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Polarized Electroproduction in an Asymptotically Free Gauge Theory

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The polarized electroproduction is investigated in the framework of a non-Abelian gauge model. The Bjorken sum rule and the Burkhardt-Cottingham superconvergence relation are rederived. The former is free from anomalous dimensions and the latter is a consequence of the less singular nature of the structure function. Also non-trivial restrictions on the expectation values of the composite operators in the Wilson expansion are derived from the positivity restrictions on the structure functions, which are crucial for the discussion.

By the recent developments in the field theories, a non-Abelian gauge theory has been found to be good for the theory of strong interactions; especially, with the success (up to the logarithmic deviations) of Bjorken scaling in the unpolarized electroproduction.[1] For the process $e^+e^-\to$hadrons,[2] further considerations might be needed. (For example, are the mass-insertion terms of Callan-Symanzik equations negligible?) Measurements of the asymmetry in the polarized deep-inelastic process will begin soon.[3] Then, it is important to investigate this process in a non-Abelian gauge model.

This process has been investigated in other framework[4] and sum rules have been derived. (These are called the Bjorken sum rule[5] and the Burkhardt-Cottingham superconvergence relation[6] ——the latter will be simply called the superconvergence relation.) The subject of this paper is to reinvestigate these sum rules in the colored quark-gluon model, whose Lagrangian is given by

$$\begin{aligned}\mathcal{L}=&-\sum_{\sigma=1}^{m}\bar{\psi}_\sigma{}^A\gamma^\mu(\delta^{AB}\partial_\mu-igT_a{}^{AB}A_\mu{}^a)\psi_\sigma{}^B-m_q\sum_A\sum_{\sigma=1}^{m}\bar{\psi}_\sigma{}^A\psi_\sigma{}^A\\&-\tfrac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}+(\text{the Fadeev-Popov ghost})\\&+(\text{weak interactions})+(\text{counter terms}).\end{aligned}$$

Then, it will be found that these sum rules are the special case of the moments of the structure functions, which are calculable up to unknown constants with the idea of asymptotic freedom.

Calculations are analogous to those in the unpolarized case.[1] We will use the conjecture that the renormalization group (or C-S) equations are approximated to the homogeneous ones in a deep-Euclidean momentum region[7] and the following two assumptions: a) The effects of the weak interactions are negligible in the present energy range. b) Infrared catastrophe of the massless Yang-Mills field is avoidable with such a mechanism as Coleman and Weinberg's.[8]

Let us consider the electroproduction in one-photon exchange. Summing up over final hadron states and the spin of a final electron, we obtain the well-known expression of the double differential cross section with the spin-dependent hadron tensor given by

$$\begin{aligned}W_{\mu\nu}(P,q,s) &= \frac{VP_0}{2\pi M}\int d^4x\, e^{-iqx}\langle P,s|J_\mu(x)J_\nu(0)|P,s\rangle \\ &= \left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\nu) \\ &\quad +\frac{1}{M^2}\left(P_\mu-\frac{(P\cdot q)}{q^2}q_\mu\right)\left(P_\nu-\frac{(P\cdot q)}{q^2}q_\nu\right)W_2(q^2,\nu) \\ &\quad +i\varepsilon_{\mu\nu\sigma\rho}q^\sigma[s^\rho d_1(q^2,\nu)+P^\rho(s\cdot q)d_2(q^2,\nu)]. \end{aligned}\tag{1}$$

The W's are the structure functions in the unpolarized case and the d's are the new ones which describe the spin-dependence of this process. In the limit of the small scattering angle θ, the longitudinal asymmetry is given by

$$d\sigma^{\uparrow\uparrow}-d\sigma^{\uparrow\downarrow}=\frac{\alpha}{\pi}\frac{E+E'}{E}\frac{1}{q^2}[\nu d_2(q^2,\nu)-d_1(q^2,\nu)].\tag{2}$$

Separation of two structure functions can be carried out by measurements of the longitudinal and transverse asymmetries.

On the other hand, from the structure of the spin-flip hadron tensor,

$$\textstyle\sum_{\mu,\nu}\sum_{s,s'}W^{s,s'}_{\mu,\nu}(P,q)\,a_\mu^{s*}a_\nu^{s'}\geqq 0$$

with arbitrary vectors a_μ^s. This condition is equivalent to a set of the inequalities:[9)]

i) $W_1\geqq 0$,

ii) $\sigma_s=\left(1+\dfrac{\nu^2}{M^2q^2}\right)W_2-W_1\geqq 0$,

iii) $W_1\geqq\left|Mq^2\left(1+\dfrac{\nu^2}{M^2q^2}\right)d_2-\dfrac{\nu}{M}d_1\right|$,

iv) $\sigma_s\left[W_1+Mq^2\left(1+\dfrac{\nu^2}{M^2q^2}\right)d_2-\dfrac{\nu}{M}d_1\right]\geqq 2q^2d_1^2$.

Now, let us consider only the spin-dependent part, namely, the antisymmetric part $W^A_{\mu\nu}$ of the hadron tensor. The calculus has been essentially given by Christ, Hasslacher and Mueller.[10)] The tensor $W^A_{\mu\nu}$ is given by the absorptive part of the forward Compton scattering amplitude defined by

$$T^A_{\mu\nu}(P,q,s)=\frac{VP_0}{2\pi M}i\int d^4x\, e^{-iqx}\langle P,s|TJ_\mu(x)J_\nu(0)|P,s\rangle$$

$$=i\varepsilon_{\mu\nu\sigma\rho}q^{\sigma}[s^{\rho}D_1(q^2,\nu)+P^{\rho}(s\cdot q)D_2(q^2,\nu)]. \tag{3}$$

So, from (1) and (3), one obtains

$$d_i(q^2,\nu)=\mathrm{Abs}\,D_i(q^2,\nu). \tag{4}$$

If we take the Bjorken limit, the product of two electromagnetic currents can be expanded in terms of the well-defined local operators, which is called the Wilson expansion.[11] In the gauge invariant form, it is written as

$$TJ_{\mu}\left(\frac{x}{2}\right)J_{\nu}\left(-\frac{x}{2}\right)=(\delta_{\mu}{}^{\alpha}\delta_{\nu}{}^{\beta}\Box-\partial^{\alpha}\partial_{\mu}\delta_{\nu}{}^{\beta}-\partial^{\beta}\partial_{\nu}\delta_{\mu}{}^{\alpha})$$
$$\times\sum_{n=0}^{N}\sum_{i=0}^{m}\frac{G_n^{(i)}(x^2+i\varepsilon)}{x^2+i\varepsilon}O^{(i)}_{\alpha\beta\mu_1\cdots\mu_n}(0)\,x^{\mu_1}\cdots x^{\mu_n}$$
$$+(\text{less singular terms}). \tag{5}$$

The operators are antisymmetric in the subscripts α, β and symmetric in the others. Then, in our model, minimum twist of the operators is zero. This is different from the fact that minimum twist of totally symmetric operators is two. Such operators are

$$O^{(0)}_{\alpha\beta\mu_1\cdots\mu_n}=\frac{1}{2!n!}[N\{G_{\alpha\mu_1}D_{\mu_2}\cdots D_{\mu_{n-1}}G_{\mu_n\beta}\}-(\text{permutation of }\alpha,\ \beta)$$
$$+(\text{permutations of }\mu_1,\cdots,\mu_n)]$$

and

$$O^{(\sigma)}_{\alpha\beta\mu_1\cdots\mu_n}=\frac{1}{2!n!}[N\{\bar{\psi}_{\sigma}\gamma_{\alpha}\gamma_{\beta}\nabla_{\mu_1}\cdots\gamma_{\mu_i}\cdots\nabla_{\mu_n}\psi_{\sigma}\}-(\text{permutation of }\alpha,\ \beta)$$
$$+(\text{permutations of }\mu_1,\cdots,\mu_n)],$$

where D_{μ} and ∇_{μ} are covariant derivatives acting on the gauge field and the fermion field and N indicates Zimmerman's normal product.[11] The expectation values of these operators taken by the single proton state are generally written as

$$\frac{VP_0}{2\pi M}\langle P,s|O^{(i)}_{\alpha\beta\mu_1\cdots\mu_n}(0)|P,s\rangle=c^{(i)}_{n+2,1}\varepsilon_{\alpha\beta\sigma\rho}s^{\sigma}P^{\rho}P_{\mu_1}\cdots P_{\mu_n}$$
$$+\frac{1}{n!}[c^{(i)}_{n+2,2}\varepsilon_{\alpha\beta\sigma\mu_1}s^{\sigma}P_{\mu_2}\cdots P_{\mu_n}+(\text{permutations of }\mu_1,\cdots,\mu_n)]$$
$$+\frac{1}{n!}[c^{(i)}_{n+2,3}\varepsilon_{\alpha\beta\sigma\mu_1}P^{\sigma}s_{\mu_2}P_{\mu_3}\cdots P_{\mu_n}+(\text{permutations of }\mu_1,\cdots,\mu_n)]$$
$$+(\text{trace terms}). \tag{6}$$

The c's are uncalculable constants in the present field theories. Substituting (5) and (6) into (3) and comparing the resulting expression with the left-hand side of (3), we obtain invariant amplitudes as

$$D_1=\sum_{n=0}^{N}\left(-\frac{\nu}{q^2}\right)^{n+1}\sum_{i=0}^{m}\widetilde{G}_n{}^{(i)}(q^2)\,(c^{(i)}_{n+2,1}+\frac{q^2}{\nu^2}c^{(i)}_{n+2,2}), \tag{7}$$

$$D_2=\frac{1}{q^2}\sum_{n=0}^{N}\left(-\frac{\nu}{q^2}\right)^{n}\sum_{i=0}^{m}\widetilde{G}_n{}^{(i)}(q^2)\,(-c^{(i)}_{n+2,1}+\frac{q^2}{\nu^2}c^{(i)}_{n+2,3}), \tag{8}$$

where $\widetilde{G}_n{}^{(i)}$ are the Fourier transforms of $G_n{}^{(i)}$ and are normalized to be dimensionless.

If one of $c^{(i)}_{n+2,1}$ is non-zero, the structure function d_1 is finite up to logarithms in the Bjorken limit since, in an asymptotically free theory, the coefficient functions $\widetilde{G}_n{}^{(i)}$ have only logarithmic dependence on q^2. However, the positivity restriction iv) requires d_1^2 to vanish as $(q^2)^{-1-\varepsilon}$, where ε is a positive number. Because the right-hand side of the inequality iv) vanishes due to the behavior of σ_s in this limit. Therefore,

$$c^{(i)}_{n+2,1}=0 \qquad \text{for all } i \text{ and } n\,. \tag{9}$$ [12]

(Let us note that these equalities are satisfied in the quark-parton model.) Also similar coefficients of the next-to-leading operators must vanish. There are contributions of the twist-two operators with the same singularity as those of the leading operators, but they will be omitted for the time being. Thus,

$$\mathcal{D}_1(q^2,\omega)=\nu D_1(q^2,\nu)=-\sum_{n=0}^{N}\omega^n\sum_{i=0}^{m}\left(-\frac{1}{2}\right)^n c^{(i)}_{n+2,2}\widetilde{G}_n{}^{(i)}(q^2), \tag{10}$$

$$\mathcal{D}_2(q^2,\omega)=\nu^2 D_2(q^2,\nu)=\sum_{n=0}^{N}\omega^n\sum_{i=0}^{m}\left(-\frac{1}{2}\right)^n c^{(i)}_{n+2,3}\widetilde{G}_n{}^{(i)}(q^2) \tag{11}$$

with the scaling variable ω. From (4), (10) and (11), $\nu d_1(q^2,\nu)$ and $\nu^2 d_2(q^2,\nu)$ are expected to converge into the non-trivial functions $V_1(\omega)$ and $V_2(\omega)$ of only the variable ω, except for logarithms, in the Bjorken limit. Taking the limit of $N\to\infty$ in (10) and (11), we obtain the analytic functions $\mathcal{D}_1{}^{AF}$ and $\mathcal{D}_2{}^{AF}$ in a complex-ω plane with the cuts on the real axis. Without difficulties, the moments of the absorptive part of these analytic functions can be expressed in terms of the coefficients of ω^n in (10) and (11). Furthermore, replacing νD_1 and $\nu^2 D_2$ with $\mathcal{D}_1{}^{AF}$ and $\mathcal{D}_2{}^{AF}$ in (4) respectively, it is found that, for odd n

$$\int_0^1 dx\, x^{n-1}V_1(q^2,x)=\pi\sum_{i=0}^{m}\left(-\frac{1}{2}\right)^{n+1}c^{(i)}_{n+2,2}\widetilde{G}_n{}^{(i)}(q^2) \tag{12}$$

and

$$\int_0^1 dx\, x^{n-1}V_2(q^2,x)=-\pi\sum_{i=0}^{m}\left(-\frac{1}{2}\right)^{n+1}c^{(i)}_{n+2,3}\widetilde{G}_n{}^{(i)}(q^2)\,. \tag{13}$$

If the right-hand sides of (12) and (13) are completely known, the scaling functions V_1 and V_2 are given by using the inverse formula.

To calculate the coefficient functions $\widetilde{G}_n{}^{(i)}$, let a renormalization point κ be larger than the quark mass m_q such that m_q is negligible. Then the renormalization group equation holds good for an arbitrary vertex function in a deep-Euclidean momentum region.[7] By applying this equation to the suitable vertex functions[10] with the product of the currents and using the Wilson expansion (5), the coupled equations for the coefficient functions $\widetilde{G}_n^{(i)}$ are obtained as

$$D\widetilde{G}_n^{(i)}(q^2)+\sum_{j=0}^{m}\gamma_{n+2}^{ij}\widetilde{G}_n^{(j)}(q^2)=0 \tag{14}$$

with the differential operator

$$D=\kappa^2\frac{\partial}{\partial\kappa^2}+\beta(g^2)\frac{\partial}{\partial g^2}\,.$$

Here, we must note the following situations: a) Since the twist-two operators give the contributions with the same singularity as those of the leading operators, one might worry about the mixing between those operators. Such mixing, however, does not change Eq. (14) as is pointed out by Callan and Gross.[13] b) For $n=1$, there are only the composite operators made of the fermion field, but not of the gauge field. In this case, the coupled equations turn out to be

$$D\widetilde{G}_1^{(\sigma)}+\sum_{\rho=1}^{m}\gamma_3^{\sigma,\rho}\widetilde{G}_1^{(\rho)}=0\,. \tag{14$'$}$$

For odd n and to second order in the coupling constant g, $\gamma_{n+2}^{i,j}$ are of the form

$$\gamma_{n+2}^{i,j}=-\frac{g^2}{16\pi^2}b_{n+2}^{i,j} \tag{15}$$

with

$$b_{n+2}^{0,0}=C_2(G)\left(\frac{1}{3}+\sum_{2}^{n}\frac{4}{k}\right)+\frac{4}{3}mT(R),$$

$$b_{n+2}^{0,\sigma}=\frac{8}{n}T(R),\quad b_{n+2}^{\sigma,0}=\frac{n+2}{n(n+1)}C_2(R)$$

and

$$b_{n+2}^{\sigma,\rho}=\delta_{\sigma\rho}\left(1-\frac{2}{n(n+1)}+\sum_{2}^{n}\frac{4}{k}\right)C_2(R),$$

where $C_2(G)$, $C_2(R)$ and $T(R)$ are the group theoretical constants of Gross and Wilczek.[1] The solutions of Eq. (14) or (14)$'$ are of the well-known form with the effective coupling constant $\bar{g}$. When q^2 is sufficiently large, by replacing γ-matrices with b-matrices and using the well-known form of $\bar{g}$, we obtain the solutions as

$$\mathcal{G}_n^{(i)}\left(\frac{q^2}{\kappa^2},g^2\right)=\mathcal{G}_n^{(i)}(1,\bar{g}(t))\,t^{-b_{n+2}^{i}/b} \tag{16}$$

with

$$b=\frac{11}{6}C_2(G)-\frac{2}{3}mT(R)>0 \quad \text{and} \quad t=\ln\frac{q^2}{\kappa^2},$$

where $\mathcal{G}_n{}^{(i)}$ are the linear combinations of $\widetilde{G}_n{}^{(i)}$ such that the b_{n+2}-matrix is diagonal and b^i_{n+2} are the eigenvalues of this matrix. As is easily found, for $n\geqq 2$, b^i_{n+2} are positive definite, so that the right-hand sides of (16) vanish in the limit $t\to+\infty$. But $b_3{}^\sigma$ are all zero and give finite contributions in this limit. One of these operators corresponds to the angular momentum density $\mathcal{M}_{\mu\sigma\rho}$, which is a conserved quantity. Thus,

$$\int_0^1 dx\, x^{n-1}V_1(q^2,x)=\sum_{i=0}^{m}A^{(i)}_{n+2,2}\mathcal{G}_n{}^{(i)}(1,\bar{g}(t))\,t^{-b^i_{n+2}/b} \tag{17}$$

and

$$\int_0^1 dx\, x^{n-1}V_2(q^2,x)=-\sum_{i=0}^{m}A^{(i)}_{n+2,3}\mathcal{G}_n{}^{(i)}(1,\bar{g}(t))\,t^{-b^i_{n+2}/b} \tag{18}$$

with the unknown constants $A^{(i)}_{n+2,2}$ and $A^{(i)}_{n+2,3}$ associated with the matrix elements of the composite operators.

Now we must remember that there are contributions of the twist-two operators, which have been omitted until now, but they cancel in the combination

$$\int_0^1 dx\, x^{n-1}(V_2(q^2,x)-V_1(q^2,x))$$
$$=-\sum_{i=0}^{m}(A^{(i)}_{n+2,2}+A^{(i)}_{n+2,3})\mathcal{G}_n{}^{(i)}(1,\bar{g}(t))\,t^{-b^i_{n+2}/b}, \tag{19}$$

as is easily found from (7) and (8).

The Bjorken sum rule and the superconvergence relation are immediately derived by using (18) and (19),

$$\lim_{q^2\to\infty}\int_0^\infty\frac{d\nu}{\nu}\left[\left(\frac{d\sigma}{dq^2dE'}\right)^{\uparrow\uparrow}-\left(\frac{d\sigma}{dq^2dE'}\right)^{\uparrow\downarrow}\right]$$
$$=-\frac{2\alpha(E+E')}{E^2q^4}\left(Z+O\left(\frac{1}{\ln q^2}\right)\right) \tag{20}$$

with

$$Z=-\sum_{\sigma=1}^{m}(A^{(\sigma)}_{3,2}+A^{(\sigma)}_{3,3})\mathcal{G}_1{}^{(\sigma)}(1,0).$$

The correction term comes from the effective coupling constant $\bar{g}$. Equation (20) means that only the operators, whose twist and spin are respectively zero and three (for example, the angular momentum density), contribute to the Bjorken sum rule. So, logarithmic dependence on q^2, which is characteristic in an asymptotically free gauge theory (AFGT), disappears in this sum rule. Also,

$$\int_0^\infty d\nu\, d_2(q^2, \nu) = -\frac{2}{q^2}\int_0^1 dx\, V_2(q^2, x).$$

The right-hand side vanishes in the limit $q^2 \to +\infty$. Thus, the superconvergence relation is a consequence of the less singular nature of the structure function d_2. A similar relation holds for another structure function d_1:

$$\lim_{q^2\to\infty}\int_0^\infty \frac{d\nu}{\nu} d_1(q^2, \nu) = 0.$$

Finally, we summarize the results. (1) In the Bjorken limit, the structure functions d_1 and d_2 satisfy the scaling laws

$$\nu d_1(q^2, \nu) \to V_1(x, \ln q^2)$$

and

$$\nu^2 d_2(q^2, \nu) \to V_2(x, \ln q^2)$$

respectively. These are the same as the predictions of the quark-parton model or the quark-light-cone algebra, up to logarithms due to anomalous dimensions. In this place, the positivity restrictions i)$\sim$iv) display the important role and give the self-consistency condition (9) to our analysis. The experimental test of AFGT as the theory of the strong interactions is to see logarithmic dependence on q^2 in the polarized case as well as in the unpolarized case. (2) The energy sum rule (20) approaches the scaling form of Bjorken as $(\ln q^2)^{-1}$. Therefore, it is experimentally important to see the behavior in the boundary of the asymptotic region. (3) The superconvergence relation is satisfied independently of the logarithms and gives little information on AFGT.

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Note added in proof: The positivity restrictions (9) are not needed for the composite operators made of the fermion field. It can be shown by using the totally antisymmetric nature of them in the subscripts α, β and μ_i which are attached to the Dirac matrices.