Polarization mode dispersion probability distribution for arbitrary distances

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The probability distribution of the differential group delay (DGD) at any fiber length is determined by use of a physically reasonable model of the fiber birefringence. We show that if the fiber correlation length is of the same order as or larger than the beat length, the DGD distribution approaches a Maxwellian in roughly 30 fiber correlation lengths, corresponding to a couple of kilometers in realistic cases. We also find that the probability distribution function of the polarization dispersion vector at the output of the fiber depends on the angle between it and the local birefringence vector on the Poincaré sphere, showing that the DGD remains correlated with the orientation of the local birefringence axes over arbitrarily long distances. © 2001 Optical Society of America

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Polarization mode dispersion (PMD) is caused by the random birefringence that is present in optical fibers. It can lead to pulse spreading and depolarization, and it is detrimental to system performance. As transmission rates continue to increase, the PMD has become an increasingly important fiber impairment, thus motivating extensive experimental and theoretical study over the past few years.

PMD is characterized by a three-component dispersion vector, $\mathbf{\Omega}$. Its magnitude, $|\mathbf{\Omega}|$, gives the differential group delay (DGD) between the principal states, and its direction gives the orientation of the slow principal state of polarization at the output on the Poincaré sphere.[1] At short distances, PMD is deterministic, and the DGD probability distribution is a δ function. At long distances, however, previous work in which a weak random birefringence model was used showed that the three components of $\mathbf{\Omega}$ are independent and Gaussian distributed, so the DGD distribution is Maxwellian.[1] Similar results are also obtained if one assumes that the fiber birefringence completely randomizes the polarization state over the Poincaré sphere.[2]

Both analysis and numerous numerical and experimental studies have lead to the generally accepted wisdom that the asymptotic (long-length limit) distribution function of the DGD resulting from PMD is Maxwellian. The transient behavior of the distribution, however, is not so well elucidated. In this Letter we study the fundamental question of determining the probability distribution of the DGD that is due to PMD at any fiber length with general fiber correlation and beat-length parameters. The results of our analytical work indicate that current systems approach a Maxwellian distribution in just a few kilometers, confirming the experimental work of Gisin *et al.*[3] and placing on a firm theoretical foundation the widely used approach of calculating the penalties on transmission that are due to PMD by assuming that the distribution of the DGD is Maxwellian. Moreover, our results indicate when this assumption breaks down, which may be of use in future systems. In addition, we find, for what is to our knowledge the first time, that the probability distribution function of the polarization dispersion vector at the output of the fiber depends on the angle between it and the local birefringence vector on the Poincaré sphere, showing that the DGD remains correlated with the orientation of the local birefringence axes over arbitrarily long distances.

In this Letter we adopt the first birefringence model of Wai and Menyuk.[4] Even though the second birefringence model of Wai and Menyuk[4] is more realistic in fibers,[5] Wai and Menyuk pointed out that both models lead to nearly the same diffusion rates for the polarization states on the Poincaré sphere. Thus, it follows that the evolution of the distribution predicted by the first model would be nearly the same as that of the second model. In the first model, the fiber is assumed to have a linear birefringence of fixed strength $2b$ but with an orientation of the birefringence axes that varies randomly with distance along the fiber. Our approach is to solve numerically the Fokker–Planck equation for the probability density function of the polarization dispersion vector $\mathbf{\Omega}$ associated with this model, which is a nontrivial task.

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We begin with the basic dynamic equation for dispersion vector $\mathbf{\Omega}$ (Ref. 1):

$$\frac{\partial \mathbf{\Omega}(z,\omega)}{\partial z} = \frac{\partial \mathbf{W}(z,\omega)}{\partial \omega} + \mathbf{W}(z,\omega) \times \mathbf{\Omega}(z,\omega)\,, \qquad (1)$$

where the vector $\mathbf{W}$ represents the local polarization state in the fiber. As stated above, the birefringence strength $2b$ is fixed. In addition, the rate of change of the polarization orientation is assumed to be driven by a white-noise process.[4] In other words, $\mathbf{W} = (2b\cos\theta, 2b\sin\theta, 0)$, and $\mathrm{d}\theta/\mathrm{d}z = g_\theta(z)$, where $\langle g_\theta(z)\rangle = 0$ and $\langle g_\theta(z) g_\theta(z')\rangle = \sigma^2\delta(z - z')$. The parameter σ^2 is related to the fiber correlation length, h_{fiber}, by the equation $\sigma^2 = 2/h_{\mathrm{fiber}}$. To help simplify the analysis, we employ two variable transformations:

$$\tilde{\mathbf{\Omega}} = \mathbb{H}_1(z)\mathbf{\Omega}\,, \qquad \hat{\mathbf{\Omega}} = \mathbb{H}_2(z)\tilde{\mathbf{\Omega}}\,. \qquad (2)$$

Here, matrix $\mathbb{H}_1$ represents a rotation through an angle θ in the (Ω_1, Ω_2) plane, and $\mathbb{H}_2$ is a rotation through an angle $2bz$ in the $(\tilde{\Omega}_2, \tilde{\Omega}_3)$ plane. Physically, $\tilde{\mathbf{\Omega}}$ is the dispersion vector measured in terms of the local axes of birefringence, and $\hat{\mathbf{\Omega}}$ is the dispersion vector measured relative to axes rotated by the birefringence. Note that $|\mathbf{\Omega}|$ is invariant under these rotations. The equation for $\hat{\mathbf{\Omega}}$ after transformation (2) is then

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\hat{\Omega}_1\\ \hat{\Omega}_2\\ \hat{\Omega}_3\end{pmatrix} = g_\theta \begin{pmatrix}\hat{\Omega}_2\cos 2bz - \hat{\Omega}_3 \sin 2bz\\ -\hat{\Omega}_1 \cos 2bz\\ \hat{\Omega}_1 \sin 2bz\end{pmatrix} + \begin{pmatrix}2b'\\ 0\\ 0\end{pmatrix}, \qquad (3)$$

where $b' = \mathrm{d}b/\mathrm{d}\omega$.

In what follows, we assume that $2b \gg \sigma^2/2$, i.e., $4\pi h_{\mathrm{fiber}} \gg L_B$, where L_B is the beat length. This assumption is correct for most modern-day fibers. For instance, a typical correlation length is 50 m, and a typical beat length is 15 m in high-PMD fiber, implying that $4\pi h_{\mathrm{fiber}}/L_B = 42 \gg 1$. With this assumption, the Fokker–Planck equation associated with Eq. (3) can be averaged over the rapidly rotating polarization states, with the result that the probability density function P for $\hat{\mathbf{\Omega}}$ satisfies the reduced equation[6,7]

$$\frac{\partial P}{\partial Z} = \frac{1}{2}\left[\left(\bar{\Omega}_1 \frac{\partial}{\partial \bar{\Omega}_2} - \bar{\Omega}_2 \frac{\partial}{\partial \bar{\Omega}_1}\right)^2 + \left(\bar{\Omega}_1 \frac{\partial}{\partial \bar{\Omega}_3} - \bar{\Omega}_3 \frac{\partial}{\partial \bar{\Omega}_1}\right)^2\right] P - \frac{\partial P}{\partial \bar{\Omega}_1}\,. \qquad (4)$$

Here, $Z \equiv z/h_{\mathrm{fiber}}$ and $\bar{\Omega}_k \equiv \hat{\Omega}_k/2b'h_{\mathrm{fiber}}$ $(k = 1, 2, 3)$ are dimensionless variables. With a PMD coefficient of $1\ \mathrm{ps}/\sqrt{\mathrm{km}}$ and a fiber correlation length of 50 m, the normalizing coefficient for the DGD $(2b'h_{\mathrm{fiber}})$ is approximately 0.16 ps. It is important to note that Eq. (4) no longer has any free parameters. Thus, the probability distribution function P is the same for all fiber parameters as long as the assumption $4\pi h_{\mathrm{fiber}}/L_B \gg 1$ is valid. We have considered the result of eliminating this restriction, and we will report the results in the future.

It is convenient to write Eq. (4) in spherical coordinates:

$$\bar{\Omega}_1 = \tau\cos\phi\,, \qquad \bar{\Omega}_2 = \tau\sin\phi\cos\psi\,,$$
$$\bar{\Omega}_3 = \tau\sin\phi\sin\psi\,, \qquad (5)$$

where the $\bar{\Omega}_1$ axis corresponds to the slow birefringence axis on the Poincaré sphere, $\tau = |\bar{\mathbf{\Omega}}|$, ϕ is the angle between $\bar{\mathbf{\Omega}}$ and the Ω_1 axis, and ψ is the azimuthal angle. In these coordinates, Eq. (4) becomes

$$\frac{\partial P}{\partial Z} = \frac{1}{2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial P}{\partial\phi}\right) + \frac{1}{2}\left(\frac{1}{\sin^2\phi} - 1\right)\frac{\partial^2 P}{\partial\psi^2} - \left(\cos\phi\frac{\partial P}{\partial\tau} - \frac{\sin\phi}{\tau}\frac{\partial P}{\partial\phi}\right). \qquad (6)$$

To model the evolution of the PMD distribution along the fiber, we solve Eq. (6) numerically, starting from a δ function initial condition. Numerically, we chose $P|_{Z=0}$ to be a very sharp δ-like function:

$$P(\tau,\phi,\psi,0) = \frac{D^3}{\pi^{3/2}}\exp(-D^2\tau^2)\,, \qquad (7)$$

where $D \gg 1$. Since the initial condition for P is independent of the azimuthal angle, ψ, it is easy to see from Eq. (6) that P will be independent of ψ for all Z. Thus, all ψ derivatives in Eq. (6) can be dropped, simplifying the numerical solution significantly.

We use a split-step method to solve Eq. (6). The terms on the right-hand side are of two types: a diffusion term and a convection term. The solution to the equation that contains just the diffusion term may be obtained by expansion of the solution in terms of Legendre polynomials. The solution to the equation that contains just the convection term is a translation that is performed numerically by use of a two-dimensional spline interpolation. A second-order (Strang-splitting) scheme is used to integrate along the Z direction.[8] This split-step method is unconditionally stable and efficient.

The results of simulations in which we set $D = 50$ are shown in Fig. 1. For larger D values, the results are almost the same, except at very short distances $(Z < 0.2)$. Figures 1(a)–1(g) show the DGD distribution averaged over all angles, $P(\tau, Z) \equiv 2\pi\tau^2\int_0^\pi P\sin\phi\,\mathrm{d}\phi$, at several distances. In Figs. 1(f) and 1(g), we also show a Maxwellian distribution for comparison. We observe that, initially, the distribution is a very sharp Maxwellian function because of our choice of initial condition (7) [see Fig. 1(a)]. As Z increases, p becomes more or less symmetric in shape [see Fig. 1(b)]. As Z increases further, p becomes skewed toward large τ values [see Figs. 1(c) and 1(d)], as observed in experiments.[3] At $Z \sim 9$, p becomes almost symmetric again [see Fig. 1(e)]. For larger values of Z, p is skewed toward smaller τ values and tends toward the asymptotic Maxwellian distribution

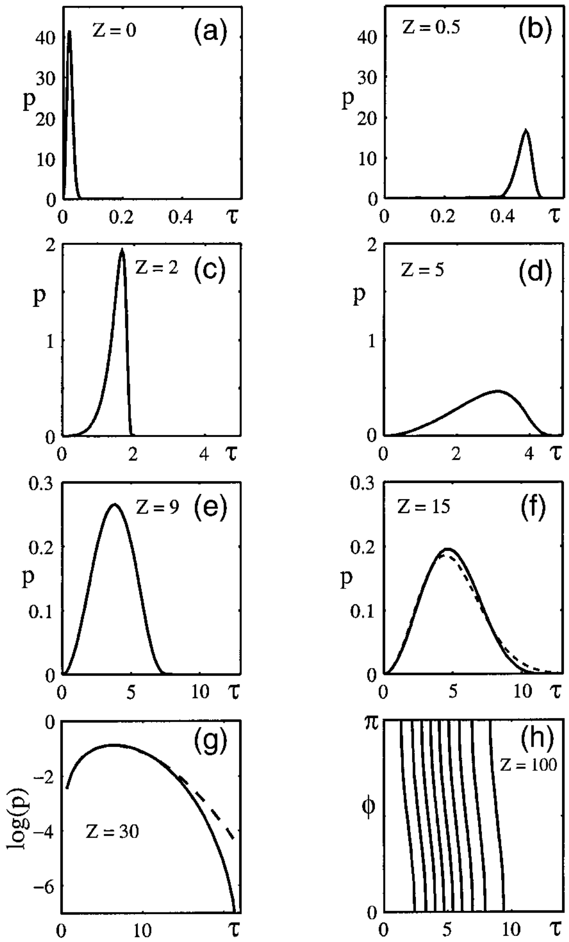

Fig. 1. (a)–(g) Probability distribution function $p(\tau, Z)$ of the DGD at various distances. The dashed curves in (f) and (g) are Maxwellian distributions. The distance is normalized to the fiber correlation length, h_{fiber}, and the DGD is normalized to $2b'h_{\text{fiber}}$. (h) Contour plot showing the angular dependence of probability distribution P for dispersion vector $\bar{\mathbf{\Omega}}$ at $Z = 100$.

[see Figs. 1(f) and 1(g)]. At $Z \sim 30$, p is already very close to the Maxwellian distribution (but deviation is still apparent at large DGD values). For larger distances, the difference is even smaller. To interpret these results, let us take a relatively large fiber correlation length of 100 m. Then, the results presented above simply indicate that, if the distance is hundreds of meters, then the DGD distribution is quite different from Maxwellian, but in a couple of kilometers the distribution becomes Maxwellian. This theoretical result is consistent with previous experiments.[3] It is noted that the DGD probability distribution at large DGD values (the tail region) is of particular interest to the prediction of outrage probabilities. Figure 1(g) indicates that this tail distribution takes much longer to approach Maxwellian. Indeed, our numerical results show that it takes over 100 fiber correlation lengths for the tail to approach Maxwellian.

Our numerical results also reveal an angular (ϕ) dependence of the probability distribution, P, that persists over arbitrarily long distances. A contour plot of P on the (τ, ϕ) plane at $Z = 100$ is shown in Fig. 1(h). We find that the DGD is larger along $\phi = 0$ and smaller along $\phi = \pi$. This angular dependence implies that at the output of the fiber the expected DGD will be correlated with the angle between polarization dispersion vector $\mathbf{\Omega}$ and the local birefringence vector on the Poincaré sphere; in particular, the expected DGD will be larger when $\mathbf{\Omega}$ is aligned with the slow axis of local birefringence on the Poincaré sphere.

The amount the probability distribution shifts when ϕ varies from 0 to π in Fig. 1(h) is 2 units in dimensionless variables. In dimensional quantities this is $4b'h_{\text{fiber}}$, which for the example parameters discussed above totals 0.32 ps. An interpretation of this follows if one thinks of the fiber as sections of length h_{fiber} randomly oriented with respect to one another and DGD accumulates at the rate $2b'$; then, the last section increases the total DGD by an additional $2b'h_{\text{fiber}}$ if $\mathbf{\Omega}$ is aligned with the slow birefringence axis but decreases it by the same amount if $\mathbf{\Omega}$ is aligned with the fast axis.

In summary, we have determined the PMD probability distribution for arbitrary distances with a realistic birefringence model. First, we have shown that for a fiber with a correlation length that is of the same order as or larger than its beat length, the Fokker–Planck equation governing the evolution of the probability density function is independent of all parameters. In addition, by numerically solving this equation, we have shown that the probability density function for the DGD approaches a Maxwellian distribution in ~30 fiber correlation lengths, i.e., a couple of kilometers for typical parameters. We also find that the probability distribution function for the polarization dispersion vector at the output of the fiber depends on the angle between it and the local birefringence vector on the Poincaré sphere, showing that the DGD remains correlated with the orientation of the local birefringence axes over arbitrarily long distances.

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