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Polar Orbits in the Kerr Space-Time

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The motion of test particles in polar orbit about the source of the Kerr field of gravity is studied, using Carter's first integrals for timelike geodesics in the Kerr space-time. Expressions giving the angular coordinates of such particles as functions of the radial one are derived, both for the case of a rotating black hole as well as for that of a naked singularity.

1. INTRODUCTION

Since Carter [1] proved that the corresponding Hamilton–Jacobi equation is separable, the orbits of particles falling freely in the Kerr field of gravity have been the object of extensive qualitative as well as quantitative analysis. Reference [2], together with the "bibliographical notes" at the end of [3, Chap. 7] comprises the most recent guide to the literature.

Generally, the analysis of such orbits tends to be quite complicated. Thus, for reasons of mathematical simplicity and or physical interest, some particular class of geodesics, for example, those corresponding to equatorial orbits, is chosen as the object of investigation. Following this trend, we focus the analysis presented in this paper on polar orbits of massive particles, i.e., timelike geodesics crossing the symmetry axis of the Kerr space-time.

The physical interest in polar orbits derives from a set of effects, such as the advance of the ascending node and the geodetic effect, which recent technological advances seem to have rendered measurable in the case of artificial satellites in polar orbit around the Earth [4, 5]. Of course, the weakness of the Earth's field of gravity and the small velocities involved in

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such experiments guarantee that an approximate analysis is satisfactory. However, the exact treatment of the corresponding orbits in the Kerr space-time gives a clearer physical insight into how the above effects arise.

This note is structured as follows: In Section 2, the equations of motion of the particles under consideration are presented along with a qualitative analysis of their basic features. In Section 3, analytic expressions for the orbit equation are obtained for characteristic cases that arise according to the results of Section 3. In the same section, the equations of motion are integrated numerically for a number of cases, whereby corroboration and easy visualization of previous results is obtained. Our analysis is based on the method of [3, Chap. 7] and extends some of the results contained in this and [6] and [7].

2. FIRST INTEGRALS FOR POLAR TIMELIKE GEODESICS

Let x^a, $a=0, 1, 2, 3$, stand for the Boyer–Lindquist (BL) [6] coordinates $(t, r, \vartheta, \varphi)$ in which the Kerr metric reads

$$\begin{aligned} ds^2 = &-(1-2Mr/\Sigma)\, dt^2 - 2(2Mr/\Sigma) a \sin^2\vartheta \, dt\, d\varphi \\ &+ (\Sigma/\Delta)\, dr^2 + \Sigma\, d\vartheta^2 + (A/\Sigma) \sin^2\vartheta \, d\varphi^2 \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Sigma &:= r^2 + a^2 \cos^2\vartheta \\ \Delta &:= r^2 + a^2 - 2Mr \\ A &:= (r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta \end{aligned} \tag{2}$$

and M, a denote the mass and specific angular momentum of the object which gives rise to the gravitational field represented by the Kerr space-time.

If $x^a(\tau)$, τ denoting proper time, is the coordinate image of the timelike geodesic $C(\tau)$ followed by a particle of rest mass μ, then the vector $u^a = \dot{x}^a := dx^a/d\tau$ satisfies the following set of equations

$$\begin{aligned} \dot{t} &= (\Delta\Sigma)^{-1}(AE - 2Mar\phi) \\ \Sigma^2 \dot{r}^2 &= [(r^2+a^2)E - a\phi]^2 - \Delta(\mu^2 r^2 + K) \\ \Sigma^2 \dot{\vartheta}^2 &= K - \mu^2 a^2 \cos^2\vartheta - (aE \sin\vartheta - \phi/\sin\vartheta)^2 \\ \dot{\varphi} &= \Delta^{-1}[(2Mr/\Sigma)\, aE + (1-2Mr/\Sigma)\, \phi/\sin^2\vartheta] \end{aligned} \tag{3}$$

where ϕ, E, and K are real constants. This was first shown by Carter [1], and the constant K which does not correspond to any obvious symmetry of the Kerr metric bears his name.[2] The constants ϕ and E, on the other hand, correspond, respectively, to the axially symmetric and stationary character of the Kerr space-time. They also represent the projection of the angular momentum along the symmetry axis and the "energy at infinity" of the particle following $C(\tau)$.

Let us now demand that the orbit represented by $C(\tau)$ be a polar one, meaning that it intersects the symmetry axis of the Kerr space-time. Since this axis consists of points where $\sin\vartheta = 0$, it follows from (3) (third line) that

$$\phi = 0 \tag{4}$$

is a necessary condition for $C(\tau)$ to be polar. Incorporating this condition in (3) and setting $\mu = 1$ for convenience we obtain

$$\dot{t} = AE/\Delta\Sigma \tag{5a}$$

$$\Sigma^2\dot{r}^2 = R(r) := (r^2 + a^2)^2[E^2 - V^2(r)] \tag{5b}$$

$$\Sigma^2\dot{\vartheta}^2 = \theta(\vartheta) := Q - a^2(1 - E^2)\cos^2\vartheta \tag{5c}$$

$$\dot{\varphi} = 2MaEr/\Delta\Sigma \tag{5d}$$

where

$$V^2 := \Delta(K + r^2)/(r^2 + a^2)^2 \tag{6}$$

and

$$Q := K - a^2E^2 \tag{7}$$

The combination of the first with the last of (5) leads to

$$d\varphi/dt = \omega(r, \vartheta) := 2Mar/A \tag{8}$$

Therefore, according to the asymptotic frame, our particle rotates about the source of the Kerr field with angular velocity $\omega(r, \vartheta)$. The latter coincides with the characteristic angular velocity of each member of the congruence of curves defined by the vector $\mathbf{e}_0$, where

$$\mathbf{e}_0 := (A/\Sigma\Delta)^{1/2}(\partial_t + \omega\partial_\varphi) \tag{9}$$

[2] Insights into the physical meaning of Carter's constant can be found in [11–13].

This congruence is orthogonal to the $t = \text{const.}$ hypersurfaces of the Kerr space-time. The frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where

$$\begin{aligned} \mathbf{e}_1 &:= (\Delta/\Sigma)^{1/2}\partial_r \\ \mathbf{e}_2 &:= (1/\Sigma)^{1/2}\partial_\vartheta \\ \mathbf{e}_3 &:= (\Sigma/A\sin^2\vartheta)^{1/2}\partial_\varphi \end{aligned} \tag{10}$$

defined along each member of the congruence is a locally nonrotating frame (LNRF) in the terminology of Bardeen [9] who first constructed this class of frames in stationary axially symmetric space-times.

Thus, (8) implies that the particle under consideration is observed to remain in the $r-\vartheta$ plane by all the LNRFs it meets as it revolves around the source of the Kerr field. On the basis of this observation, we can call the orbits considered in this paper locally planar.

The orbits considered in the following are assumed to be polar in the usual sense of the term. By this we mean that our test particle is supposed not only to cross the symmetry axis but also to sweep the whole range of the angular coordinate ϑ. Therefore, we demand that $\dot{\vartheta}$ does not vanish for any $\vartheta \in [0, \pi]$. According to (5c), this condition is equivalent to the demand that

$$Q > 0 \qquad \text{when} \quad E^2 \geqslant 1 \tag{11}$$

and

$$Q > a^2(1-E^2) \qquad \text{when} \quad E^2 < 1 \tag{12}$$

and, in conjunction with (6) and (7), it implies that

$$V^2(0) = K/a^2 > E^2 \tag{13}$$

Equation (5b) shows that $V^2(r)$ plays the role of a square potential for the r motion of our particle. Therefore, (13) implies that our particle cannot move into the negative-r region, toward which the Kerr metric is extendible [1], having started from a point where $r > 0$. Thus we do not lose in generality by restricting our considerations to the region of positive r.

From (6), (7), (11), and (12) it follows that

$$V^2(0) = K/a^2 > 1 \tag{14}$$

$$\lim_{r \to \infty} V^2(r) = 1 - 0 \tag{15}$$

and that the local extrema of $V^2(r)$ occur at the roots of the equation

$$Mr^4 - (K-a^2)r^3 + 3M(K-a^2)r^2 - (K-a^2)a^2 r - MKa^2 = 0 \tag{16}$$

Descartes' rule and (14) imply that (16) has at most three roots in the $r > 0$ region, while (14) and (15) imply that it has at least one root in the same region. Moreover, in the case of a rotating black hole, i.e., when $M^2 \geqslant a^2$, $V^2(r)$ necessarily vanishes at the roots $r_{\pm} := M \pm (M^2 - a^2)^{1/2}$ of the equation $\Delta(r) = 0$.

The above qualitative study suffices for determining the essential features of the effective square potential $V^2(r)$. This is shown by Figs. 1 and 2, where $V^2(r)$ is plotted for various values of Carter's constant K, corresponding to a rotating black hole and a naked singularity $(a^2 > M^2)$, respectively.

The general features of the r motion of a particle in polar orbit about the source of the Kerr field can be deduced from graphs of $V^2(r)$, such as those shown in Figs. 1 and 2. They make it obvious, for example, that such a particle follows a bound orbit when its specific energy at infinity E is such that $E^2 < 1$. They also show that, when $E^2 < 1$ and a black hole is involved, the particle gets dragged, disappearing behind the event horizon, i.e., the surface $r = r_+$, unless K is such that $V^2(r)$ develops a local maximum E^2_{max}

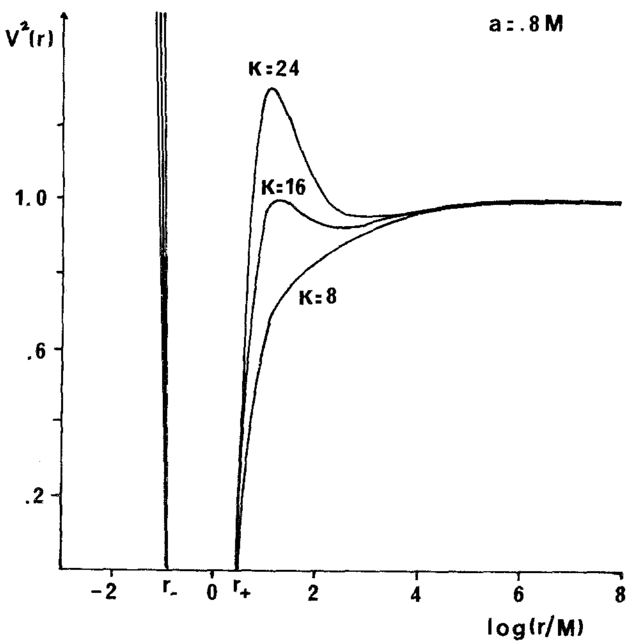

Fig. 1. The effective square potential $V^2(r)$ for polar orbits in the neighborhood of a rotating black hole with $a = 0.8\,M$. The parameter K distinguishing the three curves is Carter's constant in units of M^2.

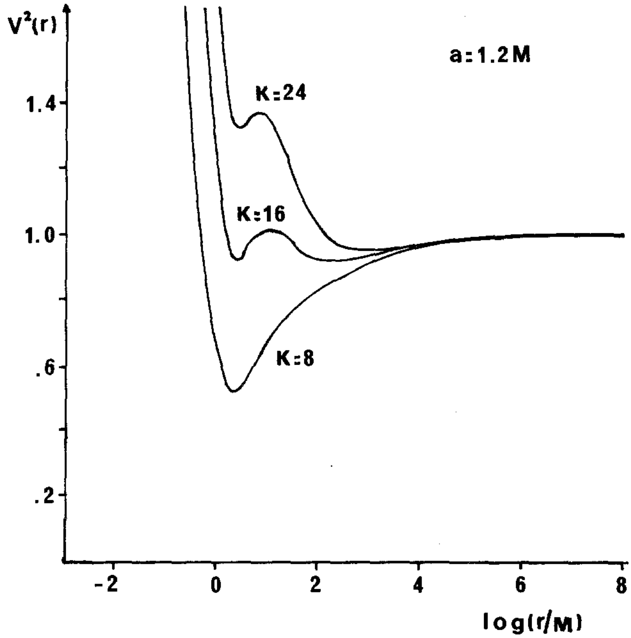

Fig. 2. The effective square potential $V^2(r)$ for polar orbits in the neighborhood of a naked singularity with $a = 1.2\,M$. Carter's constant K is given in units of M^2.

outside the event horizon and $E^2 < E^2_{\max}$. In this case, the particle will not be swallowed by the black hole, provided it is initially found in the region $r > r_0$, where r_0 is the point on the r axis where $V^2(r) = E^2_{\max}$. Similar considerations apply when $E^2 \geqslant 1$. In this case, however, a particle with $E^2 < E^2_{\max}$ is not necessarily bound. It can follow a free orbit, moving in the half-open interval $r_0 \leqslant r < \infty$, provided it is initially found in this region.

3. THE ORBIT EQUATION

Depending on whether or not the coordinate radius r is constant along a given timelike geodesic, the corresponding particle orbit is characterized as spherical or nonspherical, respectively.

Considering the case of spherical orbits first, we note that $E^2 = V^2(r_0)$, where r_0, a root of (16), is a necessary condition for a spherical orbit to obtain. Thus, (16) implies that

$$K = r(Mr^3 + a^2r^2 - 3Ma^2r + a^4)/Z \tag{17}$$

where

$$Z := r^3 - 3Mr^2 + a^2r + Ma^2 \tag{18}$$

along a spherical polar orbit. For convenience, r_0 has been replaced by r in the last two equations.

From (6) and (17) it follows that the energy of a particle following a polar orbit of radius r is determined by the equation

$$E^2 = r\Delta^2/(r^2+a^2)Z \tag{19}$$

Since E^2 must be positive, (19) implies that spherical polar orbits lie outside the interval $r_1 \leqslant r \leqslant r_2$, where r_1, r_2 are the two roots that the equation $Z(r)=0$ can have on the positive r axis. It is a simple matter to obtain $(a/M)^2 \leqslant 3(2\sqrt{3}-3) \approx 1.3924$ as the condition for such roots to obtain.

From (18) we find that

$$\Gamma^2 := 1 - E^2 = MH/(r^2+a^2)Z \tag{20}$$

where

$$H(r) := r^4 - 4Mr^3 + 2a^2r^2 + a^4 \tag{21}$$

Since $Z(r)>0$, it follows from (20) that E^2 is less, equal or greater than unity according to whether $H(r)$ is greater, equal or less than zero, respectively. Descartes' rule, on the other hand, implies that the equation $H(r)=0$ can have at most two roots for $r>0$. One finds that such roots exist as long as $(a/M)^2 \leqslant (27/16) = 1.6875$. This means that no spherical orbits with $E^2 \geqslant 1$ are possible when the ratio a^2/M^2 exceeds the above value.

Turning to (5) we see that the last two of them imply that, when $\dot{\vartheta} > 0$

$$\frac{d\varphi}{d\vartheta} = \frac{2MaEr}{\Delta\theta^{1/2}} \tag{22}$$

Let us suppose that $E^2 < 1$. Then we can use the latitude angle ψ, where

$$\psi := (\pi/2) - \vartheta \tag{23}$$

to write (22) in the form

$$\frac{d\varphi}{d\psi} = -\frac{2MaEr}{\Delta Q^{1/2}(1-k^2\sin^2\psi)^{1/2}} \tag{24}$$

where

$$k^2 := a^2\Gamma^2/Q \tag{25}$$

According to (7), (12), (17), and (20)

$$k^2 = (a/r)^2(r^4 - 4Mr + 2a^2r^2 + a^4)/H < 1 \tag{26}$$

Therefore, (24) gives

$$\varphi = \varphi_0 - \frac{2MaEr}{\Delta Q^{1/2}} F(\psi, k) \tag{27}$$

where $F(\psi, k)$ is the elliptic integral of the first kind defined by

$$F(\psi, k) := \int_0^{\psi} (1 - k^2 \sin^2\psi)^{-1/2} \, d\psi \tag{28}$$

The integration of (22) when $E^2 = 1$ is trivial. When $E^2 > 1$, one can write it in the form

$$\frac{d\varphi}{d\vartheta} = + \frac{2MaEr}{\Delta (K - a^2)^{1/2} (1 - l^2 \sin^2\vartheta)^{1/2}} \tag{29}$$

where

$$l^2 = a^2(E^2 - 1)/(K - a^2) \tag{30}$$

According to (13), $l^2 < 1$ so that

$$\varphi = \varphi_0 + \frac{2MaEr}{\Delta (K - a^2)^{1/2}} F(\vartheta, l) \tag{31}$$

In general, the bound orbits corresponding to $E^2 \geqslant 1$ do not show any qualitative difference from those with $E^2 < 1$. Therefore, we will restrict our considerations to the case $E^2 < 1$ in the following.

Equation (5d) shows that $\dot{\varphi} > 0$ for future pointing ($E > 0$) polar orbits and $a > 0$. This means that the orbit is dragged in the sense in which the source is rotating. In the case of the spherical orbits this is the main effect of the nonvanishing of a and it is reflected in the fact that the ascending node advances by $\delta\varphi$ per revolution, where, according to (27)

$$\delta\varphi = \frac{8MaEr}{\Delta Q^{1/2}} F(\pi/2, k) \tag{32}$$

when $E^2 < 1$. This is shown in Fig. 3 where the orbit of a particle with $E = 0.956$ and $K = 14.783$ orbitting a black hole with $a = 0.8$ M is pictured using the asymptotic meaning of the BL coordinates r, ϑ, and φ. Since

$$F(\pi/2, k) \approx (\pi/2)(1 + k^2/4), \qquad k^2 \ll 1 \tag{33}$$

and $(a/r)^2 = 6.4 \times 10^{-3}$ in this case, it follows from (26) and (32) that $\delta\varphi \approx 18.12^0$. The orbit shown in Fig. 3, as well as the one shown in Fig. 4

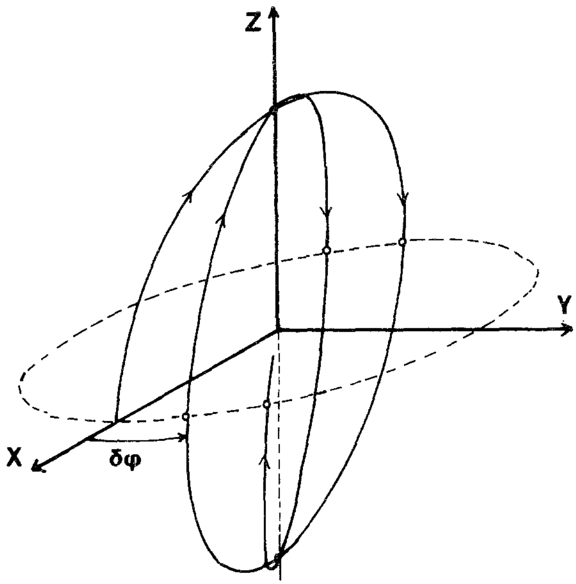

Fig. 3. A spherical polar orbit around a rotating black hole with $a = 0.8\,M$. The coordinate radius of the orbit is $r_0 = 10\,M$, $E = 0.956$, and $K = 14.783\,M^2$.

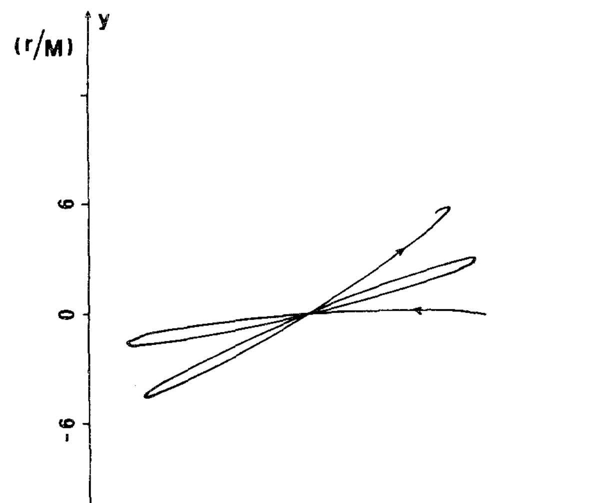

Fig. 4. The projection of the orbit shown in Fig. 3 in the x–y plane.

and a series of others corresponding to various values of E and K, was obtained by numerical integration of (5) with the purpose of checking the analytic results of this section.

In the case of the nonspherical orbits, $\dot{r} \neq 0$. Then, it follows from (5) that

$$\int d\vartheta/\Theta^{1/2} = \pm \int dr/R^{1/2} \tag{34}$$

and

$$\varphi = \pm 2MaE \int dr(r\Delta/R^{1/2}) \tag{35}$$

where

$$R(r) := -\Gamma^2 r^4 + 2Mr^3 - (a^2\Gamma^2 + Q)r^2 + 2M[a^2(1-\Gamma^2) + Q]r - Qa^2 \tag{36}$$

and the $\pm$ sign is that of $\dot{r}$.

Now

$$\int d\vartheta/\Theta^{1/2} = -\int d\psi[Q(1-k^2 \sin\psi)]^{1/2} = -Q^{-1/2}F(\psi, k) + \text{const.} \tag{37}$$

Therefore, in order to obtain ϑ and φ as functions of r, it suffices to compute the r integrals on the right-hand side of (34) and (35), respectively.

In general, the equation $R(r) = 0$ has four roots in the interval $0 < r < \infty$. Assuming that r_0 is a double root of this equation, we can write

$$R(r) = (r - r_0)^2\, G(r) \tag{38}$$

where

$$G(r) := -\Gamma^2 r^2 + 2(M - \Gamma^2 r_0)r - a^2 Q/r_0^2 \tag{39}$$

Obviously, r_0 represents the coordinate radius of a spherical orbit of the kind considered earlier. From (38) it follows that when $G(r_0) > 0$ the root r_0 lies in an open interval where $R(r) > 0$. This implies that the corresponding spherical orbit is unstable. Similarly, when $G(r_0) < 0$, the root r_0 is isolated and the corresponding spherical orbit is stable.

When the double root r_0 of the equation $R(r) = 0$ is accompanied by an open interval $r' < r < r''$ where $G(r) > 0$, a nonspherical orbit appears in this interval which is characterized by the same values of E^2 and K, or Γ^2 and Q, as the spherical one at r_0. Assuming that this is the orbit we are dealing with, we can write $R(r)$ in the form

$$R = \Gamma^2 (r - r_0)^2 (r - r')(r'' - r) \tag{40}$$

and use the substitution

$$x := (r - r_0)^{-1} \tag{41}$$

to turn the integral on the right-hand side of (34) into

$$I_0(r) := \int dr/R^{1/2} = \int dr[(r - r_0)^2 G]^{-1/2} = -\int dx/X^{1/2} \tag{42}$$

where

$$\begin{aligned} X(x) &:= \alpha + \beta x + \gamma x^2 \\ \alpha &:= -\Gamma^2 \\ \beta &:= 2(M - 2\Gamma^2 r_0) = \Gamma^2(r' + r'' - 2r_0) \\ \gamma &:= G(r_0) = \Gamma^2(r_0 - r')(r'' - r_0) \end{aligned} \tag{43}$$

Then

$$\begin{aligned} I_0 &= -(1/\gamma)^{1/2} \ln(\beta + 2\gamma x + \gamma^{1/2} X^{1/2}), \qquad \text{if} \quad G(r_0) > 0 \\ &= (-1/\gamma)^{1/2} \sin^{-1}(2\gamma x + \beta)/\Gamma^2(r'' - r') \qquad \text{if} \quad G(r_0) < 0 \end{aligned} \tag{44}$$

Actually, $G(r)$ may vanish at r_0. In this case r_0 coincides with either r' or r'', whereby

$$I_0 = \pm \frac{2}{|\Gamma| (r'' - r_0)} \left(\frac{r'' - r}{r - r_0} \right)^{1/2} \qquad \text{if} \quad r' = r_0 \tag{45}$$

When r_0 coincides with r', I_0 is obtained by replacing r'' by r' in (45).

In order to calculate the integral on the right-hand side of (35) in the case of an orbit associated with a spherical one at r_0, let us first assume that the particle is orbitting a black hole with $a^2 < M^2$. Then $\Delta(r) = (r - r_-)(r - r_+)$, and after some algebra we obtain

$$\begin{aligned} I_1(r) &:= \int (r/\Delta R^{1/2})\, dr \\ &= \frac{r_0}{\Delta(r_0)} I_0 - \frac{r_+}{(r_+ - r_-)(r_0 - r_+)} I_+ + \frac{r_-}{(r_+ - r_-)(r_0 - r_-)} I_- \end{aligned} \tag{46}$$

where

$$I_\pm(r) := \int [(r - r_\pm) G]^{1/2}\, dr \tag{47}$$

The substitution

$$x = (r - r_\pm)^{-1} \tag{48}$$

reproduces the results (42)–(44) where, now, $r_\pm$ replaces r_0 and $I_\pm$ replaces I_0, respectively. Moreover, only the first of the expressions given by (43) obtains, because $G(r_\pm) > 0$. When $r'' = r_0$, the final expression for $G(r)$ simplifies considerably and reads

$$\varphi(r) = \varphi_0 + \frac{2MaE}{|\Gamma|\,(r'' - r_+)(r'' - r_-)} \left[\frac{2r''}{r'' - r'} u(r) + \frac{r_+ u(r_+)}{r'' - r_+} \ln \left| \frac{u(r) - u(r_+)}{u(r) + u(r_+)} \right| - \frac{r_- u(r_-)}{r'' - r_-} \ln \left| \frac{u(r) - u(r_-)}{u(r) + u(r_-)} \right| \right] \tag{49}$$

where

$$u(r) = \left(\frac{r - r'}{r'' - r} \right)^{1/2} \tag{50}$$

It is clear from Fig. 1 that a particle following an orbit associated with a stable spherical one oscillates in the region $r' \leqslant r < r''$, which includes the horizons at $r = r_\pm$, since $r' < r_- < r_+ < r'' < r_0$. Of course, this does not mean that, having started from a point in the region $r > r_+$ and fallen into the black hole, the particle can return to the region outside the event horizon $r = r_+$. As a matter of fact, the particle finds itself in a new replica of the space-time region into which it enters each time it crosses one of the horizons [1]. In any case, the motion is smooth even though the functions $I_\pm(r)$, which determine $\varphi(r)$ via (35) and (46), diverge at $r = r_\pm$. The divergence of $I_\pm(r)$ at the horizons reflects the singular character of the BL coordinates at $r = r_\pm$.

From Fig. 1 it is also clear that the orbit associated with an unstable spherical one is separated into two branches by the point $r = r_0$. The branch $r \leqslant r_0$ represents a trapped orbit similar to the one described in the previous paragraph. The whole branch $r \geqslant r_0$, on the other hand, remains outside the event horizon since $r_0 > r_+$ necessarily. In this case both $\vartheta(r)$ and $\varphi(r)$ diverge as $r \to r_0$. This follows from (34), (35), (37), and (44) and is explained by the fact that the value r_0 of the coordinate radius r corresponds to a spherical orbit. Therefore, $R(r) \to 0$ as $r \to r_0$, which implies that $\dot{r} \to 0$, while, according to (5), $\dot{\vartheta}$ and $\dot{\varphi}$ tend to finite limits as the particle approaches the hypersurface $r = r_0$.

When $M^2 < a^2$, the equation $\Delta(r) = 0$ has no roots along the real r axis. Therefore, the integration of the right-hand side of (35) must in this case proceed along a line different from the one followed above for $M^2 > a^2$. Noting that

$$\frac{r}{\Delta(r - r_0)} = \frac{1}{\Delta(r_0)} \left(\frac{a^2 - r_0 r}{\Delta(r)} + \frac{r_0}{r - r_0} \right) \tag{51}$$

we conclude that (46) is now replaced by the expression

$$I_1(r) = \frac{1}{\Delta(r_0)} (r_0 I_0 + I_2) \tag{52}$$

where

$$I_2(r) := \int \frac{a^2 - r_0 r}{\Delta G^{1/2}} dr \tag{53}$$

The substitution

$$x = \frac{r - \lambda_-}{\lambda_+ - r} \tag{54}$$

where

$$\begin{aligned} \lambda_\pm &:= \lambda \pm \rho \\ \lambda &:= \frac{r'r'' - a^2}{r' + r'' - 2M} \\ \rho &:= [(\lambda - r')(\lambda - r'')]^{1/2} \end{aligned} \tag{55}$$

turns (52) into

$$I_2(r) = \frac{2\rho}{\Delta(\lambda_+)} \int \frac{(a^2 - r_0\lambda_+)x + a^2 - r_0\lambda_-}{(x^2 + p^2)[G(\lambda_+)(x^2 - q^2)]^{1/2}} dx \tag{56}$$

where

$$\begin{aligned} p^2 &:= \frac{\Delta(\lambda_-)}{\Delta(\lambda_+)} = \frac{M - \lambda_-}{\lambda_+ - M} \\ q^2 &:= -\frac{G(\lambda_-)}{G(\lambda_+)} = \frac{r' + r'' - 2\lambda_-}{r' + r'' - 2\lambda_+} \end{aligned} \tag{57}$$

By (54) the boundary points r', r'' of the orbit associated with the spherical one at r_0 are mapped on the points $x = \pm |q|$ of the x axis. Both p^2 and q^2 are positive quantities. This is obvious for the former, since $\Delta(r) > 0$, while for the latter it follows from the fact that (55) implies that $\lambda \notin [r', r'']$ and only one of $\lambda_\pm$ belongs to the same interval.

Finally, a lengthy calculation shows that

$$\begin{aligned} I_2(r) &= B_+ \tan^{-1}\left(\frac{x^2 - q^2}{p^2 + q^2}\right)^{1/2} + B_- \coth^{-1}\left[\frac{(p^2 + q^2)x^2}{p^2(x^2 - q^2)}\right]^{1/2} \quad \text{if } G(\lambda_+) > 0 \\ &= B_+ \tanh^{-1}\left(\frac{q^2 - x^2}{p^2 + q^2}\right)^{1/2} + B_- \tan^{-1}\left[\frac{(p^2 + q^2)x^2}{p^2(q^2 - x^2)}\right]^{1/2} \quad \text{if } G(\lambda_+) < 0 \end{aligned} \tag{58}$$

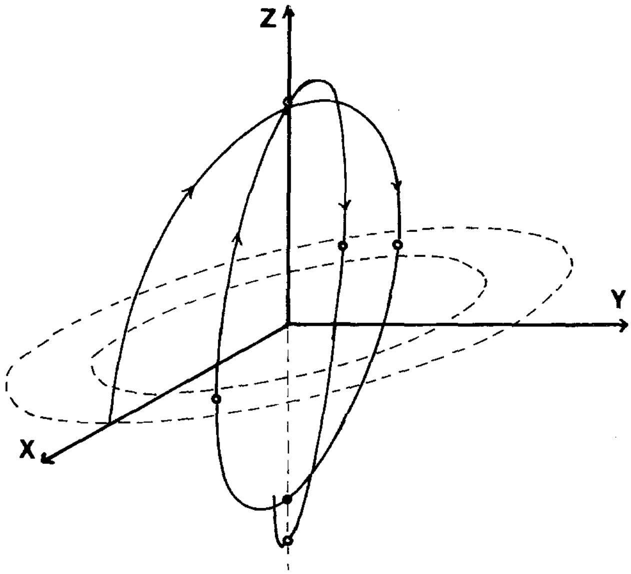

Fig. 5. The nonspherical polar orbit which results from the one shown in Fig. 3 by changing the energy of the particle to $E=0.957$. The particle oscillates between $r'=8.41\,M$ and $r''=12.10\,M$.

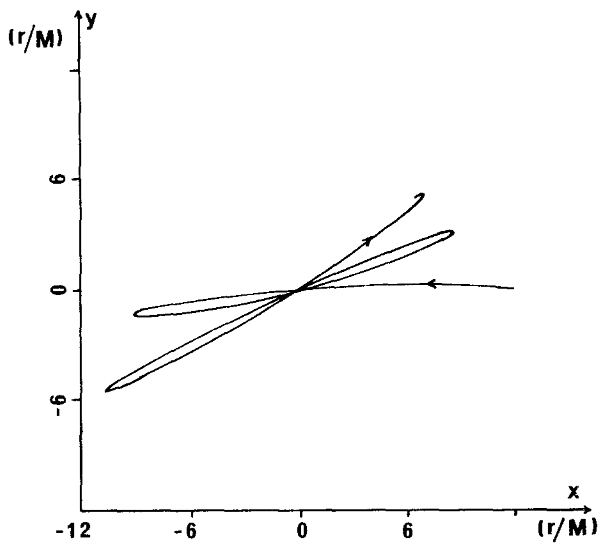

Fig. 6. The projection of the orbit shown in Fig. 5 in the x–y plane.

where

$$B_{\pm} = \frac{2\rho(a^2 - r_0\lambda_{\pm})}{\Delta(\lambda_+)|G(\lambda_+)|^{1/2}} \tag{59}$$

It remains to compute $I_1(r)$ when the equation $R(r) = 0$ has either only two or four distinct positive roots. However, the results obtained so far make it evident that the analytic expressions for $I_1(r)$ tend to be quite complicated. Moreover, it should be clear from Figs. 1 and 2 that no qualitatively new features arise in connection with the above cases. Therefore, they will not be discussed further here. In any case, one will most likely turn to numerical integration or approximation techniques for the analysis of specific cases of nonspherical polar orbits. An example of a nonspherical orbit obtained by numerical integration is illustrated in Figs. 5 and 6. The results of the approximation approach have been presented by us in a separate note [10].

REFERENCES

1. Carter, B. (1968). *Phys. Rev.*, **174**, 1559.
2. Sharp, N. A. (1979). *Gen. Rel. Grav.*, **10**, 659.
3. Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford).
4. Anderson, J. et al. (1982). *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, R. Ruffini, ed. (North-Holland, New York), p. 939.
5. van Patten, R. A., and Everitt, C. W. F. (1976). *Phys. Rev. Lett.*, **336**, 629.
6. Wilkins, D. C. (1972). *Phys. Rev. D*, **5**, 814.
7. Johnston, M., and Ruffini, R. (1974). *Phys. Rev. D*, **10**, 2324.
8. Boyer, R. H., and Lindquist, R. W. (1967). *J. Math. Phys.*, **8**, 225.
9. Bardeen, J. M. (1970). *Astrophys. J.*, **161**, 103.
10. Stoghianidis, E., and Tsoubelis, D. (1986). *Phys. Lett. A*, **116**, 213.
11. de Felice, F. (1980). *J. Phys. A: Math. Gen.*, **13**, 1701.
12. Qadir, A., Quamar, J., and Rafique, M. (1985). *Phys. Lett. A*, **109**, 90.
13. Faridi, A. M. (1986). *Gen. Rel. Grav.*, **18**, 271.