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# Polarization Optimality of Equally Spaced Points on the Circle for Discrete Potentials

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**Abstract** We prove a conjecture of Ambrus, Ball and Erdélyi that equally spaced points maximize the minimum of discrete potentials on the unit circle whenever the potential is of the form

$$\sum_{k=1}^{n} f(d(z, z_k)),$$

where $f : [0, \pi] \to [0, \infty]$ is non-increasing and convex and $d(z, w)$ denotes the geodesic distance between $z$ and $w$ on the circle.

**Keywords** Polarization · Chebyshev constants · Roots of unity · Potentials · Max-min problems

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## 1 Introduction and Main Results

Let $\mathbb{S}^1 := \{z = x + \mathrm{i}y \mid x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$ denote the unit circle in the complex plane $\mathbb{C}$. For $z, w \in \mathbb{S}^1$, we denote by $d(z, w)$ the geodesic (shortest arclength) distance between $z$ and $w$. Let $f : [0, \pi] \to [0, \infty]$ be non-increasing and convex on $(0, \pi]$

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with $f(0) = \lim_{\theta\to 0^+} f(\theta)$. It then follows that $f$ is a continuous extended real-valued function on $[0, \pi]$.

For a list of $n$ points (not necessarily distinct) $\omega_n = (z_1, \dots, z_n) \in (\mathbb{S}^1)^n$, we consider the *$f$-potential* of $\omega_n$,

$$U^f(\omega_n; z) := \sum_{k=1}^{n} f(d(z, z_k)) \quad (z \in \mathbb{S}^1), \tag{1}$$

and the *$f$-polarization* of $\omega_n$,

$$M^f(\omega_n; \mathbb{S}^1) := \min_{z\in\mathbb{S}^1} U^f(\omega_n; z). \tag{2}$$

In this note, we are chiefly concerned with the *$n$-point $f$-polarization* of $\mathbb{S}^1$ (also called the *$n$th $f$-Chebyshev constant* of $\mathbb{S}^1$),

$$M_n^f(\mathbb{S}^1) := \sup_{\omega_n\in(\mathbb{S}^1)^n} M^f(\omega_n; \mathbb{S}^1), \tag{3}$$

which has been the subject of several recent papers (e.g., [1,2,5,6]).

In the case (relating to Euclidean distance) when

$$f(\theta) = f_s(\theta) := |e^{i\theta} - 1|^{-s} = (2\sin|\theta/2|)^{-s}, \quad s > 0, \tag{4}$$

we abbreviate the notation for the above quantities by writing

$$\begin{aligned} U^s(\omega_n; z) &:= \sum_{k=1}^{n} f_s(d(z, z_k)) = \sum_{k=1}^{n} \frac{1}{|z - z_k|^s}, \\ M^s(\omega_n; \mathbb{S}^1) &:= \min_{z\in\mathbb{S}^1} \sum_{k=1}^{n} \frac{1}{|z - z_k|^s}, \\ M_n^s(\mathbb{S}^1) &:= \sup_{\omega_n\in(\mathbb{S}^1)^n} M^s(\omega_n; \mathbb{S}^1). \end{aligned} \tag{5}$$

The main result of this note is the following theorem conjectured by Ambrus et al. [2]. Its proof is given in the next section.

**Theorem 1** *Let $f : [0, \pi] \to [0, \infty]$ be non-increasing and convex on $(0, \pi]$ with $f(0) = \lim_{\theta\to 0^+} f(\theta)$. If $\omega_n$ is any configuration of $n$ distinct equally spaced points on $\mathbb{S}^1$, then $M^f(\omega_n; \mathbb{S}^1) = M_n^f(\mathbb{S}^1)$. Moreover, if the convexity condition is replaced by strict convexity, then such configurations are the only ones that achieve this equality.*

Applying this theorem to the case of $f_s$ given in (4) we immediately obtain the following.

**Corollary 2** *Let $s>0$ and $\omega_n^* := \{e^{i2\pi k/n} : k=1,2,\dots,n\}$. If $(z_1,\dots,z_n)\in(\mathbb{S}^1)^n$, then*

$$\min_{z\in\mathbb{S}^1}\sum_{k=1}^{n}\frac{1}{|z-z_k|^s}\le M^s(\omega_n^*;\mathbb{S}^1)=M_n^s(\mathbb{S}^1), \tag{6}$$

*with equality if and only if $(z_1,\dots,z_n)$ consists of distinct equally spaced points.*

The following representation of $M^s(\omega_n^*;\mathbb{S}^1)$ in terms of *Riesz s-energy* was observed in [2]:

$$M^s(\omega_n^*;\mathbb{S}^1)=\frac{\mathcal{E}_s(\mathbb{S}^1;2n)}{2n}-\frac{\mathcal{E}_s(\mathbb{S}^1;n)}{n},$$

where

$$\mathcal{E}_s(\mathbb{S}^1;n):=\inf_{\omega_n\in(\mathbb{S}^1)^n}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{|z_j-z_k|^s}.$$

Thus, applying the asymptotic formulas for $\mathcal{E}_s(\mathbb{S}^1;n)$ given in [3], we obtain the dominant term of $M_n^s(\mathbb{S}^1)$ as $n\to\infty$:

$$M_n^s(\mathbb{S}^1)\sim\begin{cases}\dfrac{2\zeta(s)}{(2\pi)^s}(2^s-1)n^s, & s>1\,,\\ (1/\pi)\,n\log n, & s=1\,,\\ \dfrac{2^{-s}}{\sqrt{\pi}}\dfrac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}n, & s\in[0,1),\end{cases}$$

where $\zeta(s)$ denotes the classical Riemann zeta function and $a_n\sim b_n$ means that $\lim_{n\to\infty}a_n/b_n=1$. These asymptotics, but for $M^s(\omega_n^*;\mathbb{S}^1)$, were stated in [2][1].

For $s$ an even integer, say $s=2m$, the precise value of $M_n^{2m}(\mathbb{S}^1)=M^{2m}(\omega_n^*;\mathbb{S}^1)$ can be expressed in finite terms, as can be seen from formula (1.20) in [3].

**Corollary 3** *We have*

$$M_n^{2m}(\mathbb{S}^1)=\frac{2}{(2\pi)^{2m}}\sum_{k=1}^{m}n^{2k}\zeta(2k)\alpha_{m-k}(2m)(2^{2k}-1),\quad m\in\mathbb{N},$$

*where $\alpha_j(s)$ is defined via the power series for* $\operatorname{sinc} z=(\sin\pi z)/(\pi z)$*:*

$$(\operatorname{sinc} z)^{-s}=\sum_{j=0}^{\infty}\alpha_j(s)z^{2j}\,,\quad \alpha_0(s)=1.$$

[1] We remark that there is a factor of $2/(2\pi)^p$ missing in the asymptotics given in [2] for the case $p:=s>1$.

*In particular,*

$$
\begin{aligned}
M_n^2(\mathbb{S}^1) &= \frac{2}{(2\pi)^2} n^2 \zeta(2) = \frac{n^2}{4},\\
M_n^4(\mathbb{S}^1) &= \frac{2}{(2\pi)^4}[n^2\zeta(2)\alpha_1(4)(2^2-1) + n^4\zeta(4)(2^4-1)] = \frac{n^2}{24} + \frac{n^4}{48},\\
M_n^6(\mathbb{S}^1) &= \frac{2}{(2\pi)^6}[n^2\zeta(2)\alpha_2(6)(2^2-1) + n^4\zeta(4)\alpha_1(6)(2^4-1) + n^6\zeta(6)(2^6-1)]\\
&= \frac{n^2}{120} + \frac{n^4}{192} + \frac{n^6}{480},
\end{aligned}
$$

The case $s = 2$ of the above corollary was first proved in [1,2] and the case $s = 4$ was first proved in [5]. We remark that an alternative formula for $\alpha_j(s)$ is

$$\alpha_j(s) = \frac{(-1)^j B_{2j}^{(s)}(s/2)}{(2j)!}(2\pi)^{2j}, \quad j = 0, 1, 2, \ldots,$$

where $B_j^{(\alpha)}(x)$ denotes the generalized Bernoulli polynomial. Asymptotic formulas for $M_n^f(\mathbb{S}^1)$ for certain other functions $f$ can be obtained from the asymptotic formulas given in [4].

As other consequences of Theorem 1, we immediately deduce that equally spaced points are optimal for the following problems:

$$\min_{\omega_n \in (\mathbb{S}^1)^n} \max_{z \in \mathbb{S}^1} \sum_{k=1}^n |z - z_k|^\alpha \quad (0 < \alpha \le 1), \tag{7}$$

and

$$\max_{\omega_n \in (\mathbb{S}^1)^n} \min_{z \in \mathbb{S}^1} \sum_{k=1}^n \log \frac{1}{|z - z_k|}, \tag{8}$$

with the solution to (8) being well-known. Furthermore, various generalizations of the polarization problem for Riesz potentials for configurations on $\mathbb{S}^1$ are worthy of consideration, such as minimizing the potential on circles concentric with $\mathbb{S}^1$.

## 2 Proof of Theorem 1

For distinct points $z_1, z_2 \in \mathbb{S}^1$, we let $\widehat{z_1 z_2}$ denote the closed subarc of $\mathbb{S}^1$ from $z_1$ to $z_2$ traversed in the counterclockwise direction. We further let $\gamma(\widehat{z_1 z_2})$ denote the length of $\widehat{z_1 z_2}$ (thus, $\gamma(\widehat{z_1 z_2})$ equals either $d(z_1, z_2)$ or $2\pi - d(z_1, z_2)$). Observe that the points $z_1$ and $z_2$ partition $\mathbb{S}^1$ into two subarcs: $\widehat{z_1 z_2}$ and $\widehat{z_2 z_1}$. The following lemma (see proof of Lemma 1 in [2]) is a simple consequence of the convexity and monotonicity of the function $f$ and is used to show that any $n$-point configuration $\omega_n \subset \mathbb{S}^1$ such that

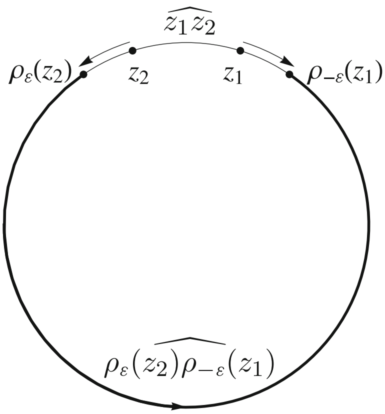


**Fig. 1** The points $z_1, z_2, \rho_{-\varepsilon}(z_1), \rho_\varepsilon(z_2)$ in Lemma 4. The potential increases at every point in the subarc $\rho_\varepsilon(\widehat{z_2)\rho_{-\varepsilon}}(z_1)$ when $(z_1, z_2) \to (\rho_{-\varepsilon}(z_1), \rho_\varepsilon(z_2))$; see (9).

$M^f(\omega_n; \mathbb{S}^1) = M_n^f(\mathbb{S}^1)$ must have the property that any local minimum of $U^f(\omega_n; \cdot)$ is a global minimum of this function (Fig. 1).

For $\phi \in \mathbb{R}$ and $z \in \mathbb{S}^1$, we let $\rho_\phi(z) := e^{i\phi} z$ denote the counterclockwise rotation of $z$ by the angle $\phi$.

**Lemma 4** ([2]) *Let $z_1, z_2 \in \mathbb{S}^1$ and $0 < \varepsilon < \gamma(\widehat{z_2 z_1})/2$. Then with f as in Theorem 1,*

$$U^f((z_1, z_2); z) \leq U^f((\rho_{-\varepsilon}(z_1), \rho_\varepsilon(z_2)); z) \tag{9}$$

*for z in the subarc $\rho_\varepsilon(\widehat{z_2)\rho_{-\varepsilon}}(z_1)$, while the reverse inequality holds for z in the subarc $\widehat{z_1 z_2}$. If f is strictly convex on $(0, \pi]$, then these inequalities are strict. If $z_1 = z_2$, then we set $\widehat{z_1 z_2} = \{z_1\}$ and $\widehat{z_2 z_1} = \mathbb{S}^1$.*

We now assume that $\omega_n = (z_1, \ldots, z_n)$ is ordered in a counterclockwise manner and also that the indexing is extended periodically so that $z_{k+n} = z_k$ for $k \in \mathbb{Z}$. For $1 \leq k \leq n$ and $\Delta \in \mathbb{R}$, we define $\tau_{k,\Delta} : (\mathbb{S}^1)^n \to (\mathbb{S}^1)^n$ by

$$\tau_{k,\Delta}(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n) := (z_1, \ldots, \rho_{-\Delta}(z_k), \rho_\Delta(z_{k+1}), \ldots, z_n).$$

If $z_{k-1} \neq z_k$ and $z_{k+1} \neq z_{k+2}$, then $\tau_{k,\Delta}(\omega_n)$ retains the ordering of $\omega_n$ for $\Delta$ positive and sufficiently small. Given $\mathbf{\Delta} := (\Delta_1, \ldots, \Delta_n)^T \in \mathbb{R}^n$, let $\tau_{\mathbf{\Delta}} := \tau_{n,\Delta_n} \circ \cdots \circ \tau_{2,\Delta_2} \circ \tau_{1,\Delta_1}$ and $\omega_n' := \tau_{\mathbf{\Delta}}(\omega_n)$. Letting $\alpha_k := \gamma(\widehat{z_k z_{k+1}})$ and $\alpha_k' := \gamma(\widehat{z_k' z_{k+1}'})$ for $k = 1, \ldots, n$, we obtain the system of $n$ linear equations:

$$\alpha_k' = \alpha_k - \Delta_{k-1} + 2\Delta_k - \Delta_{k+1} \quad (1 \leq k \leq n), \tag{10}$$

which is satisfied as long as $\sum_{k=1}^n \alpha_k' = 2\pi$ or, equivalently, if $\omega_n'$ is ordered counterclockwise. Let

$$\operatorname{sep}(\omega_n) := \min_{1 \leq \ell \leq n} \alpha_\ell.$$

Then (10) holds if

$$\max_{1\le k\le n} |\Delta_k| \le (1/4)\mathrm{sep}(\omega_n), \tag{11}$$

in which case, the configurations

$$\omega_{n,\boldsymbol{\Delta}}^{(\ell)} := \tau_{n,\Delta_\ell} \circ \cdots \circ \tau_{2,\Delta_2} \circ \tau_{1,\Delta_1}(\omega_n) \quad (\ell = 1, \ldots, n) \tag{12}$$

are all ordered counterclockwise. If the components of $\boldsymbol{\Delta}$ are nonnegative, then we may replace the '(1/4)' in (11) with '(1/2)'.

**Lemma 5** *Suppose $\omega_n = (z_1, \ldots, z_n)$ and $\omega_n' = (z_1', \ldots, z_n')$ are n-point configurations on $\mathbb{S}^1$ ordered in a counterclockwise manner. Then there is a unique $\boldsymbol{\Delta}^* = (\Delta_1^*, \ldots, \Delta_n^*) \in \mathbb{R}^n$ so that*

(a) $\Delta_k^* \ge 0, \quad k = 1, \ldots, n,$
(b) $\Delta_j^* = 0$ *for some $j \in \{1, \ldots, n\}$, and*
(c) $\tau_{\boldsymbol{\Delta}^*}(\omega_n)$ *is a rotation of $\omega_n'$.*

*Proof* The system (10) can be expressed in the form

$$A\boldsymbol{\Delta} = \boldsymbol{\beta}, \tag{13}$$

where

$$A := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}; \quad \boldsymbol{\Delta} := \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\beta} := \begin{pmatrix} \alpha_1' - \alpha_1 \\ \alpha_2' - \alpha_2 \\ \vdots \\ \alpha_n' - \alpha_n \end{pmatrix}.$$

It is elementary to verify that $\ker A = (\mathrm{range} A)^\perp = \mathrm{span}\,(\mathbf{1})$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. Since $\boldsymbol{\beta}^T\mathbf{1} = \sum_{k=1}^n (\alpha_k' - \alpha_k) = 0$, the linear system (13) always has a solution $\boldsymbol{\Delta}$. Let $j \in \{1, \ldots, n\}$ satisfy $\Delta_j = \min_{1\le k\le n} \Delta_k$. Then subtracting $\Delta_j\mathbf{1}$ from $\boldsymbol{\Delta}$, we obtain the desired $\boldsymbol{\Delta}^*$. Since $\ker A = \mathrm{span}\,\mathbf{1}$, there is at most one solution of (13) satisfying properties (a) and (b), showing that $\boldsymbol{\Delta^*}$ is unique.

Part (c) holds as a direct result of the fact that both $\omega_n$ and $\omega_n'$ are ordered counterclockwise.

**Lemma 6** *Let $\Omega_n = (z_1, \ldots, z_n)$ be a configuration of n distinct points on $\mathbb{S}^1$ ordered counterclockwise, and with f as in Theorem 1, suppose $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_n) \in \mathbb{R}^n$ is such that*

(a) $0 \le \Delta_k \le (1/2)\mathrm{sep}(\Omega_n)$ *for $k = 1, \ldots, n$, and*
(b) *there is some $j \in \{1, \ldots, n\}$ for which $\Delta_j = 0$.*

*Let* $\Omega_n' := \tau_{\mathbf{\Delta}}(\Omega_n) = (z_1', \dots, z_n')$. *Then* $\widehat{z_j' z_{j+1}'} \subset \widehat{z_j z_{j+1}}$ *and*

$$U^f(\Omega_n; z) \le U^f(\Omega_n'; z) \quad (z \in \widehat{z_j' z_{j+1}'}). \tag{14}$$

*If f is strictly convex on* $(0, \pi]$ *and* $\Delta_k > 0$ *for at least one k, then the inequality* (14) *is strict.*

We remark that $\Delta_k = 0$ for all $k = 1, \dots, n$ is equivalent to saying that the points are equally spaced.

*Proof* Recalling (12), it follows from condition (a) that $(z_1^{(\ell)}, \dots, z_n^{(\ell)}) := \omega_{n,\mathbf{\Delta}}^{(\ell)}$ are counterclockwise ordered. Since $\Delta_j = 0$ and $\Delta_k \ge 0$ for $k = 1, \dots, n$, the points $z_j^{(\ell)}$ and $z_{j+1}^{(\ell)}$ are moved at most once as $\ell$ varies from 1 to $n$ and move toward each other, while remaining in the complement of all other subarcs $\widehat{z_k^{(\ell)} z_{k+1}^{(\ell)}}$, i.e.,

$$\widehat{z_j' z_{j+1}'} = \widehat{z_j^{(n)} z_{j+1}^{(n)}} \subseteq \widehat{z_j^{(\ell)} z_{j+1}^{(\ell)}} \subseteq \widehat{z_{k+1}^{(\ell)} z_k^{(\ell)}},$$

for $k \in \{1, \dots, n\} \setminus \{j\}$ and $\ell \in \{1, \dots, n\}$. Lemma 4 implies that, for $\ell = 1, \dots, n$, we have $U^f(\omega_n^{(\ell-1)}; z) \le U^f(\omega_n^{(\ell)}; z)$ for $z \in \widehat{z_j^{(\ell)} z_{j+1}^{(\ell)}}$ (where $\omega_n^{(0)} := \omega_n$) and the inequality is strict if $\Delta_\ell > 0$. Hence, (14) holds and the inequality is strict if $f$ is strictly convex and $\Delta_k > 0$ for some $k = 1, \dots, n$. □

We now proceed with the proof of Theorem 1. Let $\omega_n = (z_1, \dots, z_n)$ be a non-equally spaced configuration of $n$ (not necessarily distinct) points on $\mathbb{S}^1$ ordered counterclockwise. By Lemma 5, there is some equally spaced configuration $\omega_n'$ (i.e., $\alpha_k' = 2\pi/n$ for $k = 1, \dots, n$) and some $\mathbf{\Delta^*} = (\Delta_1^*, \dots, \Delta_n^*)$ such that (a) $\omega_n' = \tau_{\mathbf{\Delta^*}}(\omega_n)$, (b) $\Delta_k^* \ge 0$ for $k = 1, \dots, n$, and (c) $\Delta_j^* = 0$ for some $j \in \{1, \dots, n\}$. Then (10) holds with $\alpha_k := \gamma(\widehat{z_k, z_{k+1}})$ and $\alpha_k' := 2\pi/n$. Since $\omega_n$ is not equally spaced, we have $\Delta_k^* > 0$ for at least one value of $k$.

For $0 \le t \le 1$, let $\omega_n^t := \tau_{(t\mathbf{\Delta^*})}(\omega_n) = (z_1^t, \dots, z_n^t)$ and, for $k = 1, \dots, n$, let $\alpha_k^t := \gamma(\widehat{z_k^t z_{k+1}^t})$. Recalling (10), observe that

$$\begin{aligned} \alpha_k^t &= \alpha_k - t(\Delta_{k-1} + 2\Delta_k - \Delta_{k+1}) \\ &= \alpha_k + t(2\pi/n - \alpha_k) \\ &= (1-t)\alpha_k + t(2\pi/n), \end{aligned}$$

for $0 \le t \le 1$ and $k = 1, \dots, n$, and so $\operatorname{sep}(\omega_n^t) \ge t(2\pi/n)$. Now let $0 < t < s < \min(1, t(1 + \pi/(nD)))$, where $D := \max\{\Delta_k : 1 \le k \le n\}$. Then Lemma 6 (with $\Omega_n = \omega_n^t$, $\mathbf{\Delta} = (s-t)\mathbf{\Delta^*}$, and $\Omega_n' = \tau_{\mathbf{\Delta}}(\Omega_n) = \omega_n^s$) implies that $\widehat{z_j^s z_{j+1}^s} \subseteq \widehat{z_j^t z_{j+1}^t}$ and that

$$U^f(\omega_n^t; z) \le U^f(\omega_n^s; z) \quad (z \in \widehat{z_j^s z_{j+1}^s}), \tag{15}$$

where the inequality is sharp if $f$ is strictly convex.

Consider the function

$$h(t) := \min\{U^f(\omega_n^t; z) : z \in \widehat{z_j^t z_{j+1}^t}\}, \quad (0 \le t \le 1).$$

Observe that

$$h(t) \le \min\{U^f(\omega_n^t; z) : z \in \widehat{z_j^s z_{j+1}^s}\} \le \min\{U^f(\omega_n^s; z) : z \in \widehat{z_j^s z_{j+1}^s}\} = h(s),$$

for $0 < t < s < \min(1, t(1 + \pi/(nD)))$. It is then easy to verify that $h$ is non-decreasing on $(0, 1)$. Since $\omega_n^t$ depends continuously on $t$, the function $h$ is continuous on $[0, 1]$ and thus $h$ is non-decreasing on $[0, 1]$.

We then obtain the desired inequality

$$M^f(\omega_n; \mathbb{S}^1) \le h(0) \le h(1) = M^f(\omega_n'; \mathbb{S}^1),$$

where the last equality is a consequence of the fact that $\omega_n'$ is an equally spaced configuration and so the minimum of $U^f(\omega_n'; z)$ over $\mathbb{S}^1$ is the same as the minimum over $\widehat{z_j' z_{j+1}'}$. If $f$ is strictly convex, then $h(0) < h(1)$ showing that any optimal $f$-polarization configuration must be equally spaced. This completes the proof of Theorem 1. □

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