quad (5.7)$$

Now, by using the decomposition (5.5), and by recalling that the antisymmetry of θ implies that $\langle s\eta^h, \eta^l \rangle = \{s\eta_{ij}^h \circ \eta_{jk}^l\} = -\{\eta_{ij}^h \circ s\eta_{jk}^l\} = -\langle \eta^h, s\eta^l \rangle$, we may decompose the various terms of (5.7) in different ways as follows:

$$\begin{aligned} s\eta_{ij}^2 (\dot{D}_k^1 \eta_{jk}^3 - \eta_{jk}^3 \dot{D}_j^1) &= \dot{D}_j^2 \dot{D}_k^1 \eta_{jk}^3 - \dot{D}_j^2 \eta_{jk}^3 \dot{D}_j^1 - \dot{D}_i^2 \dot{D}_k^1 \eta_{jk}^3 + \dot{D}_i^2 \eta_{jk}^3 \dot{D}_j^1 \\ &= -\eta_{ij}^2 \dot{D}_k^1 \dot{D}_k^3 + \eta_{ij}^2 \dot{D}_k^1 \dot{D}_j^3 + \eta_{ij}^2 \dot{D}_k^3 \dot{D}_j^1 - \eta_{ij}^2 \dot{D}_j^3 \dot{D}_j^1, \end{aligned}$$

and

$$\begin{aligned} s(\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1) \eta_{jk}^3 &= \dot{D}_j^1 \dot{D}_j^2 \eta_{jk}^3 - \dot{D}_j^1 \dot{D}_i^2 \eta_{jk}^3 - \dot{D}_j^2 \dot{D}_i^1 \eta_{jk}^3 + \dot{D}_i^2 \dot{D}_i^1 \eta_{jk}^3 \\ &= -\dot{D}_j^1 \eta_{ij}^2 \dot{D}_k^3 + \eta_{ij}^2 \dot{D}_i^1 \dot{D}_k^3 + \dot{D}_j^1 \eta_{ij}^2 \dot{D}_j^3 - \eta_{ij}^2 \dot{D}_i^1 \dot{D}_j^3. \end{aligned}$$

The last expression we have to compute is the following:

$$\begin{aligned} \langle [B_s(\eta^1), B_s(\eta^2)], \eta^3 \rangle &= \{ \langle ([\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i), \eta^3 \rangle \} \\ &= \{ (\dot{D}_j^1 \dot{D}_j^2 - \dot{D}_j^2 \dot{D}_j^1 - \dot{D}_i^1 \dot{D}_i^2 + \dot{D}_i^2 \dot{D}_i^1) \eta_{jk}^3 \} \\ &= \{ \dot{D}_j^1 \dot{D}_j^2 \eta_{jk}^3 - \dot{D}_j^2 \dot{D}_j^1 \eta_{jk}^3 - \dot{D}_i^1 \dot{D}_i^2 \eta_{jk}^3 + \dot{D}_i^2 \dot{D}_i^1 \eta_{jk}^3 \}. \end{aligned}$$

As for the last term, we may use the following decomposition:

$$\begin{aligned} \langle [B_s(\eta^2), B_s(\eta^3)], \eta^1 \rangle &= \langle \eta^1, [B_s(\eta^2), B_s(\eta^3)] \rangle \\ &= \{ \eta_{ij}^1 ([\dot{D}^2, \dot{D}^3]_k - [\dot{D}^2, \dot{D}^3]_j) \} \\ &= \{ \eta_{ij}^1 \dot{D}_k^2 \dot{D}_k^3 - \eta_{ij}^1 \dot{D}_k^3 \dot{D}_k^2 - \eta_{ij}^1 \dot{D}_j^2 \dot{D}_j^3 + \eta_{ij}^1 \dot{D}_j^3 \dot{D}_j^2 \}. \end{aligned}$$

Now, by collecting all the pieces, we find

$$\begin{aligned}
\tilde{d}\theta(\eta^1, \eta^2, \eta^3) &= B_s(\eta^1)(\theta(\eta^2, \eta^3)) - B_s(\eta^2)(\theta(\eta^1, \eta^3)) + B_s(\eta^3)(\theta(\eta^1, \eta^2)) \\
&\quad - \langle [B_s(\eta^1), B_s(\eta^2)], \eta^3 \rangle + \langle [B_s(\eta^1), B_s(\eta^3)], \eta^2 \rangle \\
&\quad - \langle [B_s(\eta^2), B_s(\eta^3)], \eta^1 \rangle \\
&= \{s\eta_{ij}^2(\dot{D}_k^1\eta_{jk}^3 - \eta_{jk}^3\dot{D}_j^1) - \dot{D}_j^3\dot{D}_i^1\eta_{jk}^2 - \dot{D}_j^1\dot{D}_i^3\eta_{jk}^2 \\
&\quad + \dot{D}_j^1\dot{D}_j^3\eta_{jk}^2 + \dot{D}_i^3\dot{D}_i^1\eta_{jk}^2\} \\
&= \{s\eta_{ij}^2(\dot{D}_k^1\eta_{jk}^3 - \eta_{jk}^3\dot{D}_j^1) \\
&\quad + (\dot{D}_j^1\dot{D}_j^3 - \dot{D}_j^1\dot{D}_i^3 - \dot{D}_j^3\dot{D}_i^1 + \dot{D}_i^3\dot{D}_i^1)\eta_{jk}^2\} \\
&= \{s\eta_{ij}^2(\dot{D}_k^1\eta_{jk}^3 - \eta_{jk}^3\dot{D}_j^1) + (\dot{D}_j^1(s\eta_{ij}^3) - (s\eta_{ij}^3)\dot{D}_i^1)\eta_{jk}^2\} \\
&= \{s\eta_{ij}^2(\dot{D}_k^1\eta_{jk}^3 - \eta_{jk}^3\dot{D}_j^1) + s(\dot{D}_j^1\eta_{ij}^3 - \eta_{ij}^3\dot{D}_i^1)\eta_{jk}^2\} \\
&= \{s\eta_{ij}^2 \circ \xi_{jk}^3 + s\xi_{ij}^3 \circ \eta_{jk}^2\} \\
&= 0,
\end{aligned}$$

where, for the infinitesimal deformation η_ϵ^3 of η^3 along $B_s(\eta^1)$, we have set $\eta_{\epsilon ij}^3 = \eta_{ij}^3 + \epsilon\xi_{ij}^3$. Note that the last equality follows from the antisymmetry of θ . \square

Remark 5.2. We note here that a perfectly analogous result holds for the moduli spaces of H -stable sheaves with fixed determinant; the only difference being that, in this case, the tangent (resp. cotangent) space to the moduli variety at a point E is given by $\text{Ext}_0^1(E, E)$ (resp. $\text{Ext}_0^1(E, E \otimes \omega_S)$), where the subscript 0 means that we are considering only trace-free classes.

6. The rank of θ

Let us turn now to the computation of the rank of the Poisson structure θ , i.e., of the dimension of the symplectic leaves of the Poisson variety \mathcal{M}^0 .

Let S be a Poisson surface and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on S . If s is a symplectic structure, i.e., if S is an abelian or a $K3$ surface, then θ actually defines a symplectic structure on \mathcal{M}^0 , hence it has everywhere maximal rank (equal to the dimension of \mathcal{M}^0).

If S is not symplectic, let us denote by D the divisor of s and suppose, for simplicity, that D is an irreducible nonsingular curve (note that from the adjunction formula it follows that the canonical line bundle of D is trivial, hence D is an elliptic curve). For any vector bundle E on S , we have an exact sequence

$$0 \rightarrow \mathcal{E}nd(E) \otimes \omega_S \xrightarrow{s} \mathcal{E}nd(E) \rightarrow \mathcal{E}nd(E|_D) \rightarrow 0. \quad (6.1)$$

Lemma 6.1. *If E is H -semistable, then $H^0(S, \mathcal{E}nd(E) \otimes \omega_S) = 0$.*

Proof. Note that an H -semistable sheaf is also μ - H -semistable, where a coherent sheaf E is said to be μ - H -semistable (resp. μ - H -stable) if it is torsion-free and for any proper coherent torsion-free subsheaf F of E , we have

$$\mu(F) = \frac{(c_1(F) \cdot H)}{\text{rk}(F)} \leq \frac{(c_1(E) \cdot H)}{\text{rk}(E)} = \mu(E) \quad (\text{resp. } \mu(F) < \mu(E)).$$

Now we recall that, in the proof of Proposition 3.3, we have shown that $(K_S \cdot H) < 0$, hence, if $\phi \in \text{Hom}(E, E \otimes \omega_S)$ is not zero, we have $\mu(E) \leq \mu(\phi(E)) \leq \mu(E \otimes \omega_S) = \mu(E) + (K_S \cdot H) < \mu(E)$, which is a contradiction. \square

Now, by considering the long exact cohomology sequence of (6.1), we get

$$0 \rightarrow \mathbb{C} \rightarrow H^0(D, \mathcal{E}nd(E|_D)) \rightarrow H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{B(E)} H^1(S, \mathcal{E}nd(E)) \rightarrow \dots$$

We have thus proved the following

Proposition 6.2. *The kernel of the hamiltonian morphism $B(E)$ is given by*

$$\text{Ker } B(E) = H^0(D, \mathcal{E}nd(E|_D)) / \mathbb{C}.$$

Hence

$$\text{rk}(B(E)) = \dim \mathcal{M}^0 - \dim H^0(D, \mathcal{E}nd(E|_D)) + 1.$$

This shows how the rank of the Poisson structure θ at a point $E \in \mathcal{M}^0$ is determined by the restriction $E|_D$ of E to the curve D . On the open subset of \mathcal{M}^0 consisting of vector bundles E that restrict to a simple bundle on D , the rank of θ is equal to the dimension of \mathcal{M}^0 , i.e., θ is nondegenerate, or, in other terms, it induces a symplectic structure.

Remark 6.3. If $\dim \mathcal{M}^0 = 2$, then \mathcal{M}^0 is, like S , a Poisson surface, i.e., an abelian or a $K3$ surface in the symplectic case, or a ruled surface with an effective anticanonical divisor in the general case.

Remark 6.4. By recalling that the dimension of $\mathcal{M}^0 = \mathcal{M}^0(r, c_1, c_2)$ is given by $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$ (if it is non-empty), and that the dimension of a symplectic variety is even, we see that, if r and c_1 are both even, then $\dim \mathcal{M}^0$ is odd, hence the symplectic leaves of \mathcal{M}^0 have dimension strictly less than the dimension of \mathcal{M}^0 . It follows that there are no H -stable vector bundles $E \in \mathcal{M}^0$ such that $E|_D$ is simple. The same conclusion holds, for example, if r is odd and $\chi(\mathcal{O}_S)$ is even.

Remark 6.5. If $S = \mathbb{P}^2$, much more is known on the structure of the moduli space \mathcal{M} . If we denote by \mathcal{M}_μ^0 the moduli space of μ - H -stable locally free sheaves on \mathbb{P}^2 , then we know that \mathcal{M}_μ^0 is irreducible and that, under some technical conditions on the rank and Chern classes, it is everywhere dense in $\overline{\mathcal{M}}$ (see [8] for details). It follows that, in this case, θ defines a Poisson structure on all of \mathcal{M} .

Remark 6.6. Let us denote now by \mathcal{M}_D the moduli space of semistable vector bundles on the curve D and by \mathcal{M}_S the open subset of \mathcal{M}^0 such that

$$E \in \mathcal{M}_S \Rightarrow E|_D \in \mathcal{M}_D.$$

Let us denote by $\rho : \mathcal{M}_S \rightarrow \mathcal{M}_D$ the restriction map.

By recalling the Poisson structure of \mathcal{M}_S , for every $E \in \mathcal{M}_S$ we get the following commutative diagram

$$\begin{array}{ccc}
T_E \mathcal{M}_S & \xrightarrow{T_E \rho} & T_{E|_D} \mathcal{M}_D \\
\uparrow B(E) & & \uparrow \tilde{B}(E) \\
T_E^* \mathcal{M}_S & \xleftarrow{T_E^* \rho} & T_{E|_D}^* \mathcal{M}_D,
\end{array}$$

where $\tilde{B}(E) = T_E \rho \circ B(E) \circ T_E^* \rho$.

Since the canonical bundle of the curve D is trivial, the preceding diagram is canonically identified with

$$\begin{array}{ccc}
H^1(S, \mathcal{E}nd(E)) & \longrightarrow & H^1(D, \mathcal{E}nd(E|_D)) \\
\uparrow B(E) & & \uparrow \tilde{B}(E) \\
H^1(S, \mathcal{E}nd(E) \otimes \omega_S) & \longleftarrow & H^0(D, \mathcal{E}nd(E|_D)).
\end{array}$$

Now, the long exact cohomology sequence of (6.1), gives

$$\begin{aligned}
H^0(D, \mathcal{E}nd(E|_D)) &\xrightarrow{\delta} H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{B(E)} \\
&H^1(S, \mathcal{E}nd(E)) \xrightarrow{\alpha} H^1(D, \mathcal{E}nd(E|_D)),
\end{aligned}$$

and it is easy to see that $\tilde{B}(E)$ is precisely the composition $\alpha \circ B(E) \circ \delta$, hence it is zero.

This proves that the Poisson structure of \mathcal{M}_S induces on the image $\rho(\mathcal{M}_S) \subset \mathcal{M}_D$ a Poisson structure that is identically zero.

We would like to end with some remarks on the relations between the moduli space of stable vector bundles on \mathbb{P}^2 and the moduli space of anti self-dual connections on S^4 .

Let P be a principal $SU(r)$ -bundle on $S^4 = \mathbb{R}^4 \cup \{\infty\}$, and k be minus its Pontryagin index. Let us denote by $\widetilde{M}(SU(r), k)$ the moduli space parametrizing pairs consisting of an anti self-dual $SU(r)$ -connection on P and an isomorphism $P_\infty \cong SU(r)$. This is a manifold of (real) dimension $4rk$, if k is sufficiently large. The usual moduli space of anti self-dual $SU(r)$ -connections on P is then given by $M(SU(r), k) = \widetilde{M}(SU(r), k)/SU(r)$.

Now let us fix a line $D \subset \mathbb{P}^2$ (so that we get isomorphisms $\mathbb{P}^2 \setminus D \cong \mathbb{C}^2 \cong \mathbb{R}^4$) and denote by $\widetilde{\mathcal{M}}(r, 0, k)$ the moduli space of pairs consisting of a rank r holomorphic vector bundle E on \mathbb{P}^2 with $c_1(E) = 0$ and $c_2(E) = k$ and a trivialization of $E|_D$.

Donaldson proved in [6] that there is a natural isomorphism

$$\widetilde{M}(SU(r), k) \cong \widetilde{\mathcal{M}}(r, 0, k).$$

If $\mathcal{M}^0(r, 0, k)$ denotes the moduli space of stable rank r vector bundles on \mathbb{P}^2 with Chern classes $c_1 = 0$ and $c_2 = k$, it is known that there are Zariski open subsets $U \subset \widetilde{\mathcal{M}}(r, 0, k)$ and $V \subset \mathcal{M}^0(r, 0, k)$ such that U fibers over V . As a consequence we find that an open subset of the moduli space $M(SU(r), k)$ fibers over V with fiber $SL(r, \mathbb{C})/SU(r)$. Hence, after factoring out by this group, we get a complex structure and a (complex) Poisson structure on the real moduli space of anti self-dual connections.

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