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POLAR CLASSES OF SINGULAR VARIETIES

BY RAGNI PIENE (*)

CONTENTS

Introduction

Let $X \subseteq \mathbf{P}^n$ be a smooth, projective variety of dimension r. For each integer k satisfying $0 \leqq k \leqq r$ consider a $(n-r+k-2)$-dimensional linear subspace $L_{(k)}$ of $\mathbf{P}^n$. The tangent r-space T_x to X at a point x intersects $L_{(k)}$ in a space of at least k-2 dimensions. The set of points $x \in X$ such that this intersection space has dimension at least k-1 is called a *k-th polar locus* of X and denoted $M_k (L_{(k)})$.

For example, take $X \subseteq \mathbf{P}^3$ to be a surface. Let $L_{(1)} = \{ p \}$ be a point and $L_{(2)} = L$ a line in $\mathbf{P}^3$. The 1st polar locus $M_1 (p)$ of X consists of points such that the tangent plane at the point contains p, while the 2nd polar locus $M_2 (L)$ consists of the points

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whose tangent plane contains the line L. For most choices of p, $M_1(p)$ is a *curve*, while for most choices of L, $M_2(L)$ consists of a finite number of points.

In the general situation $X \subset \mathbf{P}^n$, $\dim X = r$, the locus $M_k(L_{(k)})$ will have codimension k in X, for most choices of $L_{(k)}$. Moreover, for such $L_{(k)}$, the rational equivalence class of the cycle defined by $M_k(L_{(k)})$ does not depend on $L_{(k)}$. We let $[M_k] = [M_k(L_{(k)})]$ denote this class and call it *the k-th polar class of* X. The degree μ_k of M_k is called *the k-th class*. The top class μ_r is called *the class* of X. The $[M_k]$'s are invariant under generic projections, i. e., they are *projective invariants*. For $k < r$, μ_k is also equal to the k-th class of a general hyperplane section of X.

The general study of polar loci goes back to Severi ([S]), though the cases of curves and surfaces had been treated earlier. The ideas of Severi were taken up by Todd ([To]), who called the polar loci M_k *polar varieties* and used them to define canonical classes on the variety.

Porteous ([Pt]) showed how to recover Todd's definition of canonical classes in terms of singularities of maps. This point of view was taken by Lascoux ([Lx]) who considered also the polar varieties in this way.

Pohl ([Ph 2]) studied a more general situation: Let X be a smooth variety and $f : X \to \mathbf{P}^n$ a map which is an immersion on an open dense subset of X (but f need not be a generic projection). He defined a homology class γ_1 (what we here call the 1st polar class $[M_1]$ of X with respect to f) on X and proved a formula for γ_1 in terms of the 1st Chern class of X, the class of a hyperplane section, and the "cuspidal edges" (i. e., the divisorial part of the ramification locus of f).

When $f : X \to \mathbf{P}^2$ is a curve, the degree of γ_1 is the number of tangents (at smooth points of the image) that pass through a given point, i. e., it is the *class* of the curve. Hence Pohl's formula is a generalization of one of the Plücker formulas for a plane curve. He asked for (1) a similar generalization of the other basic Plücker formula (which gives the class in terms of the degree of the curve and the number of nodes and cusps). Moreover, he asked for (2) formulas for the higher codimension polar classes, generalizing those that exist when f is an immersion (or a generic projection of an immersion).

Teissier ([Te]) found a formula for the class of a hypersurface with *isolated* singularities which answers (1) in that case. So one could ask for (3) a generalization of Teissier's formula to hypersurfaces with *arbitrary* singularities.

The questions (1), (2), and (3) were the starting points of the present work. We succeed in answering (3). In fact we find formulas for all the polar classes of a hypersurface, so that we also answer (1) and (2) in this particular case.

Partial answers to (1) and (2) are obtained in the case of local complete intersections; we then get a formula for $[M_1]$. When the local complete intersection comes with a desingularization, this formula and Pohl's formula for $[M_1]$ yield a formula for a certain "double point class".

Our methods are heavily influenced by Pohl's ([Ph 1], [Ph 2]). We define the polar classes of a singular variety X with respect to a generic immersion $f : X \to \mathbf{P}^n$. We show that there is a bundle P on a blow up $\overline{X}$ of X such that the polar classes of X are the pushdowns of the Chern classes of P.

When f is an embedding and X is a local complete intersection, or when f is arbitrary but X is smooth, we can compute $c_1(P)$. The problem of finding $c_k(P)$, $k > 1$, remains open, except for the hypersurface case, where $c_k(P) = c_1(P)^k$ holds.

We show that the polar loci and polar classes are invariant under generic projections, as in the classical case of an embedded, smooth variety. If we consider a generic projection $p : X \to \mathbf{P}^{r+1}$ of the given map $f : X \to \mathbf{P}^n$, where $r = \dim X$, we are thus reduced to the hypersurface case and can compute the polar classes of X in terms of characters of p. When X is smooth and f is an embedding, the singularities of p are *ordinary* and have been studied ([To], [Ll], [Rb 1]). In the general case, however, in addition to the aquired ordinary singularities, the singularities of X change character when projected (e. g., the singularities of $p(X)$ are hypersurface singularities, while X need not even be a Gorenstein variety). This approach to computing the polar classes of X thus seems to require a study of the behavior of singularities under projections and it will not be pursued here.

Acknowledgment

Most of the results presented here are contained in the author's doctoral dissertation (M.I.T., 1975), written under the direction of Steven Kleiman. To him many thanks are due, for introducing the subject and for helpful discussions. Moreover, the present version was influenced by Kleiman's treatment in *The Enumerative Theory of Singularities* ([Kl 2]). In particular this caused a shift in emphasis away from numerical formulas and towards formulas for rational equivalence classes of cycles. Thus the *polar classes* became the focus of this article.

Notations

We fix an algebraically closed ground field k, of arbitrary characteristic. All schemes are assumed to be algebraic, i. e., separated and of finite type over k. A reduced (but possibly reducible) equidimensional, proper scheme is called a *variety*.

We shall use the *intersection theory* for *singular schemes* as developed by Fulton ([Fu] for quasi projective schemes; [F-M] for the general case). If X is a scheme, we denote by $A.X$ its *Chow group*, i. e., the group of cycles on X modulo rational equivalence, graded by dimension. There is a unique *theory of Chern classes* on X, which to each bundle (i. e., locally free coherent sheaf) E associates an element $c(E) = \sum c_i(E) \in A^{\cdot}X$, where $A^{\cdot}X$ denotes the *ring of Chern operators* on X ([V], [F-M]).

Let $g : X \to Y$ be a proper map. We let $\cap : A^{\cdot}Y \otimes A.X \to A.X$ denote the *cap product*; it makes $A^{\cdot}X$ into a graded $A^{\cdot}Y$-module. For $\alpha \in A^{\cdot}Y$, $\beta \in A.X$, there is the *projection formula* $g_*(g^*\alpha \cap \beta) = \alpha \cap g_*\beta$.

We write $s(E) = c(E^{\vee})^{-1}$, where $^{\vee}$ denotes dual, and call it the *Segre class* of the bundle E. We shall also use the following notation: Let $\Gamma \subseteq X$ be a closed subscheme and let $p : \tilde{X} \to X$ denote the blow up of Γ, with relatively ample line bundle $O(1)$. Then the *Segre covariant class* of Γ in X is defined to be the element

$$s(\Gamma, X) = -\sum_{j=0}^{r-1} p_*(s_{r-j}(O(1)) \cap [\tilde{X}])$$

in $A.X$. Hence

$$s_j(\Gamma, X) = -p_*(c_1(O(1))^{r-j} \cap [\tilde{X}]) \in A_j X$$

holds, for $j = 0, 1, \ldots, r-1$. Moreover (by the projection formula) $s_j(\Gamma, X)$ has support contained in the support of Γ, so that we get $s_j(\Gamma, X) = 0$ for $j >$ dim Γ.

We write $\int \alpha$ for the *degree* of the 0-dimensional component of an $\alpha \in A^{\cdot}X$. If $\beta \in A^{\cdot}X$, we write also $\int \beta$ for $\int \beta \cap [X]$. For a proper map $f : X \to Y$, $\int f_* \alpha = \int \alpha$ holds for all $\alpha \in A.X$.

By $f : X \to \mathbf{P}^n$ we shall always mean a proper map from a variety X of dimension r to projective n-space such that f is an immersion (i. e., f is unramified) *at all generic points* of X. Such a map f will be called a *generic immersion*. We let $L = f^* O_{\mathbf{P}^n}(1)$ denote the pullback to X of the tautological line bundle on $\mathbf{P}^n$.

Often we write $\mathbf{P}^n$ in the "coordinate free" way as $\mathbf{P}(V)$, with V a $(n+1)$–dimensional vector space. The dual projective space $\mathbf{P}^{n\vee}$ of hyperplanes in $\mathbf{P}^n$ is then $\mathbf{P}(V^\vee)$. By $\mathrm{Grass}_{m+1}(V)$ we denote the Grassmann variety parametrizing $m+1$-*quotients* of V (i. e., m-planes in $\mathbf{P}^n$). There is a canonical isomorphism $\mathbf{P}(V^\vee) \cong \mathrm{Grass}_n(V)$.

Given integers $0 < a_1 \ldots < a_{n-m} \leqq n+1$, then to each flag

$$F = \{ V_0 \subset V_1 \subset \ldots \subset V_{n-m} \}$$

in V, with dim $V_i = a_i$, we let $\Sigma(a; F)$ denote the corresponding *Schubert variety* of $G = \mathrm{Grass}_{m+1}(V)$: If $0 \to K \to V_G \to Q \to 0$ denotes the tautological sequence on G, $\Sigma(a; F)$ is the subscheme of G where the induced maps

$$\Lambda^{a_i - i + 1} V_{iG} \to \Lambda^{a_i - i + 1} Q$$

are 0. Equivalently, $\Sigma(a; F)$ parametrizes m-planes $\mathbf{P}^m \subseteq \mathbf{P}^n$ such that

$$\dim(\mathbf{P}^m \cap \mathbf{P}(V/V_i)) \geqq m - a_i + i$$

holds, for $i = 1, \ldots, n-m$.

In particular we shall consider the 1*st special Schubert variety* $\Sigma_k(V')$ of r-planes intersecting a given $(n-r+k-2)$-plane $\mathbf{P}(V/V')$ in a space of at least $k-1$ dimensions. We note that

$$\Sigma_k(V') = \Sigma(a; F)$$

holds, where $a_1 = r-k+2$, $a_i = r+1+i$ for $i > 1$, and where $F = \{ V_i \}$ is any flag satisfying $V_1 = V'$ and dim $V_i = a_i$.

Finally, we recall that the ith *Fitting ideal* $F^i(M)$ of a coherent module M on a scheme X is defined as the sheaf of ideals generated at each point $x \in X$ by the $(q-i)$-minors of the matrix of any presentation

$$O^p_{X,x} \to O^q_{X,x} \to M_x \to 0$$

of M at x. For the general properties of these ideals we refer to ([F]; [G-R], pp. 28-41; [Ka], p. 145; [SGA 7 I], p. 114).

1. Polar loci and the bundle P

Let X be a variety of dimension r and $f: X \to \mathbf{P}^n = \mathbf{P}(V)$ a generic immersion. Denote by U the largest open subscheme of X where X is smooth and where f is an immersion. For a given linear $(n-r+k-2)$-dimensional subspace $L_{(k)}$ of $\mathbf{P}^n$, we let $M_k(U)$ denote the locus of points $x \in U$ such that the tangent r-space to $f(X)$ at $f(x)$ intersects $L_{(k)}$ in a space of at least $k-1$ dimensions. The closure M_k of $M_k(U)$ in X is called a *polar locus* of X (with respect to f) ([Ph 2], p. 19; [Kl 2], IV. C).

We shall see below that M_k has a natural scheme structure and that, for a *general* $L_{(k)}$, M_k has codimension k in X and has no embedded components. Moreover, if char $k = 0$, M_k is also reduced.

First, recall that there is a natural map $a: V_X \to P^1(L)$, where $P^1_X(L)$ denotes the sheaf of principal parts of the line bundle $L = f^* O_{\mathbf{P}^n}(1)$, such that a represents the (projectivized) tangent spaces to X, as explained below. (See [Pi], § 2, § 6.) There is a canonical isomorphism $\alpha: V_{\mathbf{P}(V)} \xrightarrow{\sim} P^1_{\mathbf{P}(V)}(O_{\mathbf{P}(V)}(1))$ and a is the map obtained by composing $f^*\alpha$ with the map $f^* P^1_{\mathbf{P}(V)}(O_{\mathbf{P}(V)}(1)) \to P^1_X(L)$. Therefore the cokernel of a is isomorphic to the cokernel of the map

$$f^*\Omega_{\mathbf{P}(V)} \otimes L \to \Omega_X \otimes L,$$

hence to $\Omega_f \otimes L$, where $\Omega_f = \Omega_{X/\mathbf{P}(V)}$ denotes the sheaf of relative differentials.)

Choose a basis $e_0, \ldots, e_n$ of $V = H^0(\mathbf{P}^n, O_{\mathbf{P}^n}(1))$. Let $x \in X$ be a closed, smooth point, put $A = O_{X,x}$ and fix an isomorphism $L_x \cong A$. Let $\bar{e}_i$ denote the image of e_i in L_x via the map $V_X \to L$, and $x_i \in A$ the image of $\bar{e}_i$ via the fixed isomorphism. Choose a regular system of parameters $(t_1, \ldots, t_r)$ for A. For $a \in A$ we let $da \in P^1_A$ denote the class of the element $a \otimes 1 - 1 \otimes a$. Then P^1_A is a free A-module with basis $\{1, dt_1, \ldots, dt_r\}$. Let $\{1, d^1, \ldots, d^r\} \subseteq \mathrm{Diff}^1_A = P^{1\vee}_A$ denote the dual basis ([EGA IV$_4$], 16.11).

With the above notations and choices of bases the diagram

$$\begin{array}{ccc} V_{X,x} & \xrightarrow{a_x} & P^1_X(L)_x \\ \| \wr & & \| \wr \\ A^{n+1} & \xrightarrow{M} & A^{r+1} \end{array}$$

commutes, where M is the matrix $(d^j x_i)_{0 \leq i \leq n, 0 \leq j \leq r}$. The map a is surjective at a point x if and only if f is unramified at x, since $\mathrm{Coker}(a) \cong \Omega_f \otimes L$ holds.

In view of the above discussion, if $x \in U$, the $(r+1)$-quotient $a(x) = a_x \otimes k$ gives an embedding $\mathbf{P}(P^1_X(L)(x)) \subsetneq \mathbf{P}(V)$ of the tangent r-space to X (or $f(X)$) at $f(x)$.

The map $\varphi : U \to G = \mathrm{Grass}_{r+1}(V)$ defined by the quotient $a|_U$ is called the *tangent map* (or *Gauss map*, or 1*st associated map*) of *f*. Let $L_{(k)} = \mathbf{P}(W)$ be a $(n-r+k-2)$-dimensional subspace of $\mathbf{P}(V)$ and $\Sigma_k \subseteq G$ the corresponding 1st special Schubert variety, i. e., Σ_k corresponds to r-spaces of $\mathbf{P}(V)$ intersecting $L_{(k)}$ in a space of at least $k-1$ dimensions. Hence the points of $M_k(U)$ are the points of $\varphi^{-1}\Sigma_k$, and we give $M_k(U)$ the scheme structure of $\varphi^{-1}\Sigma_k$. Therefore $M_k(U)$ is the scheme of zeros of the induced map

$$\Lambda^{r-k+2} V'_U \to \Lambda^{r-k+2} P^1_X(L)\,|_U,$$

where we set $V' = \ker(V \to W)$. We shall consider the polar locus M_k as the *schematic closure* of $M_k(U)$.

Next we show that for general $L_{(k)}$, the *cycles* M_k have codimension k and are all rationally equivalent. We do this by constructing a proper, birational map $\pi : \overline{X} \to X$ and a quotient $V_{\overline{X}} \to P$ extending $a|_U$, such that the equality

$$[M_k] = \pi_*(c_k(P) \cap [\overline{X}])$$

holds.

The obstructions to extending $a|_U$ on X are, (i) if X is not smooth, $P^1_X(L)$ is not everywhere locally free with rank $r+1$, (ii) a is surjective only where f is an immersion.

First we will deal with (i). The singular subscheme of X is defined by its Jacobian ideal, the r-th Fitting ideal $F^r(\Omega_X)$. Because there is an exact sequence ([EGA IV], 16.3.1, 16.7.2):

$$(\bigstar) \qquad 0 \to \Omega_X \otimes L \to P^1_X(L) \to L \to 0,$$

$F^r(\Omega_X \otimes L) = F^{r+1}(P^1_X(L))$ holds, and $F^r(\Omega_X \otimes L) = F^r(\Omega_X)$ holds since the formation of Fitting ideals is invariant under tensor product with line bundles. Raynaud proved ([G-R],5.4.3) that if M is a coherent sheaf which generically is a $(r+1)$-bundle and if $F^{r+1}(M)$ is invertible, then the quotient of M by the annihilator in M of $F^{r+1}(M)$ is a $(r+1)$-bundle. So let

$$\tilde{\pi}: \quad \tilde{X} \to X$$

denote the blow up of $F^r(\Omega_X)$, and apply Raynaud's result to $\tilde{\pi}^* P^1_X(L)$. Then $F^{r+1}(\tilde{\pi}^* P^1_X(L))$ is invertible since it is equal to $F^{r+1}(P^1_X(L))\,O_{\tilde{X}}$ (by general properties of Fitting ideals). Set $A = \mathrm{Ann}_{\tilde{\pi}^* P^1_X(L)}(F^{r+1}(\tilde{\pi}^* P^1_X(L)))$ and $\tilde{P} = \tilde{\pi}^* P^1_X(L)/A$. Then $\tilde{P}$ is a $(r+1)$-bundle. Let $\tilde{a} : V_{\tilde{X}} \to \tilde{P}$ denote the induced map.

Next we want to make $\tilde{a}$ surjective. We need the following general result.

LEMMA (1.1). — *Let* $\alpha : E \to F$ *be a map of bundles on a scheme* Y, *of ranks* n and m, *and put* $I = F^0(\mathrm{Coker}(\alpha))$. *Then* I *is invertible if and only if* $\mathrm{Im}(\alpha)$ *is a* m*-bundle.*

Proof. – Put $F' = \mathrm{Im}(\alpha)$. If F' is a m-bundle, then I is the ideal generated (locally) by the determinant of the map $F' \subsetneq F$ between m-bundles, hence is invertible.

Conversely, suppose I is invertible. The map α induces a surjection $\bar{\alpha} : \Lambda^m E \to \Lambda^m F \otimes I$. This 1-quotient defines a map $g : Y \to \mathbf{P}(\Lambda^m E)$. Let $i : \mathrm{Grass}_m(E) \subsetneq \mathbf{P}(\Lambda^m E)$ denote the Plücker embedding and put $U = Y - V(I)$. Since $\alpha | U$ is surjective, $g | U$ factors through i, so that $U \subseteq g^{-1}(\mathrm{Grass}_m(E))$. But U is schematically dense in Y and i is closed, hence $Y = g^{-1}(\mathrm{Grass}_m(E))$ holds. Thus there exists a m-bundle F'' and surjection $\beta : E \to F''$ such that $\beta | U$ is isomorphic to $\alpha | U$. We claim that $F'' = F'$ holds. To see this, set $K' = \ker \alpha$ and $K = \ker \beta$ and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dotted] & E \arrow[r] \arrow[d, equal] & F'' \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & K' \arrow[r] & E \arrow[r] & F' \arrow[r] & 0. \end{tikzcd}$$

The dotted arrow on the left exists because the induced map $K \to F'$ of bundles is zero on U, hence on all of Y. The resulting map $F'' \to F'$ is surjective, but also injective since it is so on U and since F'' is a bundle. Hence the vertical arrows are isomorphisms.

Q.E.D.

Let $\bar{\pi} : \bar{X} \to \tilde{X}$ denote the blow up of $F^0(\mathrm{Coker}\,\alpha)$. Applying the Lemma to $\bar{\pi}^* \tilde{a} : V_{\bar{X}} \to \bar{\pi}^* \tilde{P}$ and setting $P = \mathrm{Im}(\bar{\pi}^* \tilde{a})$, we obtain a $(r+1)$-quotient

$$\bar{a} : \quad V_{\bar{X}} \to P$$

on $\bar{X}$.

PROPOSITION (1.2). – *For a general $(n-r+k-2)$-space $L_{(k)}$ the class $[M_k]$ of M_k in $A.X$ is independent of $L_{(k)}$. If $\pi : Z \to X$ is any proper, birational map such that the $(r+1)$-quotient $a|_U$ extends to a $(r+1)$-quotient $b : V_Z \to P$, there is an equality*

$$[M_k] = \pi_*(c_k(P) \cap [Z]).$$

Proof. – The first statement follows from the second. The proof of the second statement relies on the freedom to move Schubert varieties on a Grassmann variety. The general result is the following.

TRANSVERSALITY LEMMA (1.3). – *Let Z be a reduced, equi-dimensional scheme, $g : Z \to G = \mathrm{Grass}_{m+1}(V)$ a morphism. Fix a Schubert condition $(a_1, \ldots, a_{n-m})$. Then for a general flag $F = \{V_1 \subseteq \ldots \subseteq V_{n-m} \subseteq V\}$, with* $\dim V_i = a_i$, *the corresponding Schubert variety*

$$\Sigma = \Sigma(a; F)$$

satisfies the following conditions:

(i) $g^{-1}\Sigma$ *is either empty, or equi-dimensional with* $\mathrm{codim}(g^{-1}\Sigma, Z) = \mathrm{codim}(\Sigma, G)$;

(ii) $g^{-1}\Sigma$ *satisfies* (S_1) (i. e., $g^{-1}\Sigma$ *has no embedded components*). *If* char $k = 0$, $g^{-1}\Sigma$ *is reduced*;

(iii) *given an open, dense subscheme* $U \subseteq Z$, $g^{-1}\Sigma|_U$ *is dense in* $g^{-1}\Sigma$;

(iv) *the cycle* $g^*[\Sigma]$ *is defined and is equal to* $[g^{-1}\Sigma]$.

Proof. — (i) follows from ([Kl 1], 2 (i)), with the group being the general linear group acting on V (see also the proof of [Pi], 4.1).

(ii) is a version of ([Kl 1], 2 (ii)) (for other versions, see ([Kl 1], 7)), and it is proved by proving the corresponding version of ([Kl 1], 1 (ii)). We use general results on schemes and morphisms satisfying (S_1), in particular ([EGA IV_2], 6.4.1 (ii); [EGA IV_3], 9.7.6), as well as the fact that Σ satisfies (S_1). To prove that $g^{-1}\Sigma$ is reduced (in char 0) we use also that Σ is reduced. In fact, it is known that Schubert varieties are not only reduced, but that they are Cohen-Macaulay and normal ([Ho], [Lk]).

(iii) Put $Z' = Z - U$. We may clearly assume Z' is equi-dimensional (by arguing on each equi-dimensional component). Then (i) applied to $g|_{Z'} : Z' \to G$, together with (ii), shows $\operatorname{Ass}(g^{-1}\Sigma) \subseteq g^{-1}\Sigma|_U$, since the associated points are all minimal, and the statement follows.

(iv) By (i), $g^*[\Sigma]$ is defined, and its support is $g^{-1}\Sigma$. Hence it suffices to show that $g^*[\Sigma]$ and $[g^{-1}\Sigma]$ are equal on the generic points $\operatorname{Ass}(g^{-1}\Sigma)$. By (iii), applied to U = smooth locus of Z, we may assume that these points are all smooth on Z, hence we conclude by ([K-L], Lemma 9), using the fact that Σ is Cohen-Macaulay.

Let us return to the proof of the Proposition. Let $g : Z \to G = \operatorname{Grass}_{r+1}(V)$ denote the map defined by b. We apply the Transversality Lemma to the (1st special) Schubert varieties Σ_k parametrizing r-planes meeting a given $(n-r+k-2)$-plane in a space of at least $k-1$ dimensions. Let Σ_k be general, defined as the scheme of zeros of

$$\Lambda^{r-k+2} V'_G \to \Lambda^{r-k+2} Q$$

(here $V_G \to Q$ denotes the tautological quotient). Since "scheme of zeros" is compatible with pullbacks, $g^{-1}\Sigma_k$ is the scheme of zeros of

$$\Lambda^{r-k+2} V'_Z \to \Lambda^{r-k+2} P.$$

By Porteous' formula (on G, which is non singular) ([K-L], Cor. 11),

$$[\Sigma_k] = c_k(Q) \cap [G].$$

By (iv), $[g^{-1}\Sigma_k] = g^*[\Sigma_k]$, hence:

$$[g^{-1}\Sigma_k] = g^*(c_k(Q) \cap [G]) = c_k(P) \cap [Z],$$

i. e., Porteous' formula holds also for $[g^{-1}\Sigma_k]$ on (the singular) Z.

By (iii), we may assume $\operatorname{Ass}(g^{-1}\Sigma_k) \subseteq \pi^{-1} U$, and since $\pi|_{\pi^{-1}U}$ is an isomorphism, we obtain

$$\pi_*[g^{-1}\Sigma_k] = [\overline{M_k(U)}] = [M_k],$$

as desired.

Q.E.D.

The proposition shows that the general polar loci M_k belong to the same rational equivalence class $[M_k]$. We call $[M_k]$ the *k-th polar class* of X (with respect to f).

The map $\pi : \overline{X} \to X$ and the quotient $\overline{a} : V_{\overline{X}} \to P$ constructed above gives one way of extending $a|_U$, and hence also the tangent map $\varphi : U \to G = \mathrm{Grass}_{r+1}(V)$. Let $\Phi : \overline{X} \to G$ denote the map defined by $\overline{a}$. Now there is another way of extending φ, as follows.

Let $\Gamma \subset X \times G$ denote the closure of the graph of φ, and let $\gamma : \Gamma \to X$ and $\Phi_\Gamma : \Gamma \to G$ denote the projections. Then the quotient on Γ corresponding to Φ_Γ extends $a|_U$, and Γ is the *minimal* scheme on which $a|_U$ extends: Let $\pi : Z \to X$ and b be as in Proposition (1.2). The map $(\pi, g) : Z \to X \times G$ induces an isomorphism $Z|_{\pi^{-1}U} \xrightarrow{\sim} \gamma^{-1} U$, hence Z maps onto $\Gamma = \overline{\gamma^{-1} U}$, so that π factors through $\gamma : \Gamma \to X$ and g factors through $\Phi_\Gamma : \Gamma \to G$.

When f is an immersion the map γ is usually called the *Nash blowing up* of X.

It is known ([N], Remark 2, p. 300) that the Nash blowing up of a local complete intersection is the same as the blow up of the Jacobian ideal. We give a proof of this in (1.4) which differs from ([N], *loc. cit.*). An example ([N]) when these maps are *not* equal is the case of two planes in $\mathbf{P}^4$ intersecting in a point. The Nash blow up separates the planes. The Jacobian ideal is equal to the ideal defining the point of intersection, so its blow up introduces a new curve on the planes in addition to separating these.

PROPOSITION (1.4). — *If f is an immersion and $K_a = \mathrm{Ker}(a)$ is a bundle, then the maps $\overline{\pi}$ and γ are isomorphic. This holds in particular if f is an embedding and X is a local complete intersection in $\mathbf{P}^n$.*

Proof. — Given the existence of the map $\tilde{X} \to \Gamma$ it suffices to show that the Jacobian ideal $F^r(\Omega_X) = F^{r+1}(P^1_X(L))$ becomes invertible on Γ. We shall show this by applying Lemma (1.1) to the dual of the map $b : \gamma^* K_a \to V_\Gamma$.

Since $P^1_X(L)$ is generically a bundle the sequence on Γ,

$$0 \to \gamma^* K_a \xrightarrow{b} V_\Gamma \to \gamma^* P^1_X(L) \to 0,$$

is generically exact. By assumption $\gamma^* K_a$ is a bundle, therefore it has no torsion, and the sequence is everywhere exact.

Next we observe that $F^{r+1}(\mathrm{Coker}\, b) = F^0(\mathrm{Coker}(b^\vee))$ holds, because a matrix and its transpose have the same minors. Let $V_\Gamma \to \mathfrak{g}$ denote the $(r+1)$-quotient on Γ defining Φ, with kernel K_Γ. I claim that $b^\vee$ factors through the surjection $V_\Gamma^\vee \to K_\Gamma^\vee$. To see this, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \gamma^* K_a \arrow[r, "b"] \arrow[d, dotted] & V_\Gamma \arrow[r] \arrow[d, equal] & \gamma^* P^1_X(L) \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & K_\Gamma \arrow[r] & V_\Gamma \arrow[r] & \mathfrak{g} \arrow[r] & 0. \end{tikzcd}$$

The dotted arrow on the left exists because the map of bundles $\gamma^* K_a \to \mathfrak{g}$ is 0 on $\gamma^{-1} U$, hence on all of Γ. Hence the dotted arrow on the right exists. Dualizing we see that $\mathrm{Im}(b^\vee)$ is isomorphic to the image of the (necessarily injective) map $K_\Gamma^\vee \to \gamma^* K_a^\vee$. By Lemma (1.1) we conclude that $F^0(\mathrm{Coker}\, b^\vee)$ is invertible, i. e., that $F^{r+1}(\gamma^* P^1_X(L))\, O_\Gamma$ is invertible.

Suppose $X \subsetneq \mathbf{P}^n$ is a local complete intersection, with conormal bundle N. Then $K_a = N \otimes L$ holds, because there is a sequence

$$0 \to N \otimes L \to V_X \to P^1_X(L) \to 0$$

which is generically exact, hence exact everywhere since $N \otimes L$ is a bundle.

Q.E.D.

2. Formulas for the polar classes

Let $f : X \to \mathbf{P}^n$ be a generic immersion. In the previous section we constructed a map $\pi : \overline{X} \to X$, equal to the composition of two blow up's, and a $(r+1)$-quotient $\overline{a} : V_X \to P$ on X such that the polar classes of X are given by

$$[M_k] = \pi_*(c_k(P) \cap [\overline{X}]),$$

for $k = 0, 1, \ldots, r$.

Therefore we can find formulas for the polar classes whenever we know the Chern classes of P. In two cases we find expressions for c_1 (P), this is (I) when f is an embedding and X is a local complete intersection (II) when X is smooth. In either of these cases, if $f(X)$ is a *hypersurface* (i. e., $r = n-1$ holds), $c_k(P) = c_1(P)^k$ holds for all k, hence we obtain formulas for *all* the polar classes of X.

I. X IS A LOCAL COMPLETE INTERSECTION IN $\mathbf{P}^n$. — When f is an immersion, $a : V_X \to P^1_X(L)$ is surjective, so that $\pi = \tilde{\pi}$ is equal to the blow up of the Jacobian ideal $F^r(\Omega_X)$ ($\overline{\pi} = \text{id}$, $\overline{X} = \tilde{X}$, $P = \tilde{P}$). Set $K = \operatorname{Ker}(\tilde{a})$.

When f is an embedding and X is a local complete intersection in $\mathbf{P}^n$, with conormal bundle N, we saw in the proof of Proposition (1.4) that the (blown up) Jacobian ideal ideal $J = F^r(\Omega_X) O_{\tilde{X}}$ is equal to the 0-th Fitting ideal of the cokernel of the map

$$K^\vee \to \pi^*(N \otimes L)^\vee.$$

Hence we obtain an isomorphism

$$\Lambda^{n-r} K^\vee \tilde{\to} (\Lambda^{n-r} \pi^*(N \otimes L)^\vee) \otimes J.$$

The exact sequence of bundles

$$0 \to K \to V_{\tilde{X}} \to P \to 0$$

gives an isomorphism ([A-K], VII, 3.12):

$$\Lambda^{r+1} P \cong (\Lambda^{n-r} K)^\vee,$$

hence:

$$\Lambda^{r+1} P \cong (\Lambda^{n-r} \pi^*(N \otimes L)^\vee) \otimes J$$

since $\Lambda^{n-r}(K^\vee)$ and $(\Lambda^{n-r} K)^\vee$ are isomorphic.

The above isomorphism gives an equality between 1-st Chern classes

$$c_1(\mathrm{P}) = \pi^*(c_1(\mathrm{N}^\vee) - (n-r)\,c_1(\mathrm{L})) + c_1(\mathrm{J}).$$

Let $\mathrm{S} \subseteq \mathrm{X}$ denote the singular subscheme of X defined by the Jacobian ideal $\mathrm{F}^r(\Omega_\mathrm{X})$, and consider its associated cycle $[\mathrm{S}] \in \mathrm{Z}.\mathrm{X}$. Write $[\mathrm{S}]_{r-1} = \sum_\alpha m_\alpha [\mathrm{S}_{r-1,\alpha}]$ as a sum of its integral (codimension 1) components. Let e_α denote the multiplicity of the Jacobian ideal in the local ring of X at the generic point of $\mathrm{S}_{r-1,\alpha}$. Then we get ([Kl 2], II.D):

$$\pi_*(c_1(\mathrm{J}^{-1}) \cap [\tilde{\mathrm{X}}]) = \sum_\alpha e_\alpha [\mathrm{S}_{r-1,\alpha}].$$

We have proved the following proposition:

PROPOSITION (2.1). — *Let* $\mathrm{X} \subsetneq \mathbf{P}^n$ *be a local complete intersection. With the above notations, the* 1-*st polar class of* X *is given by*

$$[\mathrm{M}_1] = (c_1(\mathrm{N}^\vee) - (n-r)\,c_1(\mathrm{L})) \cap [\mathrm{X}] - \sum_\alpha e_\alpha [\mathrm{S}_{r-1,\alpha}].$$

Remark. — If X is smooth in codimension 1, the last term is 0. An example when it is not: suppose X is a surface in $\mathbf{P}^3$ with an ordinary j-multiple curve C_j. Then one shows $[\mathrm{S}]_1 = (j-1)[\mathrm{C}_j]$ and $\pi_*(c_1(\mathrm{J}^{-1}) \cap [\tilde{\mathrm{X}}]) = j(j-1)[\mathrm{C}_j]$.

COROLLARY (2.2). — *Suppose* X *is equal to the intersection of* $n-r$ *hypersurfaces, of degrees* d_i, $i = 1, \ldots, n-r$. *Then:*

$$[\mathrm{M}_1] = \sum_{i=1}^{n-r} (d_i - 1)\,c_1(\mathrm{L}) \cap [\mathrm{X}] - \Sigma\, e_\alpha [\mathrm{S}_{r-1,\alpha}].$$

Proof. — In this case the normal bundle $\mathrm{N}^\vee$ is equal to $\bigoplus_{i=1}^{n-r} \mathrm{L}^{d_i}$.

Q.E.D.

Suppose now that X *is a hypersurface in* $\mathbf{P}^n$, of degree d. Then K is invertible, hence we get

$$s(\mathrm{K}^\vee) = c(\mathrm{K})^{-1} = \sum_{k=0}^{r} c_1(\mathrm{K}^\vee)^k,$$

and hence:

$$c_k(\mathrm{P}) = c_1(\mathrm{P})^k.$$

As a result we get a formula for *all* the polar classes (as stated in [Kl 2], IV, 48).

THEOREM (2.3). — *The k-th polar class of a hypersurface of degree d in* $\mathbf{P}^n$ *is given by*

$$[\mathrm{M}_k] = [(d-1)\,c_1(\mathrm{L})]^k \cap [\mathrm{X}] - \sum_{i=0}^{k-1} \binom{k}{i} [(d-1)\,c_1(\mathrm{L})]^i \cap s_{r-k+i}(\mathrm{S}, \mathrm{X}),$$

where $s_{r-k+i}(\mathrm{S}, \mathrm{X}) = -\pi_*(c_1(\mathrm{J})^{k-i} \cap [\tilde{\mathrm{X}}])$ *denotes the* Segre covariant classes *of the singular subscheme* S *of* X.

Proof. — From the above, $c_1(\mathrm{P}) = \pi^* c_1(\mathrm{L}^{d-1}) + c_1(\mathrm{J})$ follows, since the conormal bundle is equal to L^{-d}. Apply the projection formula to each term of the expansion of the right hand side of

$$\pi_*(c_1(\mathrm{P})^k \cap [\tilde{\mathrm{X}}]) = \pi_*(\{\pi^* c_1(\mathrm{L}^{d-1}) + c_1(\mathrm{J})\}^k \cap [\tilde{\mathrm{X}}]).$$

Q.E.D.

II. X IS SMOOTH. (This is the case treated by Pohl [Ph 2].) — When X is smooth, $\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})$ is a bundle and the Jacobian ideal is trivial, so $\tilde{\mathrm{X}} = \mathrm{X}$ and $\tilde{\mathrm{P}} = \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})$ hold. Moreover, P is the image of the map $\pi^* a : \mathrm{V}_{\overline{\mathrm{X}}} \to \pi^* \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})$, so there is an induced isomorphism of line bundles

$$\Lambda^{r+1}\mathrm{P} \tilde{\to} (\Lambda^{r+1}\pi^* \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})) \otimes \mathrm{I},$$

where we have set $\mathrm{I} = \mathrm{F}^0(\mathrm{Coker}(\pi^* a))$. Recall that $\mathrm{Coker}(a) = \Omega_f \otimes \mathrm{L}$ holds, hence π is equal to the blow up of the ramification locus $\mathrm{R} \subseteq \mathrm{X}$ defined by the ideal $\mathrm{F}^0(\Omega_f)$.

Thus we face the same problems as in I when trying to get hold of the Chern classes of P, and we have to content ourselves with the following:

PROPOSITION (Pohl) (2.4). — *The* 1-*st polar class of a smooth r-dimensional variety* X, *with respect to a generic immersion* $f : \mathrm{X} \to \mathbf{P}^n$, *is given by*

$$[\mathrm{M}_1] = (c_1(\Omega_{\mathrm{X}}) + (r+1)c_1(\mathrm{L})) \cap [\mathrm{X}] - \sum_{\beta} e_\beta [\mathrm{R}_{r-1,\beta}],$$

where $\{\mathrm{R}_{r-1,\beta}\}_\beta$ *are the integral components of codimension* 1 *of the ramification locus* R *and* e_β *is the multiplicity of the ideal* $\mathrm{F}^0(\Omega_f)$ *in the local ring of* X *at the generic point of* $\mathrm{R}_{r-1,\beta}$.

Proof. — From the above isomorphism we get

$$c_1(\mathrm{P}) = \pi^* c_1(\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})) - c_1(\mathrm{I}^{-1}).$$

From the exact sequence [§ 1, (★)] $c_1(\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})) = c_1(\Omega_{\mathrm{X}} \otimes \mathrm{L}) + c_1(\mathrm{L})$ follows, hence $c_1(\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})) = c_1(\Omega_{\mathrm{X}}) + (r+1)c_1(\mathrm{L})$ holds, since Ω_{X} has rank r. Applying the projection formula we obtain

$$\pi_*(c_1(\mathrm{P}) \cap [\overline{\mathrm{X}}]) = (c_1(\Omega_{\mathrm{X}}) + (r+1)c_1(\mathrm{L})) \cap [\mathrm{X}] - \pi_*(c_1(\mathrm{I}^{-1}) \cap [\overline{\mathrm{X}}]).$$

The proof of the equality

$$\pi_*(c_1(\mathrm{I}^{-1}) \cap [\overline{\mathrm{X}}]) = \Sigma e_\beta [\mathrm{R}_{r-1,\beta}]$$

is the same as for the similar equality in the proof of Proposition (2.1) ([Kl 2], II D).

Q.E.D.

As in I, we also obtain an expression for the $[\mathrm{M}_k]$'s when f has codimension 1, i.e., when $r = n-1$, because then $c_k(\mathrm{P}) = c_1(\mathrm{P})^k$ holds. The proof is similar to the proof of Theorem (2.3).

THEOREM (2.5). — *If* $r = n-1$ *holds, the polar classes of* X *are given by*

$$[\mathrm{M}_k] = \sum_{j=0}^{k} \binom{k}{j} (r+1)^j c_1(\Omega_{\mathrm{X}})^{k-j} c_1(\mathrm{L})^j \cap [\mathrm{X}]$$
$$- \sum_{i=0}^{k-1} \binom{k}{i} \sum_{j=0}^{i} \binom{i}{j} (r+1)^j c_1(\Omega_{\mathrm{X}})^{i-j} c_1(\mathrm{L})^j \cap s_{r-k+i}(\mathrm{R},\mathrm{X}),$$

where $s_{r-k+i}(\mathrm{R}, \mathrm{X}) = -\pi_* (c_1(\mathrm{I})^{k-i} \cap [\overline{\mathrm{X}}])$ *denotes the Segre covariant classes of the ramification locus* $\mathrm{R} \subseteq \mathrm{X}$ *of* f.

Remark. — In the (classical) case when X is smooth and f is an immersion, there are formulas

$$[\mathrm{M}_k] = \sum_{i=0}^{k} \binom{r+1-k+i}{i} c_{k-i}(\Omega_{\mathrm{X}}) c_1(\mathrm{L})^i \cap [\mathrm{X}]$$

for the polar classes ([To], [E]). (These also hold when f is a generic projection of an embedding, see § 4.) One of Pohl's questions in [Ph 2] was to generalize these formulas to generic immersions. Thus (2.5) does this for a hypersurface, but again we do not have a result for the higher codimension case.

(To prove the above formula for $[\mathrm{M}_k]$, use the exact sequence $0 \to \Omega_{\mathrm{X}} \otimes \mathrm{L} \to \mathrm{P}^1_{\mathrm{X}}(\mathrm{L}) \to \mathrm{L} \to 0$ to get $c(\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})) = c(\Omega_{\mathrm{X}} \otimes \mathrm{L})\, c(\mathrm{L})$ and then use the formula for the Chern class of a bundle twisted by a line bundle to compute $c(\Omega_{\mathrm{X}} \otimes \mathrm{L})$.)

III. APPLICATION : A FORMULA FOR THE DOUBLE POINT CLASS. — Assume now that X is smooth, of dimension r, and that $f : \mathrm{X} \to \mathbf{P}^n$ is a generic immersion. Let $\mathrm{Z} = f(\mathrm{X})$ denote the scheme theoretical image of X, note that Z is reduced and r-dimensional. Assume that Z is a local complete intersection in $\mathbf{P}^n$ and let N denote its conormal bundle.

We want to apply the results of I and II to Z and X to obtain an expression for the double point class of the map f.

Let $\tilde{\pi} : \tilde{\mathrm{Z}} \to \mathrm{Z}$ denote the blow up of the Jacobian ideal $\mathrm{F}^r(\Omega_{\mathrm{Z}})$ and set $\mathrm{L} = \mathrm{O}_{\mathbf{P}^n}(1)\,|_{\mathrm{Z}}$. The map $a_{\mathrm{Z}} : \mathrm{V}_{\mathrm{Z}} \to \mathrm{P}^1_{\mathrm{Z}}(\mathrm{L})$ restricted to the smooth locus U of Z extends on $\tilde{\mathrm{Z}}$ to a quotient $\tilde{a} : \mathrm{V}_{\tilde{\mathrm{Z}}} \to \tilde{\mathrm{P}}$, and there is an isomorphism of line bundles on $\tilde{\mathrm{Z}}$,

$$\Lambda^{r+1} \tilde{\mathrm{P}} \cong \tilde{\pi}^* (\Lambda^{n-r} \mathrm{N}^\vee \otimes \mathrm{L}^{-n+r}) \otimes \mathrm{J},$$

where $\mathrm{J} = \mathrm{F}^r(\Omega_{\mathrm{Z}})\, \mathrm{O}_{\tilde{\mathrm{Z}}}$. (This follows from I.)

Let $\bar{\pi} : \overline{\mathrm{X}} \to \mathrm{X}$ denote the blow up of the ramification ideal $\mathrm{F}^0(\Omega_f)$ of f. On $\overline{\mathrm{X}}$ the image of the bundle map $a_{\mathrm{X}} : \mathrm{V}_{\mathrm{X}} \to \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})$ is a $(r+1)$-bundle, let $\bar{a} : \mathrm{V}_{\overline{\mathrm{X}}} \to \mathrm{P}$ denote this quotient. There is an isomorphism (see II):

$$\Lambda^{r+1} \mathrm{P} \cong \bar{\pi}^* (\Omega^r_{\mathrm{X}} \otimes f^* \mathrm{L}^{r+1}) \otimes \mathrm{I},$$

where we have set $\mathrm{I} = \mathrm{F}^0(\Omega_f)\, \mathrm{O}_{\overline{\mathrm{X}}}$ and $\Omega^r_{\mathrm{X}} = \Lambda^r \Omega_{\mathrm{X}}$.

PROPOSITION (2.6). — *The map* $f:\bar{\pi}:\mathrm{X}\to\mathrm{Z}$ *factors through* $\tilde{\pi}$, *via* $h:\bar{\mathrm{X}}\to\tilde{\mathrm{Z}}$ *say, and there is an isomorphism*

$$\bar{\pi}^*\Omega_{\mathrm{X}}^r \cong h^*\mathrm{J}\otimes\mathrm{I}^{-1}\otimes\bar{\pi}^*f^*(\Lambda^{n-r}\mathrm{N}^\vee\otimes\mathrm{L}^{-n-1}).$$

Proof. — Since Z is a local complete intersection, $\tilde{\pi}$ is minimal with respect to extending the quotient $a_{\mathrm{Z}}|_{\mathrm{U}}$ ($\tilde{\pi}$ is equal to the Nash blow up, see Proposition (1.4)). The map $f^*\,\mathrm{P}_{\mathrm{Z}}^1(\mathrm{L})\to\mathrm{P}_{\mathrm{X}}^1(f^*\,\mathrm{L})$ is generically an isomorphism (it is surjective where f is an immersion and it is generically injective since $\mathrm{P}_{\mathrm{Z}}^1(\mathrm{L})$ is generically a bundle). Therefore the quotient $\bar{a}:\mathrm{V}_{\bar{\mathrm{X}}}\to\mathrm{P}$ extends $a_{\mathrm{Z}}|_{\mathrm{U}}$ and the map h exists.

In order to establish the isomorphism it suffices to show there is an isomorphism $h^*\,\tilde{\mathrm{P}}\cong\mathrm{P}$, because of the isomorphisms given for $\Lambda^{r+1}\,\tilde{\mathrm{P}}$ and $\Lambda^{r+1}\,\mathrm{P}$ above. But the two $(r+1)$-quotients $\bar{a}$ and $h^*\,a$ are generically isomorphic on $\bar{\mathrm{X}}$, hence they are everywhere isomorphic. The last statement follows from the following general result.

LEMMA (2.7). — *Let* E *be a bundle on a scheme* Y. *Suppose* $a:\mathrm{E}\to\mathrm{F}$ *and* $b:\mathrm{E}\to\mathrm{F}'$ *are quotient bundles that are isomorphic on an open subscheme* U *containing* Ass (Y). *Then* a *and* b *are isomorphic on* Y.

Proof. — Put m = rank F = rank F, and G = Grass_m (E). The quotients a and b define maps f and g from Y to G. Set $h=(f,g):\mathrm{Y}\to\mathrm{G}\underset{\mathrm{Y}}{\times}\mathrm{G}$. Since G is separated over Y, the diagonal $\Delta\subseteq\mathrm{G}\underset{\mathrm{Y}}{\times}\mathrm{G}$ is closed. Hence $h^{-1}\,\Delta$ is closed in Y and contains U which contains Ass (Y). Hence $h^{-1}\,\Delta=\mathrm{Y}$ holds, and $f=g$ on Y.

Q.E.D.

COROLLARY (2.8). — *There is a formula*

$$(f^*c_1(\omega_{\mathrm{Z}})-c_1(\omega_{\mathrm{X}}))\cap[\mathrm{X}]=\bar{\pi}_*(c_1(h^*\mathrm{J}^{-1}\otimes\mathrm{I})\cap[\bar{\mathrm{X}}]),$$

relating the 1-st *Chern classes of the* dualizing sheaves ω_{X} *and* ω_{Z} *on* X *and* Z *to the (codimension* 1 *part of the* double point class *of* f.

Proof. — Since X is smooth, $\omega_{\mathrm{X}}=\Omega_{\mathrm{X}}^r$ holds. Since Z is a local complete intersection in $\mathbf{P}^n$, $\omega_{\mathrm{Z}}=\Lambda^{n-r}\,\mathrm{N}^\vee\otimes\Omega_{\mathbf{P}^n\,|\mathrm{Z}}^n=\Lambda^{n-r}\,\mathrm{N}^\vee\otimes\mathrm{L}^{-n-1}$ holds ([A-K], I, 4.5). We apply the projection formula to the equality of cycles obtained from the isomorphism of the Proposition.

Q.E.D.

Remark. — Both the Proposition and its Corollary hold when $\mathbf{P}^n$ is replaced by any smooth n-dimensional scheme Y. Instead of the maps a_{Z} and a_{Y} one uses $f^*\,\mathrm{P}_{\mathrm{Y}}^1\to\mathrm{P}_{\mathrm{X}}^1$ and $\mathrm{P}_{\mathrm{Y}}^1|_{\mathrm{Z}}\to\mathrm{P}_{\mathrm{Z}}^1$. Note that there is a canonical isomorphism $\mathrm{V}_{\mathbf{P}^n}\xrightarrow{\sim}\mathrm{P}_{\mathbf{P}^n}^1(1)$, which is used in the construction of a_{X} and a_{Z} (see [Pi], 6.3). Note also that $\mathrm{O}_{\mathbf{P}^n}(-n-1)$ is isomorphic to $\Omega_{\mathbf{P}^n}^n$; the term L^{-n-1} appearing in the Proposition should be replaced by $\Omega_{\mathrm{Y}}^n|_{\mathrm{Z}}$.

The definition of double point class suggested by Corollary (2.8) is the following (à la Segre covariant classes):

$$\mathrm{D}_i=-\bar{\pi}_*(c_1(h^*\mathrm{J}\otimes\mathrm{I}^{-1})^{r-i}\cap[\bar{\mathrm{X}}]),$$

for $i=0,1,\ldots,r-1$.

Various definitions of the double point class of a morphism of schemes have been given (for a survey, see [Kl 2], Chap. V). Here we shall not try to relate ours to these in general. We shall, however, show that in the case f is finite our (codimension 1 part) double point class is equal to the class of the double point cycle as defined by the *conductor* of X in Z. As a start, let us look at the case of *curves*.

Let Z be a Gorenstein curve and let $f: X \to Z$ denote its normalization. Set $C = \mathcal{H}om_{O_Z}(f_* O_X, O_Z)$, it is the *conductor* of X in Z. Rosenlicht ([Rs], Th. 8, p. 177) showed that there is a nonsingular pairing

$$f_* O_X/O_Z \times \omega_Z/f_* \Omega_X \to k,$$

where ω_Z is the dualizing sheaf on Z. Since we can also write $C = \mathrm{Ann}_{O_Z}(f_* O_X/O_Z)$ the above pairing shows that there is an isomorphism $f_* \Omega_X \cong C \otimes \omega_Z$, hence also $\Omega_X \cong \tilde{C} \otimes f^* \omega_Z$, where we put $\tilde{C} = CO_X$. The divisor on X defined by the conductor is an obvious candidate for the double point cycle of the map f.

If we assume that Z is not only Gorenstein but a local complete intersection in a smooth scheme Y, we are in the situation discussed above. First we observe that the ideals $J = F^1(\Omega_Z) O_X$ and $I = F^0(\Omega_f)$ are invertible on X, since any codimension 1 cycle on a smooth curve is a (Cartier) divisor. From Proposition (2.6) we get

$$\Omega_X \cong J \otimes I^{-1} \otimes f^* \omega_Z, \qquad \text{hence} \quad C = J \otimes I^{-1}.$$

This justifies the name "double point class" for $c_1(J^{-1} \otimes I)$.

The isomorphism $\Omega_X \cong \tilde{C} \otimes f^* \omega_Z$ can be generalized ([Kl], (V, 7)) by applying duality for finite maps to the case of a finite, birational map $f: X \to Z$ between Gorenstein schemes. We obtain:

$$\omega_X \cong \tilde{C} \otimes f^* \omega_Z.$$

In particular, $\tilde{C}$ is *invertible*, since the dualizing sheaf of a Gorenstein scheme is. As a digression let us use this to prove the following result.

PROPOSITION (2.9). — *Let $f: X \to Z$ be a finite, birational map between Gorenstein schemes. Then f is isomorphic to the blow up of the conductor* C *of* X *in* Z.

Proof. — Put $\tilde{C} = CO_X$. We have seen that $\tilde{C}$ is invertible, so that f factors through the blow up $g: Z' \to Z$ of C, via $h: X \to Z'$ say. It suffices to show that h is an isomorphism over each point $z \in Z$.

Let $z \in Z$, put $A = O_{Z,z}$, $A' = O_{Z', g^{-1}(z)}$, $B = O_{X, f^{-1}(z)}$. Note that the conductor $C \subseteq A$ is the largest ideal in A which is also an ideal in B. Let $c \in A$ be a generator of the invertible ideal CA'. If $b \in B$, then $cb \in C \subseteq A$. Consider $cb \in CA'$, so there is an $a \in A'$ with $cb = ca$. Since c is a non zero divisor in A' this implies $a = b$, hence $b \in A'$.

Q.E.D.

Returning to the situation of a finite map $f: X \to Y$, with $Z = f(X)$ a local complete intersection, $X \to Z$ birational, and X smooth, we obtain an isomorphism

$$\bar{\pi}^* C \cong h^* J \otimes I^{-1},$$

hence also an equality,

$$\bar{\pi}_*(c_1(h^* \mathrm{J}^{-1} \otimes \mathrm{I}) \cap [\bar{\mathrm{X}}]) = c_1(\mathrm{C}^{-1}) \cap [\mathrm{X}],$$

for (the codimension 1 part of) the double point cycle class in this case.

3. Numerical formulas

The results of the previous sections give relations among the *numerical characters* associated to a generic immersion $f : \mathrm{X} \to \mathbf{P}^n$: We define *the k-th class* μ_k of X (with respect to f) to be the degree of the k-th polar class $[\mathrm{M}_k]$. In particular, μ_0 is the *degree of* X. For $0 \leqq k \leqq r$ we have (by definition):

$$\mu_k = \int c_1(\mathrm{L})^{r-k} \cap [\mathrm{M}_k],$$

and μ_k is equal to the number of points (counted properly if char $k > 0$) on a section of X by $r-k$ general hyperplanes at which the tangent space, or a limit of such, meets a given linear $(n-r+k-2)$-space in a space of at least $k-1$ dimensions.

The top class μ_r is called *the class* of X. By the Transversality Lemma (1.3) a general M_r consists of a finite number of points contained in the open subscheme U where X is smooth and where f is an immersion. Each point occurs with multiplicity 1 if char $k = 0$; if char $k = p > 0$ holds, the multiplicity is equal to p^e for some $e \geqq 0$.

The class of X is equal to the number (counted properly) of smooth points of X at which the tangent space, embedded by f, meets a given $(n-2)$-space in a space of at least $r-1$ dimensions. This is the same as the number (counted properly) of tangent spaces contained in a hyperplane of a general pencil (i. e., a general line in the dual projective space $\mathbf{P}^{n\vee}$).

Earlier (§ 1) we considered the *tangent map* $\Phi : \bar{\mathrm{X}} \to \mathrm{G} = \mathrm{Grass}_{r+1}(\mathrm{V})$ defined by the quotient $\bar{a} : \mathrm{V}_{\bar{\mathrm{X}}} \to \mathrm{P}$. Let $i : \mathrm{G} \hookrightarrow \mathbf{P}(\Lambda^{r+1}\mathrm{V})$ denote the Plücker embedding, then $\Lambda^{r+1}\bar{a}$ is the pullback to $\bar{\mathrm{X}}$ via $i \circ \Phi$ of the tautological 1-quotient on $\mathbf{P}(\Lambda^{r+1}\mathrm{V})$. In particular, $c_1(\mathrm{P})$ represents the pullback of a hyperplane section of $\bar{\mathrm{X}}$ via $i \circ \Phi$, and $\int c_1(\mathrm{P})^r$ is thus the degree of $\bar{\mathrm{X}}$ with respect to $i \circ \Phi$. Set $\mu_\Phi = \int c_1(\mathrm{P})^r$. Whenever we have a formula for $c_1(\mathrm{P})$ we obtain one for μ_Φ, this happens for example in the situations of I or II of section 2. (In the complex analytic case, if X is smooth, there is also another interpretation of μ_Φ. Pohl ([Ph 2], Prop. 4) showed that $((2\pi)^n/n!)\,\mu_\Phi$ is equal to the volume swept out by Φ in G.)

There is another map, however, associated to the quotient $\bar{a}$; this is the dual map $\check{f}$, which we will now describe.

Set $\mathrm{K} = \mathrm{Ker}(\bar{a})$. The surjection $\mathrm{V}^\vee_{\bar{\mathrm{X}}} \to \mathrm{K}^\vee$ defines a closed embedding

$$\mathbf{P}(\mathrm{K}^\vee) \hookrightarrow \mathbf{P}(\mathrm{V}^\vee_{\bar{\mathrm{X}}}) = \bar{\mathrm{X}} \times \mathbf{P}^{n\vee}.$$

The *dual map* of f is the map

$$f^{\vee} : \mathbf{P}(\mathrm{K}^{\vee}) \to \mathbf{P}^{n\vee}$$

obtained by composing the above embedding with projection on the second factor. The *dual variety* of X is defined to be the image $\check{\mathrm{X}} = f^{\vee}(\mathbf{P}(\mathrm{K}^{\vee}))$. We let $p : \mathbf{P}(\mathrm{K}^{\vee}) \to \overline{\mathrm{X}}$ denote the structure map.

As before, let $\mathrm{U} \subseteq \mathrm{X}$ denote the largest open subscheme where X is smooth and f is an immersion. Since $\bar{a}$ extends $a|_{\mathrm{U}} : \mathrm{V}_{\mathrm{U}} \to \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})|_{\mathrm{U}}$, the fiber of p over a point $x \in \mathrm{U}$ consists of hyperplanes $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$ containing the tangent space $\mathbf{P}(\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})(x))$ to X at x. Therefore, the dual variety X is (as it should be) equal to the closure in $\mathbf{P}^{n\vee}$ of all hyperplanes tangent to X at points of U.

The map p is smooth of relative dimension $n-r-1$, so $\mathbf{P}(\mathrm{K}^{\vee})$ has dimension $n-1$. Its degree, via $f^{\vee}$, is equal to $\int c_1(f^{\vee *}\mathrm{O}_{\mathbf{P}^{n\vee}}(1))^{n-1}$, hence to $\int c_1(\mathrm{O}_{\mathbf{P}(\mathrm{K}^{\vee})}(1))^{n-1}$. This is the same as $\int p_*(c_1(\mathrm{O}_{\mathbf{P}(\mathrm{K}^{\vee})}(1))^{n-1} \cap [\mathbf{P}(\mathrm{K}^{\vee})])$, which by general properties of Chern and Segre classes is equal to $\int s_r(\mathrm{K}^{\vee}) \cap [\overline{\mathrm{X}}]$ or to $\int c_r(\mathrm{P}) \cap [\overline{\mathrm{X}}]$.

Let $d^{\vee}$ denote the degree of the dual variety $\check{\mathrm{X}}$ in $\mathbf{P}^{n\vee}$. We have shown

$$\mu_r = \begin{cases} \deg \check{f} \,.\, \check{d} & \text{if } \dim \check{\mathrm{X}} = n-1, \\ 0 & \text{if } \dim \check{\mathrm{X}} < n-1. \end{cases}$$

When X has dimension $n-1$ or 1, its class μ_r is equal to μ_Φ, the degree of $\overline{\mathrm{X}}$ via the tangent map Φ. (If $r = n-1$, $\mathrm{K}^{\vee}$ is invertible, so $c_r(\mathrm{P}) = c_1(\mathrm{P})^r$ holds, because of the exact sequence $0 \to \mathrm{K} \to \mathrm{V}_{\overline{\mathrm{X}}} \to \mathrm{P} \to 0$. If $r = 1$, both μ_1 and μ_Φ are equal to $\int c_1(\mathrm{P})$.) In general μ_r and μ_Φ are not equal.

Example (3.1). — Let $\mathrm{X} \subseteq \mathbf{P}^4$ be a smooth surface and suppose $\mathrm{X} = \mathrm{X}_1 \cap \mathrm{X}_2$, where X_i is a hypersurface of degree d_i. The characters of X are

$$\begin{aligned} \mu_0 &= d_1 d_2, \\ \mu_1 &= d_1 d_2 (d_1 + d_2 - 2), \\ \mu_2 &= d_1 d_2 ((d_1-1)^2 + (d_1-1)(d_2-1) + (d_2-1)^2), \\ \mu_\Phi &= d_1 d_2 (d_1 + d_2 - 2)^2. \end{aligned}$$

The normal bundle of X is equal to $\mathrm{L}^{d_1} \oplus \mathrm{L}^{d_2}$, where $\mathrm{L} = \mathrm{O}_{\mathbf{P}^4}(1)|_{\mathrm{X}}$. Since X is smooth, we get

$$\mathrm{K}^{\vee} = (\mathrm{L}^{d_1} \oplus \mathrm{L}^{d_2}) \otimes \mathrm{L}^{-1} = \mathrm{L}^{d_1-1} \oplus \mathrm{L}^{d_2-1},$$

hence

$$c_1(\mathrm{P}) = c_1(\mathrm{K}^{\vee}) = (d_1-1)(d_2-1)\, c_1(\mathrm{L})$$

and

$$c_2(\mathrm{P}) = s_2(\mathrm{K}^{\vee}) = c_1(\mathrm{K}^{\vee})^2 - c_2(\mathrm{K}^{\vee})$$
$$= ((d_1-1)^2 + (d_1-1)(d_2-1) + (d_2-1)^2)\, c_1(\mathrm{L}).$$

Observing that $\mu_0 = \int c_1(\mathrm{L})^2 = d_1 d_2$ holds, we obtain the above numerical expressions (which agree with [Ba], p. 175).

Example (3.2). — Let $\mathrm{X} \subseteq \mathbf{P}^3$ be a space curve, and suppose $\mathrm{X} = \mathrm{X}_1 \cap \mathrm{X}_2$, where X_i is a surface of degree d_i. Let $\overline{\mathrm{X}} \to \mathrm{X}$ denote the normalization of X and $\mathrm{J} = \mathrm{F}^1(\Omega_{\mathrm{X}})\, \mathrm{O}_{\overline{\mathrm{X}}}$ the pullback of the Jacobian ideal. Then:

$$\mu_1 = \mu_{\Phi} = (d_1 + d_2 - 2)\, d_1 d_2 - e,$$

where e is the degree of the divisor on $\overline{\mathrm{X}}$ defined by J^{-1}.

Let $\mathrm{I} = \mathrm{F}^0(\Omega_{\overline{\mathrm{X}}/\mathrm{X}})$ denote the ramification ideal and set $\varkappa = \deg(\mathrm{I}^{-1})$. Then (from II) we also get:

$$\mu_1 = 2\, d_1 d_2 + 2g - 2 - \varkappa,$$

where g denotes the (geometric) genus of X, and

$$2g - 2 = (d_1 + d_2 - 4)\, d_1 d_2 - (e - \varkappa),$$

where $e - \varkappa = 2\,\delta$ is the degree of the conductor (see § 2, III).

Again, let $f : \mathrm{X} \to \mathbf{P}^n$ be a generic immersion, and $\check{\mathrm{X}} \subseteq \mathbf{P}^{n\vee}$ the dual variety of X. Then it is known that the dual variety $\check{\check{\mathrm{X}}} \subseteq \mathbf{P}^n$ of $\check{\mathrm{X}}$ is equal to $f(\mathrm{X})$ if and only if the map $\check{f}$ is separable ([Wl], § 3, p. 326). In this case we say that *biduality holds* for X (or for f). In characteristic 0 biduality always holds. (Note that the dual variety of $f(\mathrm{X}) \subseteq \mathbf{P}^n$ is also equal to $\check{\mathrm{X}}$.)

For the rest of this section we will restrict ourselves to *the case where* $f : \mathrm{X} \subsetneq \mathbf{P}^n$ *is a hypersurface*. Using biduality together with "minimality" of the blow up $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ of the Jacobian ideal, we can prove the following Proposition, suggested by B. Teissier.

PROPOSITION (3.3). — *Let* $f : \mathrm{X} \subsetneq \mathbf{P}^n$ *be a hypersurface which satisfies biduality and such that* $\check{\mathrm{X}}$ *is also a hypersurface. Then there is a diagram*

$$\begin{tikzcd} & \tilde{\mathrm{X}} \arrow[dl, "\pi"'] \arrow[dr, "\check{\pi}"] & \\ \mathbf{P}^n \supseteq \mathrm{X} & & \check{\mathrm{X}} \subseteq \mathbf{P}^{n\vee} \end{tikzcd}$$

where $\check{\pi}$ (*resp.* π) *is the dual map of* X (*resp.* $\check{\mathrm{X}}$), *and where* π (*resp.* $\check{\pi}$) *is isomorphic to the blow up of the Jacobian ideal of* X (*resp.* $\check{\mathrm{X}}$).

Proof. — Since X is a hypersurface, $K = \ker(\bar{a})$ has rank 1, so $p : \mathbf{P}(K^\vee) \to \tilde{X}$ is an isomorphism. Moreover, the diagram

$$\begin{array}{ccc} \mathbf{P}(K^\vee) & \xrightarrow{f^\vee} & \mathbf{P}^{n\vee} \\ \downarrow\wr & & \| \wr \\ \tilde{X} & \xrightarrow{\Phi} & G = \mathrm{Grass}_n(V) \end{array}$$

commutes. The induced map $\check{\pi} : \tilde{X} \to \mathbf{P}^{n\vee}$ is thus isomorphic to the dual of $f : X \subsetneq \mathbf{P}^n$. Let $\tilde{\pi} : \tilde{\check{X}} \to \check{X}$ denote the blow up of the Jacobian ideal of $\check{X}$ and $(f^\vee)^\vee : \tilde{\check{X}} \to X$ the dual map. Since the dual map (of $\check{X}$) is also defined on $\tilde{X}$ by π, it follows from Proposition (1.4) that $\check{\pi}$ factors through $\tilde{\pi}$. The symmetric argument shows that $f^{\vee\vee}$ factors through π. We conclude that $f^{\vee\vee}$ and π are isomorphic.

Q.E.D.

COROLLARY (3.4). — *In the situation of the Proposition, suppose there exists a desingularization* $g : Z \to \tilde{X}$ *of* $\tilde{X}$. *Then the ideal* $H = F^0(\Omega_{Z/X})$ *and* $H^* = F^0(\Omega_{Z/\check{X}})$ *are invertible and there is an equality*

$$c_1(H^{*-1}) - c_1(H^{-1}) = (n+1)\, g^*(c_1(\check{\pi}^* L^*) - c_1(\pi^* L)),$$

where

$$L = O_{\mathbf{P}^n}(1)\,|_X \qquad \textit{and} \qquad L^* = O_{\mathbf{P}^{n\vee}}(1)\,|_{\check{X}}.$$

Proof. — Consider the map $b : (\pi \circ g)^* P^1_X(L) \to P^1_Z((\pi \circ g)^* L)$, and note

$$\mathrm{Coker}(b) = \mathrm{Coker}((\pi \circ g)^*(\Omega_X \otimes L) \to \Omega_Z \otimes (\pi \circ g)^* L) = \Omega_{Z/X} \otimes (\pi \circ g)^* L.$$

I claim that b factors through the quotient $c : g^* \pi^* P^1_X(L) \to g^* P$. In fact, $\mathrm{Ker}(b) = \mathrm{Ker}(c)$ holds, because $c(\mathrm{Ker}\, b)$ and $b(\mathrm{Ker}\, c)$ are both torsion submodules of bundles, hence are 0.

From the above it follows that H is equal to the 0-th Fitting ideal of an injective map of n-bundles $g^* P \to P^1_Z((\pi \circ g)^* L)$, hence is invertible. Moreover, there is an equality

$$c_1(H^{-1}) = c_1(P^1_Z((\pi \circ g)^* L) - c_1(g^* P),$$

hence:

$$c_1(H^{-1}) = c_1(P^1_Z) + n c_1((\pi \circ g)^* L) - c_1(g^* \check{\pi}^* L^*).$$

In the same way we obtain

$$c_1(H^{*-1}) = c_1(P^1_Z) + n c_1((\check{\pi} \circ g)^* L^*) - c_1(g^* \pi^* L).$$

By eleminating $c_1(P^1_Z)$ we obtain the desired equality.

Q.E.D.

Example (3.5). — Let $X \subseteq \mathbf{P}^2$ be a plane curve of degree d and class $d^\vee$, which satisfies biduality, and let $Z \to X$ denote its normalization. Then the numerical relation obtained from the above corollary is

$$\iota - \varkappa = 3(d^\vee - d),$$

where $\varkappa$ is the *number of cusps* of X and ι its *number of flexes* (the flexes of X are the cusps of its dual $\check{X}$, see ([Pi], § 5)).

We have already seen that whenever $f^\vee$ is birational, the class μ_r of X is equal to the degree $d^\vee$ of its dual variety $\check{X}$. If in addition X is a hypersurface, so that the conditions of the previous proposition are satisfied, we can conclude more.

PROPOSITION (3.6). — *In the situation of Proposition* (3.3) *the k-th class of* X *is equal to the* $(r-k)$*-th class of* $\check{X}$.

Proof. — From the existence of the commutative diagram in the proof of (3.3), it follows that the map $\check{\pi} : \tilde{X} \to \mathbf{P}^{n\vee}$ is defined by the 1-quotient $V_{\tilde{X}}^{\vee} \to K^\vee$. Therefore this 1-quotient is the pullback via $\check{\pi}$ of the tautological 1-quotient $V_{\mathbf{P}^{n\vee}}^{\vee} \to O_{\mathbf{P}^{n\vee}}(1)$. Set $L^* = O_{\mathbf{P}^{n\vee}}(1)|_{\tilde{X}}$. By definition, $\mu_k = \deg \pi_* (c_k(P) \cap [\tilde{X}])$, hence:

$$\begin{aligned}\mu_k &= \int c_1(L)^{r-k} \cap \pi_*(c_k(P) \cap [\tilde{X}]) \\ &= \int \pi^* c_1(L)^{r-k} . c_1(P)^k \\ &= \int \pi^* c_1(L)^{r-k} . \check{\pi}^* c_1(L^*)^k.\end{aligned}$$

But this is clearly the same as the $(r-k)$-th class of $\check{X}$, because the same argument applies to $\check{X}$.

Q.E.D.

We end this section with a formula for the classes of a hypersurface, and then we look at some examples. In particular we apply our results to the case of plane curves, thus obtaining the classical Plücker equations.

The numerical version of Theorem (2.3) is:

COROLLARY (3.7). — *Let* $X \subseteq \mathbf{P}^n$ *be a hypersurface of degree d. The k-th class of* X *is given by*

$$\mu_k = d(d-1)^k - \sum_{i=0}^{k-1} \binom{k}{i} (d-1)^i \sigma_{r-k+i},$$

where $\sigma_{r-k+i} = \int c_1(L)^{r-k+i} \cap s_{r-k+i}(S, X)$ *is the degree of the* $(r-k+i)$*-th Segre covariant class of the singular subscheme* S *of* X.

COROLLARY (3.8) (Teissier [T], II.7; [L]). — *If the hypersurface* X *has only isolated singularities, its class is given by*

$$\mu_{n-1} = d(d-1)^{n-1} - \sum_{x \in S} e_x,$$

where e_x denotes the multiplicity of the Jacobian ideal at a singular point x.

Proof of (3.8). — Since $s_i(\mathrm{S}, \mathrm{X}) = 0$ for $i > \dim \mathrm{S}$, the sum of (3.7) reduces to σ_0 when $\dim \mathrm{S} = 0$. The equality $\sigma_0 = \sum_{x \in \mathrm{S}} e_x$ follows from ([Kl 2], (II,50)).

Q.E.D.

Example (3.9). —*Plücker formulas for plane curves.* — Let $\mathrm{X} \subseteq \mathbf{P}^2$ be a curve of degree d, and let $\pi : \overline{\mathrm{X}} \to \mathrm{X}$ denote its normalization. If $x \in \mathrm{X}$ is a singular point, we set e_x equal to the multiplicity of the Jacobian ideal $\mathrm{F}^1(\Omega_{\mathrm{X}})$ at x. For each $y \in \overline{\mathrm{X}}$ we set $2\,\delta_y$ equal to the degree of the divisor defined by the conductor at y, and we let $\varkappa_y$ denote the degree of the divisor defined by the ramification ideal $\mathrm{F}^0(\Omega_\pi)$ at y. In particular, $e_x = \sum_{y \mapsto x} (2\,\delta_y + \varkappa_y)$ holds (§ 2, III).

From the results of section 2 (I, II, III) we obtain the following formulas for the class μ_1 and (geometric) genus g of X.

$$\mu_1 = d(d-1) - \sum_{x \in \mathrm{S}} e_x, \tag{I}$$

$$\mu_1 = 2d + 2g - 2 - \sum_{y \in \mathrm{R}} \varkappa_y, \tag{II}$$

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{y \in \overline{\mathrm{X}}} \delta_y. \tag{III}$$

When biduality holds for X, $\mu_1 = d^\vee$ is the degree of the dual curve $\check{\mathrm{X}}$ and we obtain 3 dual equations:

$$d = d^\vee(d^\vee - 1) - \sum_{x \in \mathrm{S}^\vee} \check{e}_x, \tag{I$^\vee$}$$

$$d = 2d^\vee + 2g - 2 - \sum_{y \in \mathrm{R}^\vee} \check{\varkappa}_y, \tag{II$^\vee$}$$

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{y \in \overline{\mathrm{X}}} \check{\delta}_y, \tag{III$^\vee$}$$

where we have used the obvious notations.

If X has as singularities only D ordinary double points and K simple cusps, then one shows (3.10) (char $k \neq 2, 3$):

$$\sum_{x \in \mathrm{S}} e_x = 2\mathrm{D} + 3\mathrm{K},$$

$$\sum_{y \in \mathrm{R}} \varkappa_y = \mathrm{K},$$

$$\sum \delta_y = \mathrm{K} + \mathrm{D},$$

and the formulas (I)-(III) reduces to the wellknown Plücker formulas (see the following Remark). Also, if biduality holds, one interprets the singularities of $\check{\mathrm{X}}$ in terms of *tangential singularities* of X. For example, the cusps on $\check{X}$ are the flexes of X ([Pi], § 5).

In general, local computations give the value of the integers e_x, $\varkappa_y$, $2\,\delta_y$ associated to the singular points of X. Let us show how this is done in 2 particular cases.

LEMMA (3.10). — (i) *If $x \in X$ is an ordinary j-multiple point, then $e_x = j(j-1)$. For $y \in \overline{X}$, $\pi(y) = x$, $2\,\delta_y = j-1$ and $\varkappa_y = 0$ hold.* (ii) *If X has a cuspidal branch at $x = \pi(y)$, $y \in \overline{X}$, of the form (a, b), where $(a, b) = 1$, $a, b \neq 0$ in k, and $a < b$, then $e_x = (a-1)\,b$, $\delta_y = (1/2)(a-1)(b-1)$, and $\varkappa_y = (a-1)$.*

Proof. — (i) Since π is not ramified at any $y \in \pi^{-1}(x)$, $\varkappa_y = 0$ holds. It is well known ([Hi], Th. 1) that $\sum_{y \to x} 2\,\delta_y = j(j-1)$ holds; since $\# \pi^{-1}(x) = j$, one obtains $2\,\delta_y = j-1$, hence $e_x = j(j-1)$. (Alternatively, e_x can be computed directly as follows. Write $\hat{O}_{X,x} = k[[x, y]]/(l_1 \ldots l_j)$, where the l_i's are linear forms in x and y satisfying $(l_i, l_s) = (x, y)\,\hat{O}_{X,x}$ for all $i \neq s$. The Jacobian ideal is the ideal generated by the partial derivatives of $l_1 \ldots l_j$. If we write $\hat{O}_{\overline{X},y} = k[[t]]$, it follows that the Jacobian ideal induces the ideal t^{j-1} in $\hat{O}_{\overline{X},y}$.)

(ii) (D. Grayson). By assumption $\hat{O}_{X,x} \cong k[[t^a, t^b]]$. The Jacobian ideal is the ideal $(at^{(a-1)b}, bt^{a(b-1)})$, hence it induces $(t^{(a-1)b})$ in $\hat{O}_{\overline{X},y} \cong k[[t]]$. The ramification ideal is computed by the presentation $\pi^* \Omega_{\mathbf{P}^2} \to \Omega_{\overline{X}}$ and is equal to (at^{a-1}, bt^{b-1}). Finally we get $2\,\delta_y = e_x - \varkappa_y = (a-1)\,b-(a-1) = (a-1)(b-1)$.

Q.E.D.

Remark. — The formula for the class of a plane curve (and the dual formula) were first found by Poncelet ([Po], p. 68). (Poncelet made an error in asserting that the presence of a j-multiple point diminished the class by j, instead of by $j(j-1)$. He stated that the presence of an ordinary cusp diminished the class by a number $\geqq 2$, but he did not give 3 as this number.)

The formulas were improved (and corrected) by Plücker ([Pl], p. 200), who also added a third relation, giving the number of flexes (see [Be], p. 342; [E-C], p. 122). The formulas were shortly after generalized from curves with simple singularities to curves with arbitrary singularities (see e. g. [Wk], p. 119).

Example (3.9) shows that our formula (3.7) does indeed generalize the Plücker formula (I) $\mu_1 = d(d-1)-2\,D-3\,K$ to an arbitrary hypersurface. Thus we answer Pohl's question in ([Ph 2], p. 29) in this case (but we have not succeeded in doing this for $X \subseteq \mathbf{P}^n$ of larger codimension).

Example (3.11). — Suppose X is a smooth surface, $f : X \to \mathbf{P}^3$ a proper generic immersion. As in section 1, let $\pi : \overline{X} \to X$ denote the blow up of the ramification ideal $F^0(\Omega_f)$ and $\tilde{\pi} : \tilde{X} \to f(X)$ the blow up of the Jacobian ideal $F^2(\Omega_{f(X)})$. With the notations of section 3 we obtain from Corollary (3.7):

$$\mu_1 = \mu_0(\mu_0-1)-\sigma_1,$$

$$\mu_2 = \mu_0(\mu_0-1)^2-2(\mu_0-1)\sigma_1-\sigma_0.$$

Consider the case X = union of 3 planes and $f(X)$ = 3 planes intersecting transversally. Then $f(X)$ has a double curve Γ_0 consisting of 3 lines meeting in a triple point, and $\Gamma = f^{-1}\Gamma_0$ consists of 3 copies of two crossing lines. We obtain $\mu_0 = 3$, $\sigma_1 = \deg\Gamma = 6$, hence $\mu_1 = 0$. The dual map $f^\vee : X \to \mathbf{P}^{3\vee}$ maps X to 3 points, hence $\mu_2 = 0$, and we get:

$$\sigma_0 = 12-24 = -12,$$

which agrees with the fact that σ_0 is equal to minus the self-intersection of Γ, which is $-3\cdot 4 = -12$.

Remark. — In the case that X is a smooth variety embedded in $\mathbf{P}^n$, the degree $d^\vee$ of its dual variety was computed by Katz ([SGA 7 II], p. 240) and later by Roberts ([Rb 2], Th. 1). The formula then is

$$\mu_r = (\deg f^\vee)\, d^\vee = \int c_r(\mathrm{P}^1_\mathrm{X}(\mathrm{L})) = \int s_r(\mathrm{N}^{-1}\otimes \mathrm{L}^{-1}) = \sum_{i=0}^{r}(i+1)\int c_{r-i}(\Omega_\mathrm{X})\, c_1(\mathrm{L})^i.$$

A proof, similar to the one in ([SGA 7 II]), goes as follows: Use (★) of section 3 together with $[\mathrm{M}_r] = c_r(\mathrm{P}^1_\mathrm{X}(\mathrm{L})) \cap [\mathrm{X}]$ to obtain the first two equalities. The last one follows from the equality $c(\mathrm{P}^1_\mathrm{X}(\mathrm{L})) = c(\Omega_\mathrm{X}\otimes \mathrm{L})\, c(\mathrm{L})$ and the formula for the Chern class of a bundle twisted by a line bundle.

Since the μ_k's are *projective* characters (see § 4), the above formula holds for any generic projection of $\mathrm{X} \subseteq \mathbf{P}^n$. In particular, the class of a surface with ordinary singularities is given by

$$\mu_2 = \int c_2(\Omega_\mathrm{X}) + 2\int c_1(\Omega_\mathrm{X})\, c_1(\mathrm{L}) + 3\mu_0.$$

4. Projections and hyperplane sections

In this section we study the behavior of the polar loci of a variety under projections and linear space sections.

The two main results are (1) the polar classes are *projective invariants* (i. e., they are invariant under generic projections), (2) the k-th class μ_k of a variety X is equal to the k-th class of the intersection of X with a general linear space of codimension $\leqq \dim \mathrm{X}-k$.

It follows that each μ_k can be considered as the class of a k-dimensional hypersurface derived from the given variety via sections and projections.

Again we consider an r-dimensional variety X and a proper generic immersion $f : \mathrm{X} \to \mathbf{P}^n$.

THEOREM (4.1). — *Let $p : \mathrm{X} \to \mathbf{P}^m$ be a generic projection of f, with $r+1 \leqq m \leqq n$. Then the polar classes of* X *with respect to p are the same as those with respect to f.*

Proof. — Write $\mathbf{P}^n = \mathbf{P}(\mathrm{V})$ and let $\mathrm{U} \subseteq \mathrm{X}$ denote the largest open subscheme where X is s smooth and f is an immersion. Recall (§1) that the k-th polar locus M^f_k of X, with respect to f and with respect to a $(r-k+2)$-dimensional subspace $\mathrm{V}' \subseteq \mathrm{V}$, was defined as the closure of $\mathrm{M}^f_k(\mathrm{U})$, where $\mathrm{M}^f_k(\mathrm{U})$ is the scheme of zeros of the map

$$\Lambda^{r-k+2}\mathrm{V}'_\mathrm{U} \to \Lambda^{r-k+2}\mathrm{P}^1_\mathrm{X}(\mathrm{L})\,|_\mathrm{U}$$

obtained by composing $\Lambda^{r-k+2}\, a_f\,|_{\mathrm{U}} : \Lambda^{r-k+2}\,\mathrm{V}_{\mathrm{U}} \to \Lambda^{r-k+2}\,\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})\,|_{\mathrm{U}}$

with $\Lambda^{r-k+2}\,\mathrm{V}' \subseteq \Lambda^{r-k+2}\,\mathrm{V}$.

A $(m+1)$-subspace W of V defines a (linear) *projection* $\mathbf{P}^n = \mathbf{P}(\mathrm{V}) \dashrightarrow \mathbf{P}^m = \mathbf{P}(\mathrm{W})$ from the center $\mathbf{P}(\mathrm{V}/\mathrm{W})$, and if $f(\mathrm{X}) \cap \mathbf{P}(\mathrm{V}/\mathrm{W}) = \emptyset$ holds, the projection induces a map $p : \mathrm{X} \to \mathbf{P}^m = \mathbf{P}(\mathrm{W})$, called the *projection of* f. Let $a_p : \mathrm{W}_{\mathrm{X}} \to \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})$ denote the natural map and note that the following diagram commutes ([Pi], 6.2 (iii)),

$$\begin{tikzcd} \mathrm{V}_{\mathrm{X}} \arrow[r, "a_f"] & \mathrm{P}^1_{\mathrm{X}}(\mathrm{L}) \\ \mathrm{W}_{\mathrm{X}} \arrow[u, hook] \arrow[ur, "a_p"'] & \end{tikzcd}$$

Let $\mathrm{U}' \subseteq \mathrm{X}$ denote the largest open subscheme where X is smooth and p is an immersion. Then, for a $(r-k+2)$-dimensional subspace $\mathrm{V}' \subseteq \mathrm{W}$, M^p_k is the closure of $\mathrm{M}^p_k(\mathrm{U}')$, the scheme of zeros of $\Lambda^{r-k+2}\,\mathrm{V}'_{\mathrm{U}'} \to \Lambda^{r-k+2}\,\mathrm{P}^1_{\mathrm{X}}(\mathrm{L})\,|_{\mathrm{U}'}$, and for such V' we get $\mathrm{M}^f_k(\mathrm{U})\,|_{\mathrm{U}'} = \mathrm{M}^p_k(\mathrm{U}')$.

For a generic flag $\mathrm{V}' \subseteq \mathrm{W} \subseteq \mathrm{V}$, both $\mathrm{V}' \subseteq \mathrm{V}$ and $\mathrm{W} \subseteq \mathrm{V}$ are generic (as points of corresponding Grassmannians). Therefore (1.3) we may assume that $\mathrm{M}^f_k(\mathrm{U})\,|_{\mathrm{U}'}$ is dense in $\mathrm{M}^f_k(\mathrm{U})$, so that we get $\mathrm{M}^f_k = \mathrm{M}^p_k$.

Q.E.D.

In the situation of the Theorem, let $\check{\mathrm{X}}_f \subseteq \mathbf{P}^{n\vee}$ and $\check{\mathrm{X}}_p \subseteq \mathbf{P}^{m\vee}$ denote the dual varieties of X with respect to f and p. The projection $\mathbf{P}^n \dashrightarrow \mathbf{P}^m$ corresponds to an inclusion $\mathbf{P}^{m\vee} \subseteq \mathbf{P}^{n\vee}$, and it is not hard to see that $\check{\mathrm{X}}_f \cap \mathbf{P}^{m\vee} = \check{\mathrm{X}}_p$ holds. Hence $\check{\mathrm{X}}_f$ and $\check{\mathrm{X}}_p$ have the same degree $d^\vee$, and the same codimension in $\mathbf{P}^{n\vee}$ and $\mathbf{P}^{m\vee}$ respectively. If $\check{\mathrm{X}}_f$ and $\check{\mathrm{X}}_p$ are both hypersurfaces, it follows that $\deg f^\vee = \deg p^\vee$ holds (see § 3), and hence $f^\vee$ is birational if and only if $p^\vee$ is. (Note that Wallace has proved the more general result: biduality holds for $f^\vee$ if and only if it holds for $p^\vee$ ([Wl], Th. 8).)

Let us now consider the varieties $\mathrm{X}_s \to \mathbf{P}^{n-s}$ obtained by intersecting X with s hyperplanes.

THEOREM (4.2). — *Let* X_s *denote the intersection of* X *with* s *general hyperplanes (pulled back via* f*) in* $\mathbf{P}^n$. *The* k*-th class of* X_s *is equal to the* k*-th class of* X, *for* $0 \leqq k \leqq r-s$.

Proof. — Clearly it suffices to treat the case $s = 1$.

Again, we write $\mathbf{P}^n = \mathbf{P}(\mathrm{V})$ and let $\mathrm{U} \subseteq \mathrm{X}$ denote the largest smooth open of X where f is an immersion. Let $\pi : \overline{\mathrm{X}} \to \mathrm{X}$ be any proper, birational map (see § 1) such that the quotient $a\,|_{\mathrm{U}} : \mathrm{V}_{\mathrm{U}} \to \mathrm{P}^1_{\mathrm{X}}(\mathrm{L})\,|_{\mathrm{U}}$ extends on $\overline{\mathrm{X}}$, say to $\bar{a} : \mathrm{V}_{\overline{\mathrm{X}}} \to \mathrm{P}$, and put $\mathrm{K} = \operatorname{Ker}(\bar{a})$.

Let $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$ be a hyperplane, defined by an n-quotient $V \to W$, and put $Y = f^{-1}(\mathbf{P}^{n-1})$. There is a commutative diagram of exact sequences,

$$\begin{array}{ccccccccc} 0 & \to & V'_Y & \longrightarrow & V_Y & \longrightarrow & W_Y & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle a|_Y} & & \downarrow{\scriptstyle a_Y} & & \\ 0 & \to & V'_Y & \to & P^1_X(L)|_Y & \to & P^1_Y(L|_Y) & \to & 0 \end{array}$$

where a_Y denotes the natural map ([Pi], § 2, § 6), and where we set $V' = \ker(V \to W)$. (The left exactness of the lower sequence follows since V'_Y is equal to the conormal bundle of Y in X twisted by L.)

Suppose $V \to W$ is general. Then $Y|_U$ (resp. $\pi^{-1}(Y)|_{\pi^{-1}U}$) is smooth and also dense in Y (resp. $\pi^{-1}(Y) = \overline{Y}$). Moreover, we may assume that the map on $\overline{X}$, $V'_{\overline{X}} \to P$, obtained from $V' \subseteq V$ and $V_{\overline{X}} \to P$, is locally split (apply [Pi], 4.1, to $V^{\vee}_{\overline{X}} \to K^{\vee}$). $Q = \operatorname{Coker}(V'_{\overline{X}} \to P)$ is a r-bundle, and there is a commutative diagram of exact sequences on $\overline{Y} = \pi^{-1}(Y)$,

$$\begin{array}{ccccccccc} 0 & \to & V'_{\overline{Y}} & \to & V_{\overline{Y}} & \to & W_{\overline{Y}} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V'_{\overline{Y}} & \to & P|_{\overline{Y}} & \to & Q|_{\overline{Y}} & \to & 0. \end{array}$$

Since this diagram is isomorphic to the one above (pulled back to $\overline{Y}$ via $\pi|_{\overline{Y}}$) on the dense subscheme $\pi^{-1}(Y)|_{\pi^{-1}U}$, the quotient $W_{\overline{Y}} \to Q|_{\overline{Y}}$ extends $a_Y|_{Y \cap U}$.

Now apply Proposition (1.2): The k-th polar class of $f|_Y : Y \to \mathbf{P}^{n-1}$ is given by

$$[M^Y_k] = (\pi|_{\overline{Y}})_* (c_k(Q|_{\overline{Y}}) \cap [\overline{Y}]).$$

Let $i : Y \subseteq X$ and $j : \overline{Y} \subseteq \overline{X}$ denote the inclusions. In A.X we get (using the projection formula):

$$i_*(\pi|_{\overline{Y}})_*(j^* c_k(Q) \cap [\overline{Y}]) = \pi_*(c_k(Q) \cap j_*[\overline{Y}]).$$

Since j is a hyperplane section of $\overline{X}$, $j_*[\overline{Y}] = \pi^* c_1(f^* L) \cap [\overline{X}]$ holds. Since Q is defined by the exact sequence $0 \to V'_{\overline{X}} \to P \to Q \to 0$, $c(Q) = c(P)$ holds. Thus we can compute

$$i_*[M^Y_k] = \pi_*(\pi^* c_1(f^* L) c_k(P) \cap [\overline{X}]) = c_1(f^* L) \cap \pi_*(c_k(P) \cap [\overline{X}])$$

and we have shown $i_*[M^Y_k] = c_1(f^* L) \cap [M^X_k]$, i. e., the k-th polar class of Y is equal to the intersection of the k-th polar class of X with a general hyperplane. In particular, their degrees are equal.

Q.E.D.

Corollary (4.3). — *Let $X \subseteq \mathbf{P}^n$ be a hypersurface satisfying biduality. Let $\pi : \overline{X} \to X$ and $\check{\pi} : \check{\overline{X}} \to \check{X} \subseteq \mathbf{P}^{n\vee}$ denote the blow ups of the Jacobian ideals (3.3). If $X_s \subseteq \mathbf{P}^{n-s}$ is a section of X by s general hyperplanes, which also satisfies biduality, then the dual* *variety of* X_s *is equal to the image of* $\pi^{-1}(X_s)$ *in* $\mathbf{P}^{n-s\vee}$ *via the projection* $p : \tilde{X} \to \mathbf{P}^{n-s\vee}$ *of* $\check{\pi}$ *induced by* $\mathbf{P}^{n-s} \subseteq \mathbf{P}^n$. *Moreover, the* k-th *class of* $(X_s)^\vee = p(\pi^{-1}(X_s))$ *is equal to the* $(n-1-s-k)$-th *class of* X, *and also equal to the* $(s+k)$-th *class of* $\check{X}$.

Proof. – For $s = 1$, inspection of the proof of the Theorem, especially the commutative diagrams of sequences, suffices. The general case follows by repeated applications.

Q.E.D.

The interpretation of the classes μ_k of X as the degree of the intersection of the polar class $[M_k]$ with the subspace X_s obtained by intersecting X with s general hyperplanes, admits a natural generalization.

Let $Y \subseteq X$ be an h-dimensional closed subscheme of X. The *k-th class of immersion of* Y *in* X (with respect to f) is defined as

$$\rho_k(Y) = \int c_1(\pi^* L)^{h-k} c_k(P) \cap [\overline{Y}^\pi],$$

where $\overline{Y}^\pi$ denotes the strict transform of Y with respect to π, i. e., $\overline{Y}^\pi = \overline{\pi^{-1}(Y|_U)}$.

The intuitive interpretation is that $\rho_k(Y)$ is equal to the number of tangent r-spaces to X at points of Y which meet a given $(n-r+k-2)$-dimensional space in a space of at least $k-1$ dimensions, and which lie on $h-k$ general hyperplanes.

In particular, we get:

$$\begin{aligned}\rho_k(X_s) &= \int c_1(L)^{r-s-k} \cap \pi_*(c_k(P) \cap [\overline{X}_s]) \\ &= \int c_1(L)^{r-k} \cap [M_k] \\ &= \mu_k \quad (\text{for } s \text{ satisfying } 0 \leqq s \leqq r-k).\end{aligned}$$

The h-th class of immersion $\rho_h(Y)$ of Y in X (with $h = \dim Y$) will be called just *the class of immersion of* Y *in* X and written $\rho(Y)$. This character is thus the number of tangent hyperplanes to X at points of Y that contain a given $(n-r+h-2)$-dimensional linear subspace, hence is equal to the degree in $\mathbf{P}^{n\vee}$ of the image of Y via the dual map $f^\vee$ of f. To be more precise, let $f^\vee : \mathbf{P}(K^\vee) \to \mathbf{P}^{n\vee}$ denote the dual map and $p : \mathbf{P}(K^\vee) \to \overline{X}$ the structure map (§see 3). Set $L^* = O_{\mathbf{P}(K^\vee)}(1) = f^{\vee *} O_{\mathbf{P}^{n\vee}}(1)$. The degree of $\overline{Y} = p^{-1}(\overline{Y}^\pi)$ via $f^\vee$ is (by definition) equal to

$$\int c_1(L^*)^{n-1-r+h} \cap [\overline{Y}],$$

hence to

$$\int (p|_{\overline{Y}})_*(c_1(L^*|_{\overline{Y}})^{n-1-r+h} \cap [\overline{Y}]) = \int s_h(K^\vee|_{\overline{Y}^\pi}) = \int c_h(P) \cap [\overline{Y}^\pi] = \rho_h(Y).$$

Suppose that $Y \subset X$ is obtained by intersecting X with the pullback of a hypersurface, of degree d say, in $\mathbf{P}^n$. Then it is easy to see

$$\rho_k(Y) = d \cdot \mu_k(X), \quad \text{for} \quad k = 0, \ldots, r-1.$$

Assume in addition that Y is reduced, X is smooth and f is an immersion. Then there is a formula for the classes of Y in terms of those of X and of the classes of immersion of the singular subscheme of Y.

PROPOSITION (4.4). — *In the above situation there are formulas*

$$\mu_k(Y) = \sum_{i=0}^{k} \left(d(d-1)^i \mu_{k-i}(X) - \sum_{j=1}^{i} \binom{i}{j} (d-1)^{i-j} \rho_{k-i}(s_{r-j}(S, Y)) \right),$$

for $k = 0, 1, \ldots, r-1$, *and where* $s(S, Y)$ *denotes the Segre covariant class of the singular subscheme* S *of* Y.

Proof. — Let $\pi : \overline{Y} \to Y$ denote the blow up of the Jacobian ideal $F^{r-1}(\Omega_Y)$. The ideal defining Y in X is $f^* O_{P_n}(-d) = L^{-d}$. As in the proof of (2.1) we get an exact sequence

$$0 \to K \to \pi^* P^1_X(L) \,|_Y \to P \to 0,$$

where $K \cong \pi^*(L\,|_Y)^{1-d} \otimes J^{-1}$, $J = F^{r-1}(\Omega_Y)\, O_{\overline{Y}}$, and where the composed map

$$V_{\overline{Y}} \to \pi^* P^1_X(L)_Y \to P$$

is the quotient extending the tangent map on Y. Hence we obtain the equality

$$c_k(P) = \sum_{i=0}^{k} c_{k-i}(\pi^* P^1_X(L)_Y)(c_1(\pi^* L\,|_Y)^{d-1} - c_1(J^{-1}))^i.$$

A straightforward computation yields the formula of the proposition (use the definition of ρ and the equality $\rho_k(Y) = d\,\mu_k(X)$ mentioned above).

Q.E.D.

Note that (4.4) reduces to (3.7) when $X = \mathbf{P}^n$, $f = \mathrm{id}$, since $\mu_j(\mathbf{P}^n) = 0$ for $j > 0$ and $\mu_0(\mathbf{P}^n) = 1$, and similarly $\rho_j = 0$ for $j > 0$ and $\rho_0(Z) =$ degree of Z.

The above definition of classes of immersion is (for smooth varieties) the one given by Severi ([S], § 4, § 8). It seems to have been mostly applied to the case where Y is a curve on a surface (see [Ba], p. 225) or on a threefold (see [Rt]).

Example (4.5). — Let X be a surface, $f : X \to \mathbf{P}^3 = \mathbf{P}(V)$ a proper generic immersion, $\pi : \overline{X} \to X$ a proper, birational map on which the dual map $f^\vee$ is defined.

Let $W \subseteq V$ be a general, 3- dimensional subspace and $p : X \to \mathbf{P}(W)$ the corresponding projection, and put $\overline{p} = p \circ \pi$. The induced map $W_{\overline{X}} \to P$ is injective (reason as in [Pi], 4.1); if Q denotes its cokernel, the ideal $F^0(Q)$ is thus invertible and defines a divisor $\overline{C}$ on $\overline{X}$. The curve $C = \overline{\pi_*(\overline{C} \cap \pi^{-1} U)}$ on X is called the *curve of contact* of X with respect to the projection p. Of course C is nothing but the 1-st polar locus of X with respect to $W \subseteq V$.

Consider the hyperplane $\mathbf{P}(W^\vee)$ in $\mathbf{P}^{3\vee} = \mathbf{P}(V^\vee)$. Then $\overline{C}$ is also equal to the pullback $f^{\vee -1}(\mathbf{P}(W^\vee))$. The degree of the curve $C \to \mathbf{P}^2 = \mathbf{P}(W)$ is equal to μ_1, and by Corollary (4.3) its class is equal to the class μ_2 of X.

If $D \subseteq X$ is a curve, its class of immersion in X is given by

$$\rho(D) = \int c_1(P) \cap [\overline{D}^\pi],$$

where $\overline{D}^\pi$ denotes the strict transform of D by π. In particular, for a general curve of contact,

$$\rho(C) = \int c_1(P) \cap [\overline{C}] = \int c_1(P)^2 = \mu_2,$$

since we may assume that $\overline{C} = \pi^{-1} C$ is the strict transform of C.

Assume now that X is *smooth* and that the ramification locus of f is finite. Let $D \subseteq X$ be a curve on X such that $f|_D$ is a generic immersion. Set $\lambda_D = \int [D]^2$, $\rho_D = \rho(D)$, $\zeta_0 = \int c_1(L) \cap [D]$ (the degree of D), g_a = arithmetic genus of D,

$$\nu = \int c_1(P) \cap ([\pi^{-1} D] - [\overline{D}^\pi]).$$

Then the following formula holds:

$$\rho_D + \lambda_D = 2 g_a - 2 + 3 \zeta_0 - \nu.$$

For we obtain

$$\begin{aligned}
\rho_D + \lambda_D &= \int c_1(P) \cap [\overline{D}^\pi] + \int [D]^2 \\
&= \int c_1(P) \cap [\pi^{-1} D] + \int [D]^2 - \int c_1(P) \cap ([\pi^{-1} D] - [\overline{D}^\pi]) \\
&= \int (c_1(\Omega_X) + 3 c_1(L) + [D]) \cap [D] - \nu \\
&= \int (c_1(\Omega_X) + [D]) . [D] + 3 \xi_0 - \nu \\
&= 2 g_a - 2 + 3 \zeta_0 - \nu
\end{aligned}$$

by the adjunction formula for a curve on a smooth surface.

Finally, assume in addition that $Z = f(X)$ has only *ordinary singularities*. Let $\Gamma = f^{-1}(\Gamma_0)$ denote the preimage of the double curve Γ_0 of Z, let ε_0 denote the degree of Γ_0 and ν_2 the number of pinch points of Z. Put $\rho = \rho_\Gamma$, $\lambda = \lambda_\Gamma$. Then the above formula becomes:

$$\rho + \lambda = 2 \varepsilon_0 (\mu_0 - 1) - \nu_2.$$

In fact, we observe $[\Gamma] = (\mu_0 - 4) c_1(L) - c_1(\Omega_X)$, hence

$$2 g_a - 2 = (\mu_0 - 4) \int c_1(L) \cap [\Gamma] = (\mu_0 - 4) 2 \varepsilon_0.$$

Moreover, $[\pi^{-1}(\Gamma)]-[\Gamma^{\pi}]$ is equal to $\sum_{\nu_2} E_\alpha$, where $\{E_\alpha\}$ are the exceptional curves of π, hence its intersection with $c_1(P)=\pi^*(c_1(\Omega_X)+3\,c_1(L))-\Sigma E_\alpha$ is equal to $\nu_2=-(\Sigma E_\alpha)^2$.

Let us compute the classes μ_1 and μ_2 of X. By (3.7) we get, since $\sigma_1=\deg\Gamma=2\,\varepsilon_0$ holds,

$$\mu_1=\mu_0(\mu_0-1)-2\,\varepsilon_0.$$

For the class μ_2 we get:

$$\mu_2=\mu_0(\mu_0-1)^2-4\,\varepsilon_0(\mu_0-1)+\lambda-\nu_2,$$

since $\sigma_0=-\int([\pi^{-1}\Gamma]+\sum[E_\alpha])^2=-\lambda+\nu_2$ holds. Using the formula relating λ and ρ we obtain the classical formula

$$\mu_2=\mu_1(\mu_0-1)-\rho-2\,\nu_2 \quad ([\text{Ba}],\ (\text{II})+(\text{IV}),\ \text{p. } 159).$$

REFERENCES

[A-K] A. ALTMAN and S. L. KLEIMAN, *Introduction to Grothendieck Duality Theory* (*Lecture Notes in Math.*, No. 146, Springer-Verlag, 1970).

[Ba] H. F. BAKER, *Principles of Geometry*, Vol. VI, *Introduction to the Theory of Algebraic Surfaces and Higher Loci*, Cambridge Univ. Press, 1933.

[Be] L. BERZOLARI, *Allgemeine Theorie der höheren ebenen algebraischen Kurven* (*Enzyklopädie der Math.* Wissenschaften, Vol. III, C. 4. Leipzig, 1906).

[E] M. EGER, *Sur les systèmes canoniques d'une variété algébrique* (*C. R. Acad. Sc.*, Paris, T. 204. 1937, pp. 217-219).

[EGA IV] A. GROTHENDIECK et J. DIEUDONNÉ, *Éléments de géométrie algébrique*, IV (*Publ. Math. I.H.E.S.*, Vol. 20, 24, 28, 32, Paris, 1964-1967).

[E-C] F. ENRIQUES et O. CHISINI, *Teoria Geometrica delle Equazioni*, Vol. II, Bologna, 1918.

[Fi] H. FITTING, *Die Determinentalideale eines Moduls* (*Jber. Deutsch. Math.-Verein*, Vol. 46, 1936, pp. 195-228).

[Fu] W. FULTON, *Rational Equivalence on Singular Varieties* (*Publ. Math. I.H.E.S.*, Vol. 45, Paris, 1976).

[F-M] W. FULTON and R. MACPHERSON (to appear).

[G-R] L. GRUSON et M. RAYNAUD, *Critères de platitude et de projectivité* (*Inv. Math.*, Vol. 13, 1971, pp. 1-89).

[Hi] H. HIRONAKA, *On the Arithmetic Genera and the Effective Genera of Algebraic Curves* (*Mem. Coll. Sc.*, Univ. Kyoto, Ser. A, Vol. XXX, Math. No. 2, 1956, pp. 177-194).

[Ho] M. HOCHSTER, *Grassmannians and their Schubert Subvarieties are Arithmetically Cohen-Macaulay* (*J. Algebra*, Vol. 25, 1973, pp. 40-57).

[Ka] I. KAPLANSKY, *Commutative rings*, The Univ. of Chicago Press, 1974 (rev. ed.).

[K-L] G. KEMPF and D. LAKSOV, *The Determinantal Formula of Schubert calculus* (*Acta Math.*, Vol. 132, 1974, pp. 153-162).

[Kl 1] S. L. KLEIMAN, *The Transversality of a General Translate* (*Comp. Math.*, Vol. 38, 1974, pp. 287-297).

[Kl 2] S. L. KLEIMAN, *The Enumerative Theory of Singularities* (in *Real and Complex Singularities*, Oslo 1976 (*Sijthoff and Noordhoff*)).

[Lk] D. LAKSOV, *The Arithmetic Cohen-Macaulay Character of Schubert Schemes* (*Acta Math.*, Vol. 129, 1972, pp. 1-9).

[Lx] A. LASCOUX, *Puissances extérieures, déterminants et cycles de Schubert* (*Bull. Soc. math. Fr.*, T. 102, 1974, pp. 161-179).

[Lm] G. LAUMON, *Degré de la variété duale d'une hypersurface à singularités isolées* (*Bull. Soc. math. Fr.*, T. 104, 1976, pp. 51-63).

[Ll] E. LLUIS, *De las singularidades que aparacen al proyectar variedades algebraicas* (*Bol. Soc. Mat. Mexicana*, Ser. 2, Vol. 1, 1956, pp. 1-9).

[N] A. NOBILE, *Some Properties of the Nash blowing-up* (*Pac. J. Math.*, Vol. 60, 1975, pp. 297-305).

[Pi] R. PIENE, *Numerical characters of a Curve in Projective n-space* in *Real and Complex Singularities*, Oslo 1976 (*Sijthoff and Noordhoff*)).

[Pl] J. PLÜCKER, *Theorie der algebraischen Kurven*, Bonn, 1839.

[Ph 1] W. F. POHL, *Differential Geometry of Higher Order* (*Topology*, Vol. 1, 1962, pp. 169-211).

[Ph 2] W. F. POHL, *Extrinsic Complex Projective Geometry* (*Proc. Conf. Complex Analysis*, Minneapolis, Springer-Verlag, Berlin, 1965).

[Po] J. V. PONCELET, *Traité des propriétés projectives des figures, ouvrage utile à ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain*, Vol. 2, 2nd éd., Paris, 1865-1866.

[Pt] I. R. PORTEOUS, *Todd's Canonical Classes, Liverpool singularities symposium I* (*Lecture Notes in Math.*, No. 192, Springer-Verlag, 1971).

[Rb 1] J. ROBERTS, *Generic Projections of Algebraic Varieties* (*Amer. J. Math.*, Vol. 93, 1971, pp. 191-215).

[Rb 2] J. ROBERTS, *A Stratification of the Dual Variety* (Summary of results with indications of proof), Preprint, 1976.

[Rs] M. ROSENLICHT, *Equivalence Relation on Algebraic curves* (*Ann. of Math.*, Vol. 56, 1952, pp. 169-191).

[Rt] L. ROTH, *Some Formulas for Primals in Four Dimensions* (*Proc. London Math. Soc.*, Ser. 2, Vol. 35, 1933, pp. 540-550).

[SGA 7 I] D. RIM, *Formal Deformation Theory*, Exposé VI in *Groupes de Monodromie en Géométrie algébrique* (SGA 7) (*Lecture Notes in Math.*, No. 288, Springer-Verlag, 1972).

[SGA 7 II] N. KATZ, *Pinceaux de Lefschetz : théorème d'existence*, Exp. XVII in *Groupes de Monodromie...* (SGA 7) (*Lecture Notes in Math.*, No. 340, Springer-Verlag, 1973).

[S] F. SEVERI, *Sulle intersezioni delle varietà algebriche e sopra i loro caratteri e singolarità proiettive* (*Mem. R. Acc. Sc. Torino*, S. II, Vol. 52, 1902, pp. 61-118).

[Te] B. TEISSIER, *Sur diverses conditions numériques d'équisingularité des familles de courbes*, preprint, 1975.

[To] J. A. TODD, *The Arithmetical Invariants of Algebraic loci* (*Proc. London Math. Soc.*, Vol. 43, 1937, pp. 190-225).

[V] J.-L. VERDIER, *Le théorème de Riemann-Roch pour les variétés algébriques éventuellement singulières* (d'après P. BAUM, W. FULTON et R. MACPHERSON) (*Séminaire Bourbaki*, 27e année, No. 464, 1974-1975).

[Wk] R. J. WALKER, *Algebraic Curves*, Dover Publ., New York, 1962.

[Wl] W. WALLACE, *Tangency and Duality over Arbitrary Fields* (*Proc. London Math. Soc.*, (3), Vol. 6, 1956, pp. 321-342).

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