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# Polarization-ring-switching for nonlinearity-tolerant geometrically-shaped four-dimensional formats maximizing generalized mutual information

Bin Chen, *Member, IEEE*, Chigo Okonkwo, *Senior Member, IEEE*, Hartmut Hafermann, and Alex Alvarado, *Senior Member, IEEE*.

**Abstract**—In this paper, a new four-dimensional 64-ary polarization ring switching (4D-64PRS) modulation format with a spectral efficiency of 6 bit/4D-sym is introduced. The format is designed by maximizing the generalized mutual information (GMI) and by imposing a constant-modulus on the 4D structure. The proposed format yields an improved performance with respect to state-of-the-art geometrically shaped modulation formats for bit-interleaved coded modulation systems at the same spectral efficiency. Unlike previously published results, the coordinates of the constellation points and the binary labeling of the constellation are *jointly* optimized. When compared against polarization-multiplexed 8-ary quadrature-amplitude modulation (PM-8QAM), gains of up to 0.7 dB in signal-to-noise ratio are observed in the additive white Gaussian noise (AWGN) channel. For a long-haul nonlinear optical fiber system of 8,000 km, gains of up to 0.27 bit/4D-sym ( 5.5% data capacity increase) are observed. These gains translate into a reach increase of approximately 16% (1,100 km). The proposed modulation format is also shown to be more tolerant to nonlinearities than PM-8QAM. Results with LDPC codes are also presented, which confirm the gains predicted by the GMI.

**Index Terms**—Achievable information rates, binary labeling, generalized mutual information, mutual information, forward error correction, multidimensional constellations, signal shaping.

## I. INTRODUCTION, MOTIVATION AND STATE OF THE ART

Signal shaping can be used in optical fiber communications to close the gap to the channel capacity, either via probabilistic shaping (PS) or geometric shaping (GS). In the former, long coded sequences induce certain nonuniform probability distribution on the constellation points [1]–[5]. This can be achieved, e.g., by using trellis shaping [6], shell mapping [7], [8], or distribution matching (DM) techniques [9]. In geometrical shaping, non-equidistant constellation points are used

with the same probabilities. Recently, geometrical shaping has received considerable attention in the optical communications literature [10]–[15]. For the additive white Gaussian noise (AWGN) channel, both PS and GS achieve Shannon’s channel capacity when the number of constellation points tends to infinity. Hybrid approaches have been investigated in [16], [17].

Although PS has superior achievable information rates (AIR) performance for a finite number of constellation points with respect to GS [18, Sec. 4.2], [19], PS requires the use of sophisticated coding techniques. DM encoders are typically used in PS and they offer very good performance for long block lengths. However, DM encoders for long block lengths are difficult to realize in practice and hardware implementation of the DM remains a significant challenge [20]. Furthermore, for short block lengths, DM encoders suffer from rate losses. Recently, partition-based DM has been proposed to overcome these limitations [21].

The main advantage of GS over PS is that it does not require sophisticated coding techniques. GS relies only on properly placing the constellation points and straightforward modifications of the mapper and demapper. The main drawbacks of GS are tighter requirement on the resolution of the digital-to-analog and analog-to-digital converters and the need for a modified equalizer. Just like PS, GS also increases the nonlinear effects of the fiber. Different studies have shown that these nonlinear effects can be mitigated by combining multi-dimensional modulations with GS [12], [22]–[24]. In this paper, we consider this approach using AIRs as a design metric.

AIRs such as mutual information (MI) and generalized mutual information (GMI) have emerged as practical tools to design optical fiber communication systems [25], [26]. AIRs have also been used to design modulation formats and to predict the performance of forward error correction (FEC) [27], [28]. GMI can be directly connected to modern binary soft-decision forward error correction (SD-FEC) based on bit-interleaved coded modulation (BICM). The practical relevance of GMI makes it the preferred alternative for optical fiber communication systems design. On the other hand, the MI can be used as an upper bound on the performance of binary SD-FEC performance (when combined with a nonbinary modulation schemes). However, the multidimensional modulations optimized for MI have been shown to be non-optimal in terms of GMI [25]. Hence, using the GMI as a figure of merit is

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essential to design a modulation, which is well-suited for bit-wise decoders.

As mentioned above, both PS and GS typically increases the (modulation-dependent) nonlinear effects in the fiber. In particular, in multi-span long-haul transmission systems, nonlinear interference noise (NLIN) modelling and numerical simulations have shown that both PS and GS decrease the effective signal-to-noise ratio (SNR) due to the increasing contribution of modulation format dependent NLIN. At the same time, PS and GS increase the AIRs. If the shaping is properly designed, the effective SNR decrease is less than the AIR increase, and therefore, an overall performance improvement is observed [2], [29].

In order to cover a wide range of channel conditions in flexible networks with multiple modulation formats for efficient network usage, multiple GS-based modulation formats with different spectral efficiency have been investigated in the optical communications literature. This includes for example GS-16QAM [10], [14], GS-32QAM [11], GS-64QAM [14], [15], GS-256QAM [14] and GS-APSK [12], [13]. GS has been reported to outperform PS constellations in numerical simulations [13], which is particularly noticeable for dispersion managed links [12], [13].

PM-8QAM with 3 bit/2D-sym plays an important role in filling the gap between quaternary phase shift keying (QPSK) and 16QAM in terms of bit rates and reach. To achieve the same spectral efficiency (3 bit/2D-sym) with improved performance, advanced 2D modulation formats, such as Circular-8QAM [30] and optimized-8QAM [31], have been reported to outperform regular PM-8QAM. As an alternative, multidimensional (MD) modulation formats have also attracted a lot of attention in optical communications by transmitting information jointly in each degree of freedom (polarization, time slot, wavelength, modes, spatial channels, etc.). The most popular 4-dimensional (4D) modulation formats were proposed to exploit the natural 4 dimensions in the optical field: in-phase and quadrature from two orthogonal polarizations. In this paper, we target this 4D design approach for the important case of constellations with 6 bit/4D-sym.

Recently, three 4D modulation formats 4D-64SP-12QAM [32], 4D-8QAM [23] and 4D-2A8PSK [12] have been shown to be superior to PM-8QAM and other previous modulation formats for BICM system. These 4D modulation formats are designed by either increasing the Euclidean distances [32] and minimizing the uncoded bit error rate (BER) [23] in 4D space (improved linear performance), or by increasing the tolerance to nonlinearity (improved nonlinear performance) [12]. In [32], 4D-64SP-12QAM was proposed, which is a subset of PM-16QAM, and where the points selection of constellation points was made to maximize Euclidean distances. 4D-64SP-12QAM outperforms regular 16QAM and also provides 0.2 dB gain over PM-8QAM (linear gain). 4D-8QAM was optimized in [23] by minimizing the uncoded BER for the AWGN channel, providing a 0.5 dB gain over PM-8QAM. 4D-2A8PSK by Kojima et al. [12] has been shown to be superior to many other formats in both the linear and nonlinear regimes. This is due to its large Euclidean distance, a 4D constant-modulus constraint, and Gray labeling. The 4D constant-modulus is

obtained by forcing the two polarization to have complementary amplitudes from two 8-ary phase shift keying (PSK) with two different amplitudes. In [12], the GMI was optimized in terms of set-partitioning coding and two ring ratios only. The obtained linear performance improvement of 4D-2A8PSK over PM-8QAM was 0.3 dB.

From the literature review above, it is still unclear what is the largest GMI gain that can be achieved by designing a 4D modulation format. Since a 4D constant-modulus property is known to be effective in reducing the modulation-dependent NLIN (especially for dispersion-managed links), the focus of this paper is on maximizing GMI performance under a constant-modulus constraint. We first reveal that the constant-modulus property can significantly reduce the modulation-dependent nonlinear noise through theoretical analysis of the effective SNR, which we believe also applies to 4D modulation formats. Secondly, we use GMI to design a 4D modulation format with a spectral efficiency of 6 bit/4D-sym for BICM system. The coordinates of the constellation points and the binary labeling of the constellation are *jointly* optimized. Thirdly, we show that the proposed 4D-64PRS modulation outperforms all previously known formats of the same spectral efficiency, in both the linear and nonlinear regimes. Shaping gains of 0.7 dB and 0.4 dB can be achieved with respect to PM-8QAM and 4D-2A8PSK for the AWGN channel, respectively. Nonlinear optical channel simulations confirm that constant-modulus 4D-64PRS modulation formats increase the effective SNR with respect to that of regular QAM modulation and that the proposed 4D modulation format can further increase AIRs. This gain includes a 0.16 dB nonlinear gain that is additional to the linear gains. These gains translate into reach increases in the range of 16%.

This paper is organized as follows. In Sec. II, the system model, the design methodology and the proposed modulation format are introduced. In Sec. III, numerical results for the proposed 4D modulation format are shown after both AWGN and nonlinear optical fiber simulations. The importance of joint optimization of coordinates and labeling in 4D modulation design is discussed in Sec. III-C. Finally, conclusions are drawn in Sec. IV.

## II. AIR-BASED GEOMETRIC SHAPING OPTIMIZATION

Fig. 1 shows the model under consideration, representing a generic multi-span optical fiber communication system. At the transmitter side, a FEC encoder generates  $m$  binary code bits  $c_{1,k}, \dots, c_{m,k}$ , where  $k = 1, 2, \dots, N_{\text{sym}}$  represents discrete time and  $N_{\text{sym}}$  is the number of symbols transmitted over the optical channel. These code bits are then mapped to symbols with  $N$  real dimensions drawn from a constellation with  $M = 2^m$  constellation points. In coherent fiber optical communications, the most natural selection is  $N = 4$ , which corresponds to communication using the two polarizations (each polarization includes two real dimensions, in-phase  $I$  and quadrature  $Q$ ). For a conventional 2D modulation format, i.e. 8QAM, the symbols transmitted over two polarization are statically independent of each other and can be considered as (polarization-multiplexed) 4D symbols. However, a true 4D

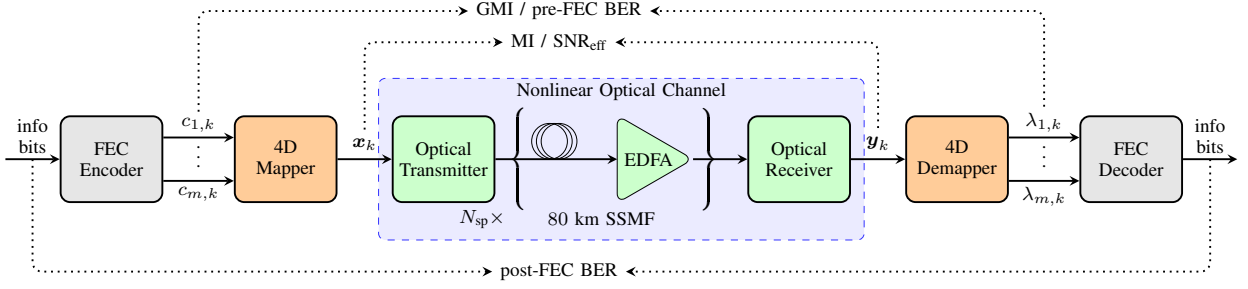


Fig. 1. System model under consideration. The nonlinear optical channel is modeled using a channel with an optical fiber link comprising multiple spans ( $N_{sp}$  is the number of spans). Each span is followed by an EDFA. The MI defined in (1) and GMI in (2) respectively, are also shown.

symbol will be jointly modulated in the 4D space created by the two polarizations. The resulting 4D symbols  $\mathbf{x}_k$  are transmitted through a nonlinear optical channel. This channel consists of  $N_{sp}$  spans of optical fibers, each of them followed by an erbium-doped fiber amplifier (EDFA). At the receiver side, after opto-electronic conversion and digital signal processing, the symbols  $\mathbf{y}_k$  are obtained. These symbols are then demapped into soft bits (represented as logarithmic likelihood ratios, LLRs)  $\lambda_{1,k}, \dots, \lambda_{m,k}$ , which are then passed to the SD-FEC decoder.<sup>1</sup>

The largest AIR for a MD memoryless channel with input  $\mathbf{X}$  and output  $\mathbf{Y}$  is given by the MI:

$$I(\mathbf{X}; \mathbf{Y}) = \mathbb{E} \left[ \log_2 \frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{Y}|\mathbf{X})}{f_{\mathbf{Y}}(\mathbf{Y})} \right], \quad (1)$$

where  $f_{\mathbf{Y}|\mathbf{X}}$  is the channel law. The most popular AIR for a bit interleaved coded modulation (BICM) is the GMI:

$$G = \sum_{i=1}^m I(C_i; \mathbf{Y}) = \sum_{i=1}^m \mathbb{E} \left[ \log_2 \frac{f_{\mathbf{Y}|C_i}(\mathbf{Y}|C_i)}{f_{\mathbf{Y}}(\mathbf{Y})} \right], \quad (2)$$

where  $\mathbf{C} = [C_1, C_2, \dots, C_m]$  is a random vector representing the transmitted bits  $[c_{1,k}, c_{2,k}, \dots, c_{m,k}]$  at time instant  $k$ , which are mapped to the corresponding symbol  $\mathbf{x}_k$ . LLRs are simply another way of representing (bit-wise) information, and thus, the GMI can also be analyzed in terms of these LLRs, as schematically shown in Fig. 1.

The transmitted symbols  $\mathbf{X}$  are assumed to be 4D symbols with 4 real dimensions drawn uniformly from a discrete constellation. The  $i$ th constellation point is denoted by  $\mathbf{s}_i = [s_{i,1}, s_{i,2}, s_{i,3}, s_{i,4}] \in \mathbb{R}^4$  with  $i = 1, 2, \dots, M$ . We use the  $M \times 4$  matrix  $\mathbb{S} = [\mathbf{s}_1^T, \mathbf{s}_2^T, \dots, \mathbf{s}_M^T]^T$  to denote the 4D constellation. The  $i$ th constellation point  $\mathbf{s}_i$  is labeled by the length- $m$  binary bit sequence  $\mathbf{b}_i = [b_{i,1}, \dots, b_{i,m}] \in \{0, 1\}^m$ . The binary labeling matrix is denoted by a  $M \times m$  matrix  $\mathbb{B} = [\mathbf{b}_1^T, \mathbf{b}_2^T, \dots, \mathbf{b}_M^T]^T$  which contains all unique length- $m$  binary sequences. The 4D constellation and its binary labeling are fully determined by the pair of matrices  $\{\mathbb{S}, \mathbb{B}\}$ .

Both the MI and GMI depend on the channel law. The MI also depends on the coordinates of the constellation

points  $\mathbb{S}$ . Thus, designing constellations to maximize the MI corresponds to finding the coordinates of  $M$  points, that satisfy a power constraint. The GMI depends also on the binary labeling  $\mathbb{B}$ . The GMI-based optimization problem is therefore more complex than the one for MI, as the binary labeling also needs to be taken into account. In [25], 4D constellations optimized for uncoded systems were shown to give high MI. Those constellations, however, were shown not to be well-suited for bit-wise decoders.

In what follows, we describe the methodology used for the design of the proposed 4D modulation format and its binary labeling. The main approach taken is to jointly optimize the binary labeling and the coordinates of the 4D constellation points by taking into account both linear and nonlinear performance. We briefly describe the theoretical foundation for this approach. Later in the section, details of the optimization procedure are presented.

### A. Theoretical Background

We consider a multi-span wavelength division multiplexing (WDM) transmission scenario where several channels are launched into a nonlinear optical fiber. The 4D noisy received signal for the channel of interest  $\mathbf{Y}_k$  can be modeled using the classic GN model as [33]

$$\mathbf{Y} = \mathbf{X} + \mathbf{Z}^{\text{NLI}} + \mathbf{Z}^{\text{ASE}}, \quad (3)$$

where  $\mathbf{X}$  are the 4D symbols transmitted over the channel of interest, and  $\mathbf{Z}^{\text{NLI}}$  is the interference induced by the nonlinearity of the fiber. We call this effect nonlinear interference (NLI), which includes intra-channel and inter-channel distortions. The (accumulated) amplified spontaneous emission (ASE) noise in the system is represented by  $\mathbf{Z}^{\text{ASE}}$ . In (3), both the NLI and ASE are modeled as additive Gaussian noise.<sup>2</sup> Furthermore, (3) is a memoryless channel, which is why we dropped the time index  $k$ .

The effective SNR (denoted by  $\text{SNR}_{\text{eff}}$ ) represents the SNR after fiber propagation and the receiver digital signal processing (DSP) and is defined as [34, Eq. (4)], [2, Eq. (16)]

$$\text{SNR}_{\text{eff}} \triangleq \frac{\mathbb{E} [\|\mathbf{X}\|^2]}{\mathbb{E} [\|\mathbf{Y} - \mathbf{X}\|^2]} = \frac{\sigma_x^2}{\sigma_z^2} = \frac{\sigma_x^2}{\sigma_{\text{ASE}}^2 + \sigma_{\text{NLI}}^2}, \quad (4)$$

<sup>2</sup>This assumption is well justified for dispersion uncompensated long-haul transmission systems.

<sup>1</sup>Throughout this paper we use the following notation. Matrices are denoted by blackboard bold letters  $\mathbb{X}$ . Random variables are denoted by capital letters  $X$  and random vectors by boldface letters  $\mathbf{X}$ . Their corresponding realizations are denoted by  $x$  and  $\mathbf{x}$ . All vectors are row vectors, where  $(\cdot)^T$  denotes transpose. Expectations are denoted by  $\mathbb{E}[\cdot]$  and the  $p$ -norm is defined as  $\|\mathbf{X}\|^p = |X_1|^p + |X_2|^p + \dots + |X_N|^p$ . Conditional probability density functions (PDFs) are denoted by  $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{Y}|\mathbf{X})$ .

where  $\sigma_x^2$  and  $\sigma_z^2$  represent the transmitted power and total noise power per two real dimensions, respectively. In (4),  $\sigma_{\text{ASE}}^2$  represents the variance of the ASE noise and  $\sigma_{\text{NLI}}^2$  the variance of the NLI that includes both intra- and inter-channel distortions.

The standard Gaussian Noise (GN) model [34] ignores dependency of nonlinear effects on the modulation format. Other more advanced models such as the NLIN model in [35] (see also [36]–[38]) allow more accurate analysis of non-conventional modulation formats, such as modulation with PS and GS. In these NLI models, it is assumed that the symbols transmitted over two polarizations are statistically independent of each other and each polarization uses identical 2D modulation formats. This corresponds to the so-called “polarization-multiplexed” systems, which includes for example PM-8QAM which we study in this paper. The NLI noise term  $\sigma_{\text{NLI}}^2$  for polarization multiplexed systems in (4) can be approximated as [2], [35], [38]

$$\sigma_{\text{NLI}}^2 \approx \underbrace{\eta_1 \cdot P^3}_{\text{Modulation-independent}} + \underbrace{P^3 [\eta_2 \cdot (\mu_4 - 2) + \eta_3 \cdot (\mu_4 - 2)^2 + \eta_4 \cdot \mu_6]}_{\text{Modulation-dependent}}, \quad (5)$$

where  $P$  is the optical launch power,  $\eta_1$ ,  $\eta_2$ ,  $\eta_3$  and  $\eta_4$  are constants (for a given system configuration) linked to the contributions of the modulation-independent and modulation-dependent nonlinearities, respectively. The expression in (5) includes traditional inter- and intra-channel effects as well as additional intra-channel terms for signal with dense WDM, as discussed in [38]. The four nonlinear terms  $\eta_1$ ,  $\eta_2$ ,  $\eta_3$  and  $\eta_4$  can be calculated via Monte Carlo integration from the ready-to-use MATLAB code presented in [39].

The coefficient  $\mu_4$  and  $\mu_6$  in (5) are the fourth and sixth standardized moments of the constellation used. In general, the  $p$ -th standardized moment  $\mu_p$  of the channel input  $\mathbf{X}$  is defined as

$$\mu_p = \frac{\mathbb{E}[\|\mathbf{X} - \mathbb{E}[\mathbf{X}]\|^p]}{(\mathbb{E}[\|\mathbf{X} - \mathbb{E}[\mathbf{X}]\|^2])^{\frac{p}{2}}}. \quad (6)$$

From (5) we can observe that NLIN is dependent on the modulation format. [37] shows that MPSK with constant-modulus can significantly reduce the NLI noise, which yield  $\mu_4 = 1$  and  $\mu_6 = 1$ . Constant-modulus modulation formats such as PSK yield  $\mu_4 = 1$  and  $\mu_6 = 1$ . However, performance of MPSK in terms of AIRs for the AWGN is poor because the amplitude of the symbols does not carry information. Half of the degrees of freedom in 2D space are therefore lost. On the other hand, either PS or GS with enough constellation points offer shaping gains for the AWGN channel of up to 1.53 dB. This is achieved by using a continuous Gaussian distribution, which in turn yields the larger modulation-dependent NLI ( $\mu_4 = 2$  and  $\mu_6 = 6$ ). Gaussian distributions are therefore optimal for the AWGN channel but result in a larger penalty in the nonlinear optical channel. PS with different input distributions have been investigated to trade-off between linear and nonlinear shaping gain [40]–[44]. The obtained gains are limited because there are limited degrees of freedom GS

optimization in 2D space [45].

To achieve larger shaping gains, more than two dimensions need to be used. The design should also aim at “shaping-out” the modulation-dependent NLIN term, as explained in [46]. To correctly shape-out the NLIN, a precise multidimensional model is required. However, current NLIN models available in the literature can only predict the modulation-dependent NLIN term of a polarization-multiplexed system with independent modulation formats over two polarization [47]. One exception to this is the general expression presented in [47, Eq. (3)], which gives an intuition that constant-modulus across two polarization can in fact reduce the induced NLIN due to self-phase modulation (SPM) and cross-phase modulation (XPM). At the same time, 4D constant-modulus formats have also been reported to show a significant reduction in SPM and XPM by separating nonlinear components. This was done via numerical simulations in [12, Sec. IV]. In this paper, the constant-modulus constraint for a 4D modulation will be taken into account.

## B. Design Strategy and Optimization

The theoretical analysis of the previous sections suggests that the MD modulation format design should maximize linear shaping gain (i.e., approach a Gaussian-like modulation) and at the same time minimize the shaping penalty due to the increased NLI (i.e., approach a constant-modulus modulation). Based on these observations, our optimized 4D modulation formats are designed in two rules:

- [R1] **Maximize Linear Performance:** Numerically optimize 4D modulation formats so that the GMI is maximized. This is achieved by jointly optimizing the 4D coordinates and the binary labeling.
- [R2] **Maximize Nonlinear Performance:** Add a constraint of constant modulus in 4D to reduce the nonlinear penalty.

In this paper, we consider constellations with SEs of 6 bit/4D-sym, which are comparable with PM-8QAM. The modulation format reported below includes a constant-modulus constraint, which cause a small loss with respect to the best achievable linear performance. This constraint, however, results in considerably better nonlinear performance. We believe this methodology can be also extended to designing modulation formats with SEs between 7 and 10 bit/4D-sym by using quasi-constant modulus to trade-off shaping gain and nonlinear penalty.

The GMI-based optimization problem we solve in this paper corresponds to find a constant-modulus constellation  $\mathbb{S}$  and labeling  $\mathbb{B}$  for a given noise variance  $\sigma_z^2$  with under an energy constraint, i.e.,

$$\{\mathbb{S}^*, \mathbb{B}^*\} = \underset{\mathbb{S}, \mathbb{B}}{\operatorname{argmax}} \{G(\sigma_z^2, \mathbb{S}, \mathbb{B})\}, \quad (7)$$

$$\text{subject to : } \|s_i\|^2 = E_s \quad \forall i \in \{1, 2, \dots, M\} \quad (8)$$

$$E_s \leq \sigma_x^2$$

where  $\mathbb{S}^*$  and  $\mathbb{B}^*$  indicate the optimal constellation and labeling, resp.

In this paper, we solve (7)–(8) using the pairwise optimization algorithm (POA) [11], [48] to optimize the coordinates

of the formats with a randomly chosen labeling. After a given number of POA iterations, the binary switch algorithm (BSA) [49] and the GMI approximation in [50] is applied to optimize the labeling and to avoid the optimization to converge to a local optimum. This optimization process is implemented in an iterative fashion and is repeated until the algorithm has converged or a maximum number of iterations is reached.

Our optimization was done without any symmetry considerations. However, in order to speed up the optimization, symmetry constraints can be used. In particular, we forced all 16 orthants to be identical.<sup>3</sup> Furthermore, 4 out of the 6 bits were assigned to determine the orthants, and the labeling optimization was done only over the remaining two bits. The resulting optimization problem is therefore simplified to a joint optimization of four 4D coordinates and its two labeling bits.<sup>4</sup> Interestingly, both the unconstrained optimization and the one with the symmetries gave the same result.

### C. Proposed Format: 4D-64PRS

The proposed 6 bit/4D-sym modulation format with  $M = 2^m = 64$  points is labeled by  $m = 6$  bits. We call this new modulation format with its binary labeling four-dimensional polarization-ring-switching 4D-64PRS. The coordinates of this family of modulation formats and the corresponding binary labeling are given in Table I. As shown in Table I, the coordinates of 4D-64PRS are such

$$\mathbf{s}_i \in \{[\pm\nu_1, \pm\nu_3, \pm\nu_2, \pm\nu_2], [\pm\nu_3, \pm\nu_1, \pm\nu_2, \pm\nu_2], [\pm\nu_2, \pm\nu_2, \pm\nu_1, \pm\nu_3], [\pm\nu_2, \pm\nu_2, \pm\nu_3, \pm\nu_1]\}, \quad (9)$$

and therefore, the proposed format is highly symmetric.

The 4D-64PRS format has 64 non-overlapping points in 4D space. For better visualization, these points can be projected on the first two and last two dimensions (i.e., on the two polarizations). This projection results in 12 distinct points in each 2D space, as shown in Fig. 2. In order to clearly show the 4D-64PRS modulation projected on 2D, and also to emphasize the inter-polarization dependency, we use in Fig. 2 the following color coding strategy: 2D projected symbols in the first and second polarization are valid 4D symbols only if they share the same color. Consider for example the 4 blue points in first 2D and the 4 blue points in second 2D, each of the  $4 \times 4 = 16$  combinations correspond to a 4D symbol. However, a blue point in the first 2D cannot be transmitted in combination with a green/red/light blue point in the second 2D. The same rules applies to light blue, green, and red points, giving a total of  $16 \times 4 = 64$  4D points.

The color coding scheme used in Fig. 2 also shows the constant-modulus constraint we imposed on the format, where  $R_1$  and  $R_2$  are the ring radii. The 4D-64PRS format can be seen as the transmission of alternating rings in both polarizations. In other words, when an inner ring ( $R_2$ ) is

<sup>3</sup>An orthant in  $N$ -dimensions is the intersection of  $N$  mutually orthogonal half-spaces. By independent selections of half-space signs, there are  $2^N$  orthants in  $N$ -dimensional space.

<sup>4</sup>Note that other optimization methods can also be used to find the optimal solution. This includes for example genetic algorithms [51] and machine learning approaches [43], [52].

TABLE I  
COORDINATES AND BINARY LABELING OF THE 4D-64PRS FORMAT.

Coordinates	Labeling	Coordinates	Labeling
$(+\nu_1, +\nu_3, +\nu_2, +\nu_2)$	000000	$(+\nu_1, +\nu_3, -\nu_2, +\nu_2)$	000010
$(+\nu_1, +\nu_3, -\nu_2, -\nu_2)$	000110	$(+\nu_1, +\nu_3, +\nu_2, -\nu_2)$	000100
$(-\nu_1, +\nu_3, +\nu_2, +\nu_2)$	010000	$(-\nu_1, +\nu_3, -\nu_2, +\nu_2)$	010010
$(-\nu_1, +\nu_3, -\nu_2, -\nu_2)$	010110	$(-\nu_1, +\nu_3, +\nu_2, -\nu_2)$	010100
$(-\nu_3, +\nu_1, +\nu_2, +\nu_2)$	011001	$(-\nu_3, +\nu_1, -\nu_2, +\nu_2)$	011011
$(-\nu_3, +\nu_1, -\nu_2, -\nu_2)$	011111	$(-\nu_3, +\nu_1, +\nu_2, -\nu_2)$	011101
$(-\nu_3, -\nu_1, +\nu_2, +\nu_2)$	111001	$(-\nu_3, -\nu_1, -\nu_2, +\nu_2)$	111011
$(-\nu_3, -\nu_1, -\nu_2, -\nu_2)$	111111	$(-\nu_3, -\nu_1, +\nu_2, -\nu_2)$	111101
$(-\nu_1, -\nu_3, +\nu_2, +\nu_2)$	110000	$(-\nu_1, -\nu_3, -\nu_2, +\nu_2)$	110010
$(-\nu_1, -\nu_3, -\nu_2, -\nu_2)$	110110	$(-\nu_1, -\nu_3, +\nu_2, -\nu_2)$	110100
$(+\nu_1, -\nu_3, +\nu_2, +\nu_2)$	100000	$(+\nu_1, -\nu_3, -\nu_2, +\nu_2)$	100010
$(+\nu_1, -\nu_3, -\nu_2, -\nu_2)$	100110	$(+\nu_1, -\nu_3, +\nu_2, -\nu_2)$	100100
$(+\nu_3, -\nu_1, +\nu_2, +\nu_2)$	101001	$(+\nu_3, -\nu_1, -\nu_2, +\nu_2)$	101011
$(+\nu_3, -\nu_1, -\nu_2, -\nu_2)$	101111	$(+\nu_3, -\nu_1, +\nu_2, -\nu_2)$	101101
$(+\nu_3, +\nu_1, +\nu_2, +\nu_2)$	001001	$(+\nu_3, +\nu_1, -\nu_2, +\nu_2)$	001011
$(+\nu_3, +\nu_1, -\nu_2, -\nu_2)$	001111	$(+\nu_3, +\nu_1, +\nu_2, -\nu_2)$	001101
$(+\nu_2, +\nu_2, +\nu_1, +\nu_3)$	001000	$(+\nu_2, +\nu_2, -\nu_1, +\nu_3)$	001010
$(+\nu_2, +\nu_2, -\nu_3, +\nu_1)$	000011	$(+\nu_2, +\nu_2, -\nu_3, -\nu_1)$	000011
$(+\nu_2, +\nu_2, -\nu_1, -\nu_3)$	001110	$(+\nu_2, +\nu_2, +\nu_1, -\nu_3)$	001100
$(+\nu_2, +\nu_2, +\nu_3, -\nu_1)$	000101	$(+\nu_2, +\nu_2, +\nu_3, +\nu_1)$	000001
$(-\nu_2, +\nu_2, +\nu_1, +\nu_3)$	011000	$(-\nu_2, +\nu_2, -\nu_1, +\nu_3)$	011010
$(-\nu_2, +\nu_2, -\nu_3, +\nu_1)$	010011	$(-\nu_2, +\nu_2, -\nu_3, -\nu_1)$	010111
$(-\nu_2, +\nu_2, -\nu_1, -\nu_3)$	011110	$(-\nu_2, +\nu_2, +\nu_1, -\nu_3)$	011100
$(-\nu_2, +\nu_2, +\nu_3, -\nu_1)$	010101	$(-\nu_2, +\nu_2, +\nu_3, +\nu_1)$	010001
$(-\nu_2, -\nu_2, +\nu_1, +\nu_3)$	111000	$(-\nu_2, -\nu_2, -\nu_1, +\nu_3)$	111010
$(-\nu_2, -\nu_2, -\nu_3, +\nu_1)$	110011	$(-\nu_2, -\nu_2, -\nu_3, -\nu_1)$	110111
$(-\nu_2, -\nu_2, -\nu_1, -\nu_3)$	111110	$(-\nu_2, -\nu_2, +\nu_1, -\nu_3)$	111100
$(-\nu_2, -\nu_2, +\nu_3, -\nu_1)$	110101	$(-\nu_2, -\nu_2, +\nu_3, +\nu_1)$	110001
$(+\nu_2, -\nu_2, +\nu_1, +\nu_3)$	101000	$(+\nu_2, -\nu_2, -\nu_1, +\nu_3)$	101010
$(+\nu_2, -\nu_2, -\nu_3, +\nu_1)$	100011	$(+\nu_2, -\nu_2, -\nu_3, -\nu_1)$	100111
$(+\nu_2, -\nu_2, -\nu_1, -\nu_3)$	101110	$(+\nu_2, -\nu_2, +\nu_1, -\nu_3)$	101100
$(+\nu_2, -\nu_2, +\nu_3, -\nu_1)$	100101	$(+\nu_2, -\nu_2, +\nu_3, +\nu_1)$	100001

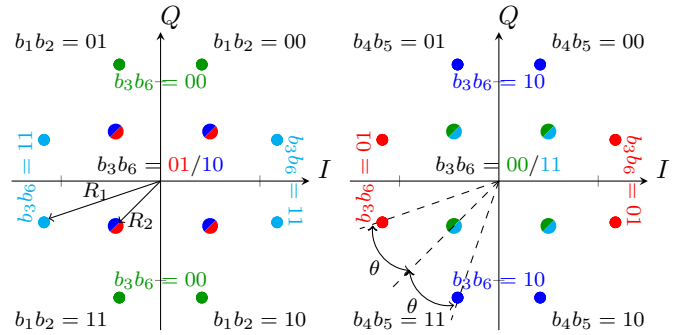


Fig. 2. 2D-projections of the designed 4D-64PRS modulation and associated bits/polarization bits mapping. The rings are given by  $R_1^2 = \nu_1^2 + \nu_3^2$  and  $R_2^2 = 2\nu_2^2$ .

transmitted in the first 2D (first polarization), the outer ring ( $R_1$ ) is used in the second 2D (second polarization). This case corresponds to blue or red points. The opposite alternation (outer ring, inner ring) is used for light blue or green points. This “polarization-ring-switching” explains the name we gave to this format. As mentioned above, a feature of the 4D-64PRS is its symmetry. Fig. 2 shows that the projections of 4D-64PRS on both polarizations are identical. Furthermore, the points in each quadrant are simply rotated versions of the points  $(\nu_2, \nu_2)$ ,  $(\nu_2, \nu_2)$ , and  $(\nu_2, \nu_2)$  in the first quadrant.

The family of 4D-64PRS formats we propose assumes that the symmetries in terms of the orthants and angle between

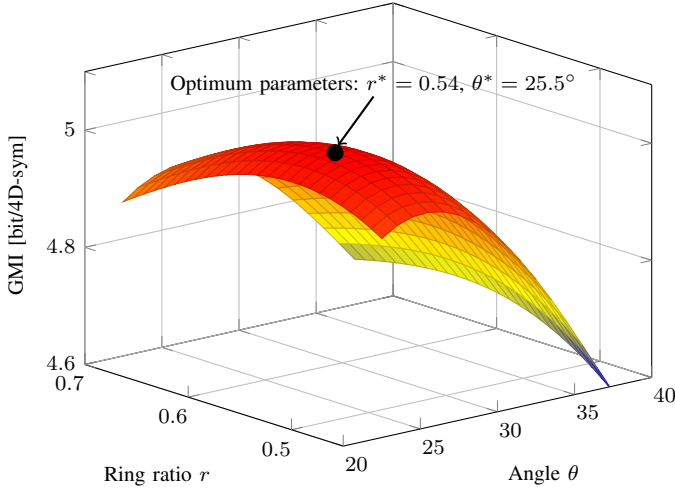


Fig. 3. GMI as a function of ring ratio  $r$  and angle  $\theta$ , SNR=8 dB in AWGN for 4D-64PRS. The optimum parameters are highlighted.

the points are maintained. The format can then be optimized in terms of two parameters: the ring ratio  $r = R_2/R_1$  and angle  $\theta$ , shown in Fig. 2. The GMI as a function of  $r$  and  $\theta$  at SNR=8 dB are shown in Fig. 3. The GMI for this SNR is clearly concave on  $r$  and  $\theta$ , and thus, its optimization is trivial. The optimum values for SNR=8 dB are  $r^* = 0.54$  and  $\theta^* = 25.5^\circ$ , which are highlighted in Fig. 3. The optimization for SNR=8 dB can be understood as an optimization for an effective SNR in (4) of 8 dB which ignores the NLI. The resulting AIR is approximately 5 bit/4D-sym, i.e., we are targeting a FEC rate of  $R = 5/6$ . The coordinates of the constellation in Table I are  $\nu_1 = 0.87, \nu_2 = 1, \nu_3 = 2.47$  (rounded to two decimal points).

We have also optimized the 4D-64PRS format for a range of SNR, as shown in Fig. 4. This figure shows that at different SNR regions, different  $r$  and  $\theta$  should be chosen to maximize the GMI. In particular, we observe small variations of the optimal angle (between  $27.2^\circ$  and  $23.4^\circ$ ), with a general tendency to decrease as the SNR increases. The ring ratio also shows small variations, decreasing from 0.61 to 0.53 as the SNR increases.

Regarding the binary labeling of 4D-64PRS, we show in Fig. 2 that the quadrants in each polarization are defined by 4 out of 6 bits, namely,  $[b_1, b_2]$  and  $[b_4, b_5]$  for the first and second polarization, respectively. These 4 bits can be seen as the ones determining which of the 16 orthants is to be chosen. The remaining 2 bits  $[b_3, b_6]$  determine the symbol from the 4 possible constellation points in the same orthant. In other words,  $[b_3, b_6]$  determine the color of the transmitted points in Fig. 2 while  $[b_1, b_2, b_4, b_5]$  determine the coordinate of the point in the same color.

We conclude this section by discussing the differences between 4D-64PRS and other existing 4D modulation formats. The key design difference with respect to set-partitioned 16QAM [32], [53] and 4D-2A8PSK [12], is that we jointly optimize the labeling and coordinates of the constellation points to achieve higher GMI (see next Sec. for details). When projected on 2D, 4D-64PRS has a similar shape to

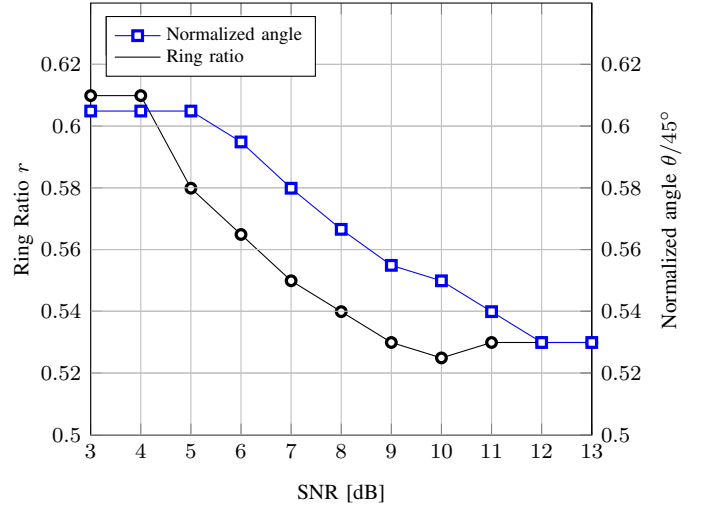


Fig. 4. Optimized ring ratio  $r$  and normalized angle  $\theta/45^\circ$  vs SNR in AWGN for 4D-64PRS modulation.

4D-64SP-12QAM [32]. However, 4D-64PRS has a different 4D geometrical structure and different labeling, which cannot be obtained by set-partitioning from PM-16QAM. As we will see in the next section, 4D-64PRS and its optimized labeling provides significant linear shaping gains with respect to 8QAM, 4D-64SP-12QAM and 4D-2A8PSK. Furthermore, the constant-modulus property of 4D-64PRS yields excellent nonlinear transmission performance.

### III. NUMERICAL RESULTS

In the following, we numerically evaluate the AIRs for both the AWGN channel and a multi-span fiber system. Different modulation formats with 6 bit/4D-sym are considered. The analysis focuses on linear shaping gains and the effect of shaping on the fiber nonlinearities and effective SNR.

#### A. Results for AWGN Channel

Fig. 5 shows the linear performance in terms of AIRs for three 4D modulation formats with 6 bit/4D-sym: 4D-2A8PSK, 4D-64SP-12QAM and the proposed 4D-64PRS. The 4D-64PRS format we use was optimized for SNR=8 dB. An optimized ring ratio of 0.65 was used for 4D-2A8PSK. The results in Fig. 5 show that the three 4D modulation formats have a similar MI performance (solid lines). This figure also shows that the GMI of 4D-28PSK and 4D-64SP-12QAM is considerably lower than their corresponding MI, as previously shown for other 4D formats in [25]. On the other hand, the gap between the GMI and MI of 4D-64PRS is relatively small. This indicates that the labeling of the proposed 4D-64PRS is optimal.

Considering the GMI of PM-8QAM as a baseline, 4D-64PRS can provide gains of 0.7 dB at an AIR between 4.8 and 5.2 bit/4D-sym. AIR gains of approximately 0.24 bit/4D-sym are observed for this AIR range. The 4D-64PRS format also performs better than 4D-2A8PSK [12] and 4D-12QAM [32] (in terms of GMI) by at least 0.4 dB.

In order to verify the accuracy of GMI prediction, we simulated the low-density parity-check (LDPC) codes used

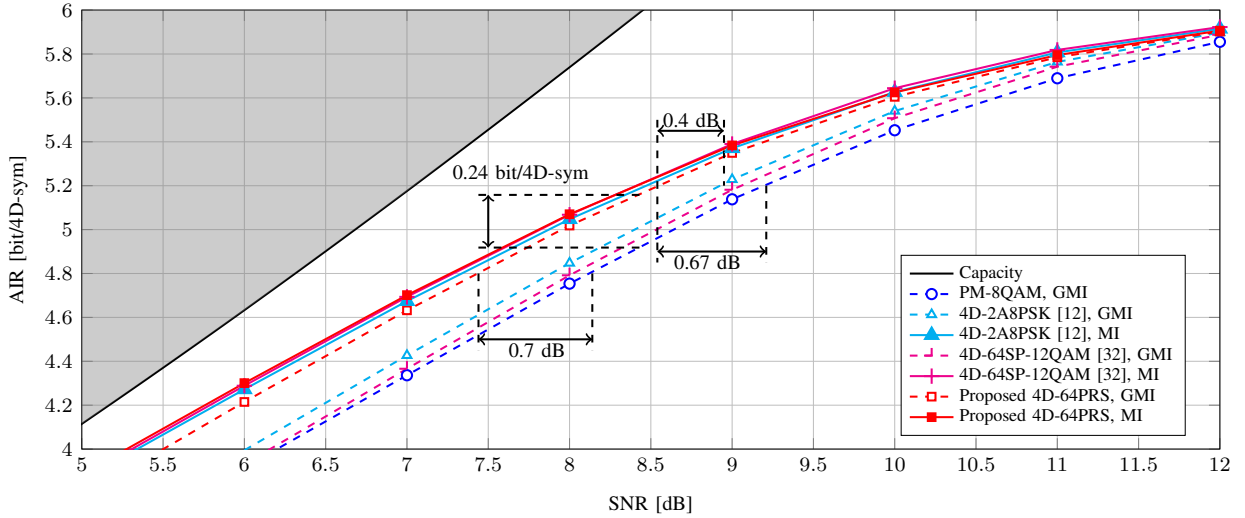


Fig. 5. MI and GMI as a function of SNR for various modulation formats with 6 bit/4D-sym. The PM-8QAM, 4D-64SP-12QAM and 4D-2A8PSK are shown as a baseline.

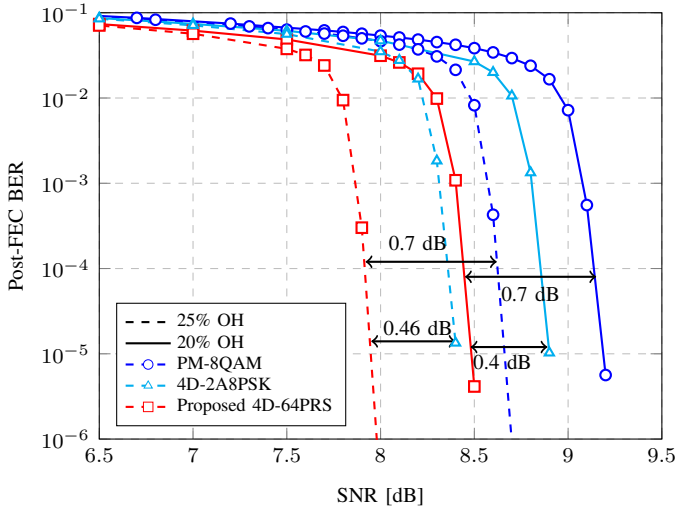


Fig. 6. Post-FEC BER performance of formats with 6 bit/4D-sym and LDPC with 20% and 25% FEC overhead.

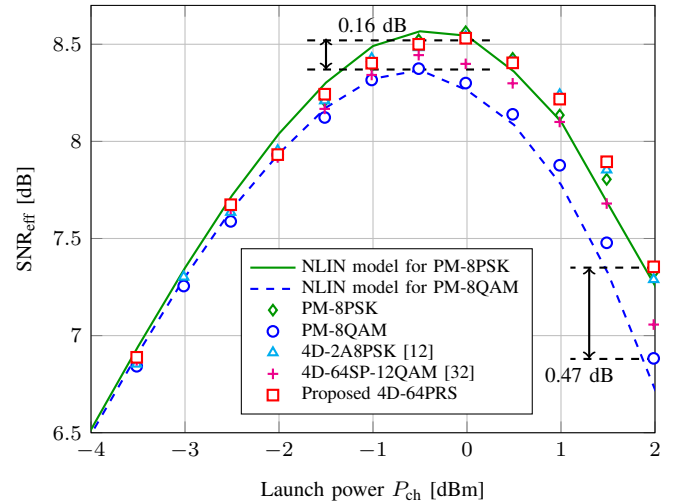


Fig. 7. Effective SNRs  $SNR_{eff}$  as function of the transmitted power for different modulation formats with 6 bit/4D-sym over a 8000 km link.

in DVB-S2 standard. FEC overheads of 20% and 25% with 64800 bits per frame for PM-8QAM, 2A8PSK and 4D-64PRS were used. The results are shown in Fig. 6. We can observe that the gains for 4D-64PRS in terms of post-FEC BER coincide well to the GMI prediction in Fig. 5.

### B. Results for Nonlinear Fiber Optical Channel

The 4D modulation formats 4D-2A8PSK, 4D-64SP-12QAM and 4D-64PRS were implemented in two polarizations and simulated over the optical fiber channel. The optical fiber channel comprising multiple standard single-mode fiber spans with  $\alpha = 0.21$  dB/km,  $D = 16.9$  ps/nm/km,  $\gamma = 1.3175$  (W·km)<sup>-1</sup>. A dual-polarization multi-span WDM system with 11 co-propagating channels was transmitted at a symbol rate of 45 GBaud, a WDM spacing of 50 GHz and a root-raised-cosine (RRC) filter roll-off factor of 0.1. Each WDM channel carries independent data with  $N_{sym} = 2^{16}$  and are transmitted

at the same power. Each span of length 80 km was followed by an erbium-doped fiber amplifier (EDFA) with a noise figure of 5 dB. Note that polarization mode dispersion (PMD) was not considered. At the receiver, an ideal receiver is used for detection: chromatic dispersion is digitally compensated, then the signal is matched filtered and downsampled. Potential constant phase rotations are ideally compensated.

We compare the effective SNR (after fiber propagation and receiver DSP) for an 8000 km link. The results are shown in Fig. 7. Results for the numerical simulations (markers) and the estimated SNR by NLIN model [39] for PM-8QAM (dashed lines) and PM-8PSK (solid lines) are shown. A good agreement between the simulation and NLIN model in [39] is observed. The simulation results in Fig. 7 show that the two 4D modulation formats under consideration (4D-2A8PSK and 4D-64PRS) provide an effective SNR 0.16 dB higher than PM-8QAM. Since the design of 4D-64SP-12QAM does



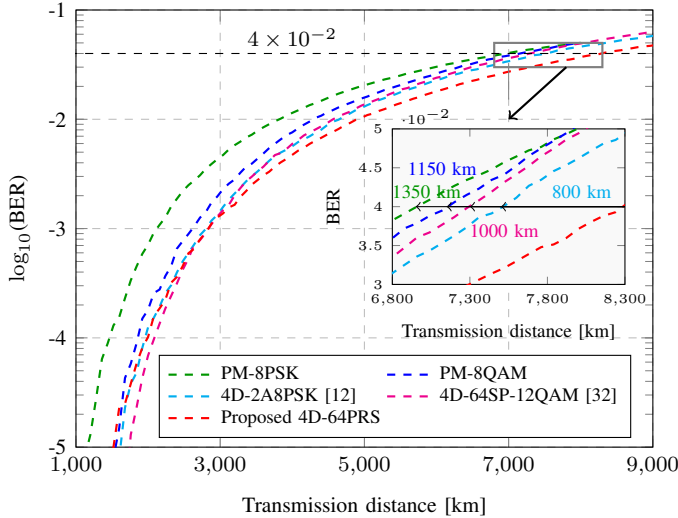


Fig. 8. BER as function of the transmission distance for different 4D modulation formats with 6 bit/4D-sym using the optimal launch power at each distance.

not consider the modulation-dependent NLI, its effective SNR is worse than 4D-2A8PSK and the proposed 4D-64PRS, but it is a little bit higher than PM-8QAM. In addition, Fig. 7 also shows the effective SNR of PM-8PSK with constant-modulus as a baseline, which has the smallest modulation-dependent NLI. We observe that the three constant-modulus modulation formats have similar effective SNR, and PM-8QAM and 4D-64SP-12QAM with power fluctuation between time slots have an effective SNR penalty due to fiber nonlinearities. The penalty becomes 0.47 dB when the launch power increase. At the optimum launch power, the proposed modulation format therefore the modulation-dependent NLI is reduced. The total shaping gain is linear shaping gain plus NLI shaping gain. Below we will show how this translates into a reach increase.

Fig. 8 shows pre-FEC BER as a function of the transmission distance for different modulation formats using the optimal launch power at each distance. At  $\text{BER} = 4 \times 10^{-2}$ , which is the typical BER threshold of the state-of-the-art SD-FEC having a code rate of 0.8, the achievable transmission distance is 8300 km for 4D-64PRS. 4D-64PRS yields a 1350 km and 1150 km reach increases with respect to PM-8PSK and PM-8QAM, resp. 4D-64PRS also provides 1000 km and 800 km gain compared to 4D-64SP-12QAM and 4D-2A8PSK, resp. For pre-FEC BER below  $1 \times 10^{-3}$ , 4D-64SP-12QAM provides the best performance. It is not only because 4D-64SP-12QAM has largest minimum Euclidean, but also the 4D-64PRS is not designed for such higher SNR at short distance.

Fig. 9 shows AIRs as a function of the transmitted power for different modulation formats with 6 bit/4D-sym over a 8000 km link. Around the optical launch power regime where linear and nonlinear propagation effects are comparable, 4D-64PRS modulation gives 0.27 bit/4D-sym GMI gain with respect to PM-8QAM. For higher launch power regime where nonlinearity is dominant, performance improvements become more significant, and this GMI gain is around 0.37 bit/4D-sym.

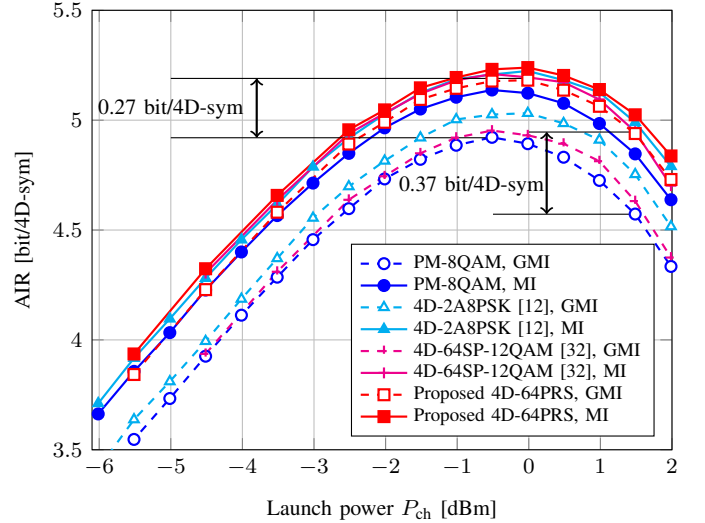


Fig. 9. AIRs as function of the transmitted power for different modulation formats with 6 bit/4D-sym over a 8000 km link.

It is important to note that the GMI gain in the nonlinear channel (at the optimum launch power, 0.27 bit/4D-sym) is higher than the linear channel in Fig. 5 (0.24 bit/4D-sym). This additional gain (or equivalently, the almost absence of NLIN penalty) validates the efficiency of the NLI tolerant design of the proposed 4D-64PRS format. In Fig. 9, we also plot the AIRs of other 4D modulation formats (4D-2A8PSK and 4D-64SP-12QAM). We can observe that 4D-64PRS outperforms both of them in terms of GMI due to its excellent linear performance and also the nonlinear-tolerant property.

To conclude this section, Fig. 10 shows AIRs as a function of the transmission distance using the optimal launch power at each distance. 4D-64PRS yields a 1100 km increase in reach relative to PM-8QAM at GMI of 5.2 bit/4D-sym, and more than 610 km relative to 4D-2A8PSK. At a transmission distance of 6840 km, 4D-64PRS also provides 0.22 bit/4D-sym and 0.1 bit/4D-sym gain compared to PM-8QAM and 4D-2A8PSK, resp.

The AIR gains from shaping translate into sensitivity improvements in nonlinear optical channel, which is linear gain plus the nonlinear gain. In particular, the reach increases in Fig. 10 come from effective SNR gains of 4D-64PRS are 0.83 dB and 0.4 dB and relative to PM-8QAM and 4D-2A8PSK, resp. These gains can be compared to those in Fig. 5, where no NLI is present (0.67 dB and 0.4 dB for PM-8QAM and 4D-2A8PSK respectively.) For the PM-8QAM case, an effective SNR gain is observed. The relative additional shaping gain in terms of GMI for the nonlinear channel with respect to the gain in the linear channel is  $(0.27 - 0.24)/0.24 = 12.5\%$ . For the 4D-2A8PSK case, the effective SNR remains the same, which comes from the fact that both 4D-64PRS and 4D-2A8PSK have the constant-modulus property. The gains of 4D-64PRS in the nonlinear channel with respect to 4D-2A8PSK is almost the same as those in the AWGN channel ( $\approx 0.4$  dB).

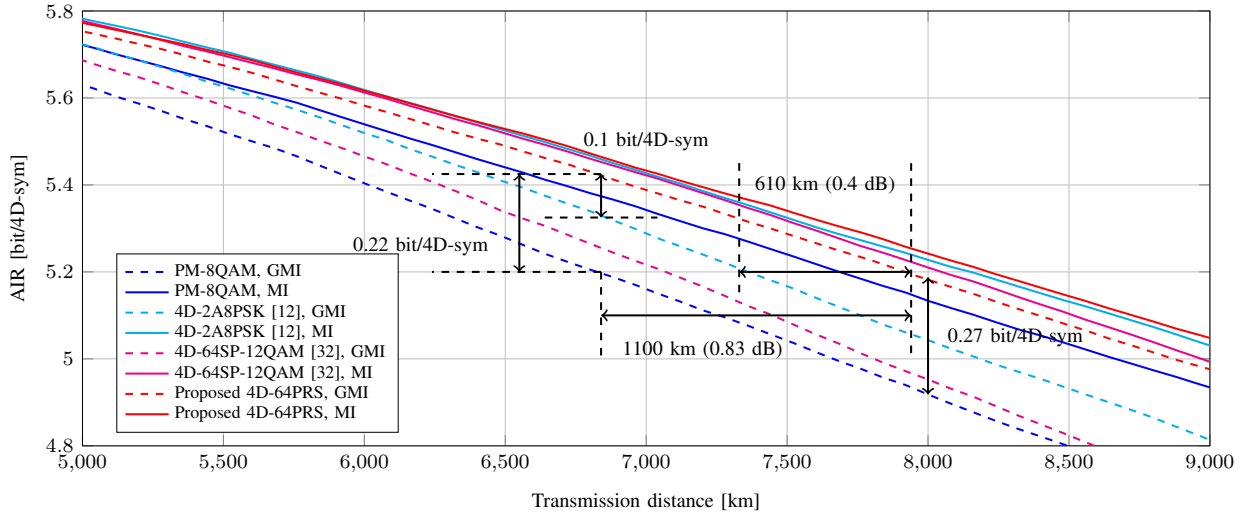


Fig. 10. AIRs as function of the transmission distance for different 4D modulation formats with 6 bit/4D-sym.

### C. Analysis of the 6 bit/4D-sym modulation formats

In this section, we analyze the structure of the proposed format (its coordinates and binary labeling are presented in Table I) and give some intuition on why it outperforms other formats available in the literature. We first study the structure of the formats in terms of minimum squared Euclidean distance (MSED), which we denote by  $d^2$ . We will also look at the number of pairs of constellation points at MSED, which we denote as  $n$ . A large  $d^2$  and small  $n$  should in principle result in high MI in the high-SNR regime, as recently proved in [54]<sup>5</sup>.

From now on, we assume all the constellations are normalized to  $E_s = 2$  (i.e., unit energy per polarization). Under this assumption, the MSED of 4D-64PRS is  $d^2 = 0.69$  and  $n = 32$ . On the other hand, 4D-2A8PSK has a MSED of  $d^2 = 0.88$  and  $n = 128$ , while 4D-64SP-12QAM has the highest MSED  $d^2 = 1$  but a rather large  $n = 272$ . Based on these properties of the formats, 4D-64SP-12QAM and 4D-2A8PSK should be better than 4D-64PRS in terms of MI. Our numerical results, however, show that all these three formats have a very similar MI (see MI results in Fig. 5). We believe that there are two reasons for this. Firstly, MSED might not be the only quantity to look at when moderate SNRs are considered, but instead, the whole SED “spectrum” should be studied. Secondly, a large number of pairs of constellation points at the MSED reduces the AIR. 4D-64SP-12QAM and 4D-2A8PSK have 8.5 and 4 times more pairs at MSED than 4D-64PRS, resp., which could explain the relatively low MI values obtained by those constellations.

The first three columns of Table II show a summary of these parameters, where we also include PM-8QAM (not analyzed in terms of MI because of the very low MI it provides). The spectrum of possible SEDs for the three constellations can also be seen in the x-axes of the histograms presented in Fig. 11.

We now analyze GMI at medium SNR values. In this case, the quantities to investigate are not only MSED and the

TABLE II  
COMPARISON OF 6 BIT/4D-SYM MODULATION FORMATS

	$d^2$	$n$	Gray-labeled	$(d_{HD=1}^2, n_{HD=1})$
PM-8QAM	0.84	768	No	(0.84, 128) (3.15, 64)
4D-64SP-12QAM	1	272	No	(1, 128) (2, 32) (5, 32)
4D-2A8PSK	0.88	128	Yes	(0.88, 128) (3.46, 64)
4D-64PRS	0.69	32	Yes	(0.69, 32) (0.90, 64) (0.98, 64) (5.50, 32)

number of pairs at MSED, but also the Hamming distances (HDs) of the binary labels of the constellation points at MSED. The lower the Hamming distance the higher the GMI. In the extreme case where all the pairs of constellations points at MSED are at Hamming distance one, the binary labeling is said to be Gray. Interestingly, both 4D-2A8PSK and 4D-64PRS constellations fulfill this property, i.e., both constellations are Gray-labeled.<sup>6</sup>

The discussion on the GMI performance is facilitated by the results in Fig. 11, where histograms of the SEDs of the formats are shown. This figure also shows a classification of the pairs at a given SED: blue bars for pairs at HD larger than one, and red bars for pairs at HD one. This information is also shown in the last column of Table II, where the pairs  $(d_{HD=1}^2, n_{HD=1})$  are presented. These pairs show the number of pairs at HD one and the corresponding SEDs associated to those.

Fig. 11 shows that 4D-64SP-12QAM has the highest MSED, however, it is worse in terms of GMI than 4D-2A8PSK and 4D-64PRS (see GMI results in Fig. 5). This is due to the high number of pairs at MSED. We can observe that,

<sup>5</sup>The results in [54] hold for 1D constellations only. However, the authors in [54] conjectured the results to hold verbatim to any number of dimensions.

<sup>6</sup>Note that a Gray-labeled 4D constellation is not necessarily Gray-labeled when considering the projection in two dimensions.

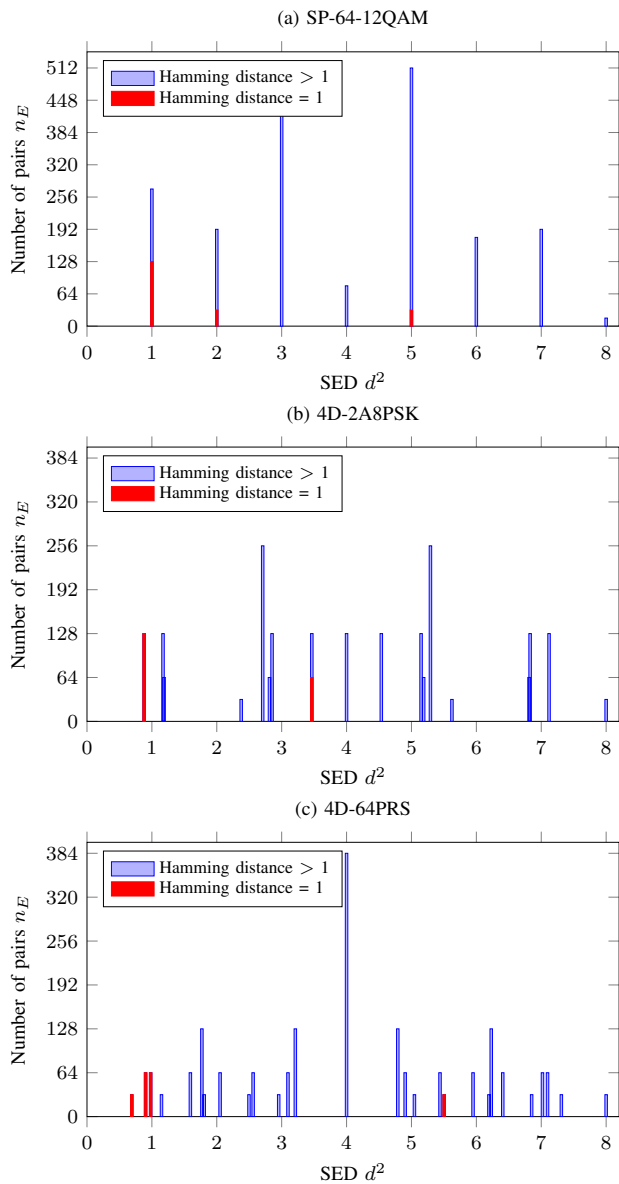


Fig. 11. Histograms of SEDs of three 4D formats: (a) 4D-64SP-12QAM, (b) 4D-2A8PSK, and (c) 4D-64PRS. The red bars show the number of pairs with Hamming distance of 1 at the SED  $d^2$ .

even the MSED of 4D-64PRS is smaller than 4D-2A8PSK and 4D-64SP-12QAM, most of the points for 4D-64PRS with Hamming distance one have a larger SED than 4D-2A8PSK and PM-8QAM. Therefore, in terms of MSED, the performance of 4D-64PRS reduces, but it increases gain by reducing the number of pairs at MSED and also increases the SED for these points.

#### IV. CONCLUSIONS

This paper proposed and studied a new four-dimensional geometrically-shaped nonlinearity-tolerant modulation format with 6 bit/4D-sym named four-dimensional polarization-ring-switching (4D-64PRS). Based on the idea of imposing a constant-modulus on the 4D structure, the coordinates of the format and its binary labeling were *jointly* optimized. The

proposed format is optimal in terms of generalized mutual information. To the best of our knowledge, 4D-64PRS is the best performing nonlinearity-tolerant format proposed with this spectral efficiency.

We believe 4D-64PRS is a promising candidate for upgrading submarine transmission systems. In this paper, we concentrated on the AWGN channel and dispersion unmanaged links to show the superiority of the proposed 4D modulation format. A performance advantage on dispersion managed (highly nonlinear) links is also expected and left for further investigation.

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