POLARIZED ENDOMORPHISMS OF COMPLEX NORMAL VARIETIES

NOBORU NAKAYAMA AND DE-QI ZHANG

Abstract. It is shown that a complex normal projective variety has non-positive Kodaira dimension if it admits a non-isomorphic quasi-polarized endomorphism. The geometric structure of the variety is described by methods of equivariant lifting and fibrations. Endomorphisms of the projective spaces are also discussed and some results on invariant subvarieties under the pullback of the endomorphism are obtained.

1. Introduction

We work over the complex number field $\mathbb{C}$. Much progress has been recently made in the study of endomorphisms of smooth projective varieties from the algebro-geometric viewpoint. Especially, the following cases of varieties are well studied: projective surfaces ([44], [17]), homogeneous manifolds ([49], [11]), some special Fano threefolds ([28]), projective bundles ([2]), and projective threefolds with non-negative Kodaira dimension ([16], [18]). Additionally, étale endomorphisms are investigated in [48] from the viewpoint of the birational classification of algebraic varieties. However, there is neither a classification of endomorphisms of singular varieties even when they are of dimension two, nor any reasonably fine classification of non-étale endomorphisms of smooth threefolds, which are then necessarily uniruled.

Let V be a normal projective variety of dimension n. An endomorphism $f\colon V \to V$ is called *polarized* if there is an ample divisor H such that f^*H is linearly equivalent to qH ($f^*H \sim qH$) for a positive number q. In this case, f is a finite surjective morphism, q is an integer, and $\deg f = q^n$ (cf. Lemma 2.1 below). A surjective endomorphism of a variety of Picard number one is always polarized. Polarized endomorphisms of smooth projective varieties are studied in papers [14] and [55]. In this paper, we shall study the polarized endomorphisms of normal projective varieties (not only smooth ones). The following Theorems 1.1, 1.2 and 1.4 are our main results.

Theorem 1.1. *Let $f\colon X \to X$ be a non-isomorphic polarized endomorphism of a normal projective variety X. Then there exist a finite morphism $\tau\colon V \to X$ from a normal*

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projective variety V, a dominant rational map $\pi\colon V \cdots\to A \times S$ for an abelian variety A and a weak Calabi–Yau variety S (cf. Definition 1.6 below), *and polarized endomorphisms $f_V\colon V \to V$, $f_A\colon A \to A$, $f_S\colon S \to S$ satisfying the following conditions*:

(1) $\tau \circ f_V = f \circ \tau$, $\pi \circ f_V = (f_A \times f_S) \circ \pi$.

(2) *τ is étale in codimension one.*

(3) *If X is not uniruled, then $\kappa(X) = 0$ and π is an isomorphism.*

(4) *If X is uniruled, then, for the graph $\Gamma_\pi \subset V \times A \times S$ of π, the projection $\Gamma_\pi \to A \times S$ is an equi-dimensional morphism birational to the maximal rationally connected fibration* (MRC fibration in the sense of [37]) *of a nonsingular model of V.*

(5) *If $\dim S > 0$, then $\dim S \geq 4$ and S contains a non-quotient singular point.*

It is conjectured that $\dim S = 0$ for S in Theorem 1.1 (cf. Conjecture 3.4). If X is smooth and $\kappa(X) \geq 0$, Theorem 1.1 (with S being a point) is proved in [14], Theorem 4.2. For uniruled X, there is a discussion on endomorphisms and maximal rationally connected fibrations in [55], Section 2.2, especially in Proposition 2.2.4 (cf. Remark 4.2 below).

Applying Theorem 1.1 and more, we have the following classification result:

Theorem 1.2. *Let $f\colon X \to X$ be a polarized endomorphism of a normal projective variety X of dimension n. Then $\kappa(X) \leq 0$ and $q^\natural(X) \leq n$ for the invariant $q^\natural(X)$ defined in* Definition 1.5. *Furthermore, X is described as follows*:

(1) *If $\dim X \leq 3$ and $q^\natural(X) = 0$, then X is rationally connected.*

(2) *$q^\natural(X) = n$ if and only if X is Q-abelian* (cf. Definition 1.7 below).

(3) *If $q^\natural(X) \geq n - 3$, then there exist a finite covering $V \to X$ étale in codimension one, a birational morphism $Z \to V$ of normal projective varieties, and a flat surjective morphism $\varpi\colon Z \to A$ onto an abelian variety A of dimension $q^\natural(X)$ such that*
   - *any fiber of ϖ is irreducible and reduced,*
   - *a general fiber of ϖ is rationally connected.*

   Moreover, the fundamental group $\pi_1(X)$ has a finite index subgroup which is a finitely generated abelian group of rank at most $2q^\natural(X)$.

(4) *If $q^\natural(X) = n - 1$, then there is a finite covering $V \to X$ étale in codimension one from a normal projective variety V satisfying one of the following conditions*:
   (a) *V is a $\mathbb{P}^1$-bundle over an abelian variety.*
   (b) *There exist a $\mathbb{P}^1$-bundle W over an abelian variety and a birational morphism $W \to V$ whose exceptional locus is a section of the $\mathbb{P}^1$-bundle.*

For a polarized endomorphism $f\colon X \to X$, by [14], Corollary 2.2, we have a closed immersion $i\colon X \to \mathbb{P}^N$ into a projective space $\mathbb{P}^N$ together with an endomorphism

$g\colon \mathbb{P}^N \to \mathbb{P}^N$ such that $g \circ i = i \circ f$. Therefore, g preserves $i(X)$, i.e., $g(i(X)) = i(X)$. Thus, it is important to study endomorphisms of projective spaces and their subvarieties preserved by the endomorphisms. As a corollary of Theorem 1.1, we have: If $V \subset \mathbb{P}^n$ is an irreducible subvariety satisfying $g(V) = V$, then either V is uniruled or $\kappa(V) = 0$. In fact, $g|_V \colon V \to V$ is a polarized endomorphism of degree > 1.

We can also consider subvarieties V which is preserved by g^{-1}, i.e., $g^{-1}(V) = V$. Such subvarieties are called *exceptional*, or *completely invariant*, in the study of dynamical systems, and the following conjecture is studied (cf. [15], [12], [8]):

Conjecture 1.3. *Let f be an non-isomorphic surjective endomorphism of $\mathbb{P}^N$. If an irreducible subvariety $V \subset \mathbb{P}^n$ satisfies $f^{-1}(V) = V$ as a subset, then V is a linear subspace.*

The paper [8] asserted the conjecture to be affirmative, but unfortunately, one of the authors of the paper informed us that the proof contains a gap and the assertion there should be regarded as a conjecture. We consider Conjecture 1.3 for a *hypersurface* V (i.e., a reduced divisor of $\mathbb{P}^n$). If V is a smooth hypersurface, then the conjecture is almost solved by [12] or by a result in [4] (cf. [15] for earlier approach in the case of $n = 2$). We shall complete the case of smooth hypersurfaces, and moreover, prove the following on the conjecture:

Theorem 1.4. *Let f be a non-isomorphic surjective endomorphism of a projective space $\mathbb{P}^n$ of dimension $n \geq 2$. Let $V \subset \mathbb{P}^n$ be a hypersurface satisfying $f^{-1}(V) = V$ as a subset.*

(1) $\deg(V) \leq n+1$ *and V has only normal crossing singularities in codimension one.*
(2) *Every irreducible component V_i of V is uniruled with* $\deg(V_i) \leq n$. *If V_i is smooth, then V_i is a hyperplane.*
(3) *If V is a union of hyperplanes, then V is normal crossing.*
(4) *If $n = 2$, then V is a union of at most three lines.*
(5) *If $n = 3$, then any irreducible component V_i is a hyperplane or a cubic rational surface.*

Notation and Conventions. For things related to the birational classification theory of algebraic varieties and the minimal model theory of projective varieties, we follow the notation in standard references such as [33], [36], etc. One remark is that the *Kodaira dimension* of a projective variety is defined as that of its non-singular model. The *linear equivalence* relation is denoted by the symbol $\sim$, the $\mathbb{Q}$-*linear equivalence* relation by $\sim_{\mathbb{Q}}$, and the *numerical equivalence* relation by $\approx$.

An endomorphism $f\colon X \to X$ is called *polarized* (resp. *quasi-polarized*) if $f^*H \sim qH$ for an ample divisor (resp. a nef and big divisor) H for some positive integer q (cf. Lemma 2.1 below).

For a projective variety Z, the singular locus is denoted by $\operatorname{Sing} Z$ and the smooth locus $Z \setminus \operatorname{Sing} Z$ by Z_{reg}. Note that, for a normal variety Z, a finite morphism $Z' \to Z$ étale in codimension one from a normal variety Z' corresponds to a finite index subgroup of $\pi_1(Z_{\mathrm{reg}})$.

Definition 1.5. Let X be a normal projective variety. Then the irregularity $q(X)$ is defined as $\dim \mathrm{H}^1(X, \mathcal{O}_X)$. We define a new invariant $q^\circ(X)$ to be the supremum of $q(X')$ for normal projective varieties X' admitting a finite surjective morphism $X' \to X$ which is étale in codimension one. We define another invariant $q^\natural(X)$ to be $q^\circ(T)$ for the special MRC fibration $X \dashrightarrow T$ defined in [46], Section 4.3; see also Lemma 4.1.

If X is a smooth projective variety, then $q^\circ(X)$ equals $q^{\max}(X)$ defined in [48]. If X has only canonical singularities and $\kappa(X) = 0$, then $q^\circ(X) \le \dim X$ by [30].

Definition 1.6. A normal projective variety Y with only canonical singularities is called a *weak Calabi–Yau variety* if $K_Y \sim 0$ and $q^\circ(Y) = 0$.

Note that this definition is slightly different from that in [48]. A weak Calabi–Yau variety has dimension at least two. A two-dimensional weak Calabi–Yau variety is nothing but a normal projective surface such that the minimal resolution of singularities is a K3 surface and that there is no finite surjective morphism from any abelian surface.

Definition 1.7. A normal projective variety W is called *Q-abelian* if there are an abelian variety A and a finite surjective morphism $A \to W$ which is étale in codimension one.

In the definition, we may choose $A \to W$ to be Galois by taking the Galois closure. If W is Q-abelian, then $q^\circ(W) = \dim W$. A similar notion "Q-torus" is introduced in [43], which is a Kähler version and is restricted to étale coverings.

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2. Some basic properties

A surjective endomorphism of a normal projective variety is a finite morphism by the same argument as in [16], Lemma 2.3. In fact, such an endomorphism $f\colon X \to X$ induces an automorphism $f^*\colon \mathsf{N}^1(X) \to \mathsf{N}^1(X)$ of the real vector space $\mathsf{N}^1(X) := \mathrm{NS}(X) \otimes \mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$, whence the pullback of an ample divisor is ample.

Lemma 2.1. *Let $f\colon X \to X$ be an endomorphism of an n-dimensional normal projective variety X such that $f^*H \approx qH$ for a positive number q and for a nef and big divisor H. Then q is a positive integer and $\deg f = q^n$. Moreover, the absolute value of any eigenvalue of $f^*\colon \mathsf{N}^1(X) \to \mathsf{N}^1(X)$ is q.*

Proof. Comparing the self intersection numbers $(f^*H)^n$ and H^n, we have $\deg f = q^n$. In particular, q is an algebraic integer. Since f^*H and H belong to the Néron-Severi subgroup, q should be a rational number. Hence, q is an integer. Let λ be the spectral radius of $f^*\colon \mathsf{N}^1(X) \to \mathsf{N}^1(X)$, i.e., the maximum of the absolute values of eigenvalues of f^*. Then, there is a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D such that $D \not\approx 0$ and $f^*D \approx \lambda D$ by [6]. Suppose that $\lambda \neq q$. Then $DH^{n-1} = 0$ by

$$\lambda q^{n-1} DH^{n-1} = f^*D(f^*H)^{n-1} = (\deg f) DH^{n-1} = q^n DH^{n-1}.$$

This is a contradiction, since we can derive $D \approx 0$ from $DH^{n-1} = 0$ as follows (cf. Lemma 2.5 below): If Γ is a prime divisor, then $D\Gamma H^{n-2} = 0$. In fact, there exist a positive rational number a and an effective $\mathbb{Q}$-divisor E such that $H \sim_{\mathbb{Q}} a\Gamma + E$, which induces

$$0 \leq aD\Gamma H^{n-2} = -DEH^{n-2} \leq 0.$$

Applying the argument above successively, we infer that $DA^{n-1} = 0$ for any ample divisor A and that $D\Gamma A^{n-2} = 0$ for any prime divisor Γ. Thus, by induction on dimension, $D|_\Gamma \approx 0$. Therefore, $DC = 0$ for any irreducible curve C, since there is a prime divisor containing C. Thus, $D \approx 0$. Therefore, we have $\lambda = q$. For the spectral radius λ' of $(f^*)^{-1}$, λ'^{-1} is the minimum of the absolute values of eigenvalues of f^*. We also have a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D' such that $D' \not\approx 0$ and $f^*D' = \lambda'^{-1}D'$ by [6]. Then, $\lambda' = q^{-1}$ by the same reason above. Hence, the absolute value of any eigenvalue is q. $\square$

The endomorphism in Lemma 2.1 is shown to be quasi-polarized by the following:

Lemma 2.2. *Let $f\colon X \to X$ be an endomorphism of an n-dimensional normal projective variety X such that $f^*H \approx qH$ for a positive number q and for a nef and big divisor H. Then the following conditions are satisfied*:

(1) *The absolute value of any eigenvalue of $f^*\colon \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \mathcal{O}_X)$ is $\sqrt{q}$.*

(2) *There is a nef and big divisor H' such that $H' \approx H$ and $f^*H' \sim_{\mathbb{Q}} qH'$. Thus, f is quasi-polarized by H'.*

Proof. (1): There exist birational morphisms $\mu\colon M \to X$ and $\nu\colon Z \to X$ from smooth projective varieties M and Z, and a generically finite surjective morphism $h\colon Z \to M$ such that $\mu \circ h = f \circ \nu$. We may assume that the birational map $\psi := \mu^{-1} \circ \nu\colon Z \to M$ is holomorphic. Then, we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^1(X, \mathcal{O}_X) & \xrightarrow{f^*} & \mathrm{H}^1(X, \mathcal{O}_X) & = & \mathrm{H}^1(X, \mathcal{O}_X) \\
{\scriptstyle \mu^*}\downarrow & & {\scriptstyle \nu^*}\downarrow & & {\scriptstyle \mu^*}\downarrow \\
\mathrm{H}^1(M, \mathcal{O}_M) & \xrightarrow{h^*} & \mathrm{H}^1(Z, \mathcal{O}_Z) & \underset{\simeq}{\xleftarrow{\psi^*}} & \mathrm{H}^1(M, \mathcal{O}_M).
\end{array}$$

Let $\phi(x)$ be the image of $x \in \mathrm{H}^1(X, \mathcal{O}_X)$ by the composition

$$\mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\mu^*} \mathrm{H}^1(M, \mathcal{O}_M) \xrightarrow{\simeq} \mathrm{H}^{0,1}(M) \subset \mathrm{H}^1(M, \mathbb{C}),$$

where $\mathrm{H}^{0,1}(M)$ is the $(0,1)$-part of the Hodge decomposition of $\mathrm{H}^1(M, \mathbb{C})$. Then, for $x \in \mathrm{H}^1(X, \mathcal{O}_X)$, we have $\psi^*\phi(f^*(x)) = h^*\phi(x)$ by the diagram above. We consider the following hermitian form on $\mathrm{H}^1(X, \mathcal{O}_X)$:

$$\langle x, y \rangle = -\sqrt{-1} \int_M \phi(x) \cup \overline{\phi(y)} \cup (\mu^* c_1(H))^{n-1} \in \mathbb{C}.$$

This is positive definite by Lemma 2.3 below. We have the equality

$$\langle f^*(x), f^*(y) \rangle = q \langle x, y \rangle$$

for x, $y \in \mathrm{H}^1(X, \mathcal{O}_X)$ by the calculation

$$\begin{aligned}
(\deg h)\langle x, y \rangle &= -\sqrt{-1} \int_Z h^*(\phi(x)) \cup \overline{h^*(\phi(y))} \cup (h^*\mu^* c_1(H))^{n-1} \\
&= -\sqrt{-1} \int_Z \psi^*\phi(f^*(x)) \cup \overline{\psi^*\phi(f^*(x))} \cup (\nu^* f^* c_1(H))^{n-1} \\
&= -\sqrt{-1} \int_M \phi(f^*(x)) \cup \overline{\phi(f^*(y))} \cup (\mu^* f^* c_1(H))^{n-1} \\
&= q^{n-1} \langle f^*(x), f^*(y) \rangle,
\end{aligned}$$

where $\deg h = \deg f = q^n$. Therefore, $q^{-1/2} f^*$ is a unitary transformation with respect to $\langle\ ,\ \rangle$. Thus, the absolute value of any eigenvalue of $q^{-1/2} f^*$ is 1.

(2): Let m be the order of $c_1(f^*H - qH)$ in $\mathrm{H}^2(X, \mathbb{Z})$. By the exponential exact sequence

$$\mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\epsilon} \mathrm{H}^1(X, \mathcal{O}_X^\star) \to \mathrm{H}^2(X, \mathbb{Z})$$

we can find an element $x \in \mathrm{H}^1(X, \mathcal{O}_X)$ with $\mathcal{O}_X(m(f^*H - qH)) = m\epsilon(x)$. There is an element $y \in \mathrm{H}^1(X, \mathcal{O}_X)$ such that $f^*y - qy = x$ by (1). Let H' be a divisor such that $\mathcal{O}_X(H - H') = \epsilon(y)$. Then $m(f^*H' - qH') \sim 0$. Thus, we are done. $\square$

The following is used in Lemma 2.2:

Lemma 2.3. *Let M be an n-dimensional smooth projective variety and H a nef and big divisor. Then the Hermitian form $\langle\ ,\ \rangle$ on $\mathrm{H}^{0,1}(M)$ defined by*

$$\langle \xi, \eta \rangle = -\sqrt{-1} \int_M \xi \cup \overline{\eta} \cup c_1(H)^{n-1}$$

is positive definite.

Proof. If H is ample, then it is positive definite by the hard Lefschetz theorem. Thus, the bilinear form is positive semi-definite even if we replace H with a nef divisor. Let W be a prime divisor of M. Then

$$-\sqrt{-1} \int_M \xi \cup \overline{\xi} \cup c_1(H)^{n-2} \cup c_1(W)$$

is non-negative for any $\xi \in \mathrm{H}^{0,1}(M)$. In fact, it is equal to

$$-\sqrt{-1} \int_{\widetilde{W}} \varphi^*(\xi) \cup \overline{\varphi^*(\xi)} \cup c_1(\varphi^* H)^{n-2}$$

for a resolution of singularities $\varphi\colon \widetilde{W} \to W$, and it is non-negative by the reason above.

There exist a positive integer m, a smooth ample divisor A, and an effective divisor $E = \sum e_i E_i$ such that $m\mu^* H \sim A + E$. Then

$$\begin{aligned} m\langle x, x \rangle &= -\sqrt{-1} \int_M \phi(x) \cup \overline{\phi(x)} \cup m c_1(\mu^* H)^{n-1} \\ &= -\sqrt{-1} \int_A \phi(x)|_A \cup \overline{\phi(x)|_A} \cup c_1(\mu^* H|_A)^{n-2} \\ &\quad + \sum e_i(-\sqrt{-1}) \int_{E_i} \phi(x)|_{E_i} \cup \overline{\phi(x)|_{E_i}} \cup c_1(\mu^* H|_{E_i})^{n-2}. \end{aligned}$$

Hence, if $\langle x, x \rangle = 0$, then

$$-\sqrt{-1} \int_A \phi(x)|_A \cup \overline{\phi(x)|_A} \cup c_1(\mu^* H|_A)^{n-2} = 0.$$

Since $\mu^* H|_A$ is nef and big, we can consider the induction on $\dim M$. Then, we infer that $\phi(x)|_A = 0$ as an element of $\mathrm{H}^{0,1}(A)$. Hence $\phi(x) = 0$, since $\mathrm{H}^1(M, \mathcal{O}_M(-A)) = 0$ by the Kodaira vanishing theorem, and since $\mathrm{H}^1(M, \mathcal{O}_M) \to \mathrm{H}^1(A, \mathcal{O}_A)$ is injective. Thus, we are done. $\square$

Remark. The proof of Lemma 2.2 is similar to that of [55], Theorem 1.1.2, where X is assumed to be smooth.

The following is a property of Galois closures of powers $f^k = f \circ \cdots \circ f$ (cf. [47]):

Lemma 2.4. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective variety X. Let $\theta_k\colon V_k \to X$ be the Galois closure of $f^k\colon X \to X$ for $k \geq 1$ and let $\tau_k\colon V_k \to X$ be the induced finite Galois covering such that $\theta_k = f^k \circ \tau_k$. Then* *there exist finite Galois morphisms* $g_k, h_k \colon V_{k+1} \to V_k$ *such that* $\tau_k \circ g_k = \tau_{k+1}$ *and* $\tau_k \circ h_k = f \circ \tau_{k+1}$.

Proof. The composition $f^k \circ \tau_{k+1} \colon V_{k+1} \to X \to X$ is Galois since so is $f^{k+1} \circ \tau_{k+1} = \theta_{k+1}$. Hence, $f^k \circ \tau_{k+1}$ factors the Galois closure θ_k of f^k. Thus, $\tau_{k+1} = \tau_k \circ g_k$ for a morphism $g_k \colon V_{k+1} \to V_k$. Let H_i be the Galois group of $f^i \circ \tau_{k+1} \colon V_{k+1} \to X$ for $0 \le i \le k+1$. Then V_k is regarded as the Galois closure of $V_{k+1}/H_1 \to V_{k+1}/H_{k+1}$, thus $V_k \simeq V_{k+1}/H$ for the maximal normal subgroup H of H_{k+1} contained in H_1. Hence, we have a morphism $h_k \colon V_{k+1} \to V_k$ with $\tau_k \circ h_k = f \circ \tau_{k+1}$. □

The following is a generalization of a part of the Hodge index theorem:

Lemma 2.5. *Let* X *be a smooth projective variety of dimension* n *and let* D *a pseudo-effective* $\mathbb{R}$*-divisor. Suppose that* $DH_1H_2 \cdots H_{n-1} = 0$ *for nef and big* $\mathbb{R}$*-divisors* $H_1, \ldots, H_{n-1}$*. Then the positive part* $P_\sigma(D)$ *of the* σ*-decomposition of* D *in the sense of* [45] *is numerically trivial.*

Proof. We may assume the negative part $N_\sigma(D)$ to be zero. We consider the induction on $n = \dim X$. If $n = 2$, then D is nef and $DH = 0$; thus $D \approx 0$ by the Hodge index theorem. Suppose that $n \ge 3$. Let A be a non-singular ample divisor of X. Since H_{n-1} is big, there exist a positive rational number a and an effective $\mathbb{R}$-divisor E such that $H_{n-1} \approx aA + E$. Then

$$0 \le aDAH_1 \cdots H_{n-2} = -DEH_1 \cdots H_{n-2} \le 0.$$

Here, we use the property that $D|_\Gamma$ is pseudo-effective for any prime divisor Γ. Then we have $D|_A \approx 0$ by induction on n. Since $DA^{n-1} = D^2A^{n-2} = 0$, we have $D \approx 0$ by the hard Lefschetz theorem. □

The following is proved for smooth varieties in [48], Proposition 4.3:

Lemma 2.6. *Let* V *be a normal projective variety with only canonical singularities and with* $K_V \sim_{\mathbb{Q}} 0$*. Then there exists a finite morphism* $\tau \colon V^\sim \to V$ *satisfying the following conditions, uniquely up to isomorphism over* V:

(1) τ *is étale in codimension one.*
(2) $q^\circ(V) = q(V^\sim)$.
(3) τ *is Galois, and* $\deg \tau$ *is minimal among finite coverings satisfying the conditions* (1), (2).

We call τ the *Albanese closure* of V in codimension one.

Proof. The same argument as in [48] works as follows: We may assume that $q^\circ(V) > 0$. There is a Galois covering $W \to V$ étale in codimension one with $q(W) = q^\circ(V)$. Then $K_W \sim_{\mathbb{Q}} 0$ and W has only canonical singularities. Let $W \to \mathrm{Alb}(W)$ be the Albanese map of W; this is holomorphic since W has only rational singularities. Let $\mathrm{Gal}(W/V)$ be the Galois group of W/V. Then we have a natural homomorphism $\mathrm{Gal}(W/V) \to \mathrm{Aut}(\mathrm{H}_1(\mathrm{Alb}(W), \mathbb{Z}))$. Let W_0 be the quotient space of W by the kernel of the homomorphism. Then the Galois covering $W_0 \to V$ also satisfies the conditions (1), (2). Let $W' \to V$ be any covering satisfying the conditions (1), (2). Then there exist finite morphisms $W'' \to W$ and $W'' \to W'$ over V such that the composite $W'' \to V$ is Galois and étale in codimension one. Then $W_0'' \simeq W_0$ and there is a morphism $W' \to W_0$ over V. Hence, $V^\sim := W_0$ satisfies the required conditions and $V^\sim \to V$ is unique up to non-canonical isomorphism. $\square$

A surjective endomorphism of the direct product of certain varieties is split. The following gives an example:

Lemma 2.7. *Let A be an abelian variety and S a normal projective variety with $q(S) = 0$ and at most rational singularities. Suppose that S is not uniruled. Let $f \colon S \times A \to S \times A$ be a surjective morphism. Then $f = f_S \times f_A$ for suitable endomorphisms f_S and f_A of S and T, respectively.*

Proof. Note that f induces a surjective endomorphism f_A of $A = \mathrm{Alb}(S \times A)$. We can write $f(s, a) = (\rho_a(s), f_A(a))$, where $\rho \colon A \to \mathrm{Sur}(S)$, $a \mapsto \rho_a$, is a morphism into

$$\mathrm{Sur}(S) := \{g \colon S \to S \mid g \text{ is a surjective morphism}\}.$$

By [26], Theorem 3.1, the compact subvariety $\mathrm{Im}(\rho)$ is contained in the orbit of some $f_S \in \mathrm{Sur}(S)$ by the action of $\mathrm{Aut}^0(S)$. For a nonsingular model S' of S, the birational automorphism group $\mathrm{Bir}(S')$ contains $\mathrm{Aut}^0(S)$ as a subgroup. By [24], Theorem (2.1), $\mathrm{Bir}(S')$ is a disjoint union of abelian varieties of dimension equal to $q(S') = q(S) = 0$. Thus $\mathrm{Im}(\rho)$ is a single element, say $\{f_S\}$. Then $f = f_S \times f_A$. $\square$

The following is proved in [46], Section 4.3:

Lemma 2.8. *Let $\pi \colon X \to Y$ be an equi-dimensional surjective morphism of normal projective varieties with connected fibers. Let $f_X \colon X \to X$ and $f_Y \colon Y \to Y$ be endomorphisms such that $\pi \circ f_X = f_Y \circ \pi$. If f_X is polarized (resp. quasi-polarized), then so is f_Y. Here, if $\deg f_X = q^{\dim X}$, then $\deg f_Y = q^{\dim Y}$.*

3. The non-uniruled case

We shall study non-isomorphic quasi-polarized endomorphisms of non-uniruled normal projective varieties in this section.

Example 3.1. Let A be an abelian variety of dimension n and let H be a symmetric ample divisor, i.e., H is ample and $\iota^*H \sim H$ for the involution $\iota\colon x \mapsto -x$. Then the multiplication map $\mu_m\colon A \ni x \mapsto mx = x + \cdots + x \in A$ by an integer m is polarized by H as $\mu_m^* H \sim m^2 H$ (cf. [41], Chapter II, § 6, Corollary 3). Let $X = A/\iota$ be the quotient variety by the involution ι. Then μ_m descends to a polarized endomorphism f_m of X of degree $\deg \mu_m = m^{2n}$. If $\dim A$ is even, then $K_X \sim 0$ and X has only canonical singularities. In particular, X is non-uniruled and admits a non-isomorphic polarized endomorphism. If $\dim X = 2$, then X is birational to a K3 surface, hence f_m for $m > 1$ is not nearly étale in the sense of [48], Definition 3.2 (cf. [48], Example 3.14).

The following result is fundamental:

Theorem 3.2. *Let $f\colon V \to V$ be a surjective endomorphism of a normal projective variety V and let H be a nef and big Cartier divisor on V such that $f^*H \sim qH$ for a positive integer $q > 1$. Suppose that V is not uniruled. Then, there exist a projective birational morphism $\sigma\colon V \to X$ onto a normal projective variety X, an endomorphism f_X of X, and an ample divisor A on X such that*

1. *X has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$,*
2. *$f_X^* A \sim qA$,*
3. *$f_X \circ \sigma = \sigma \circ f$, and*
4. *$H \sim \sigma^* A$.*

Proof. From the ramification formula $K_V = f^*K_V + R$ and $n = \dim V$, we have

$$(q-1)K_V H^{n-1} + RH^{n-1} = 0.$$

Thus, $K_V H^{n-1} \le 0$. Let $\mu\colon Y \to V$ be a birational morphism from a smooth projective variety Y. Since Y is not uniruled, K_Y is pseudo-effective by [7] (cf. [39], §11.4.C). Then $K_Y(\mu^*H)^{n-1} = K_V H^{n-1} = 0$. Hence $K_Y \approxeq N_\sigma(K_Y)$ by Lemma 2.5, and $\kappa_\sigma(Y) = \kappa(Y) = 0$ by [45], Chapter V, Corollary 1.12 and Theorem 4.8. In particular, $K_Y \sim_{\mathbb{Q}} E$ for an effective $\mathbb{Q}$-divisor E such that $E(\mu^*H)^{n-1} = 0$. Therefore, $K_Y + \mu^*H$ has a Zariski-decomposition whose negative part is E and whose positive part is $\mathbb{Q}$-linearly equivalent to μ^*H by [45], Chapter III, Proposition 3.7. Then, the positive part is semi-ample by a version of base point free theorem (cf. [20], (A.5); [31], Theorem 1; [42], Theorem 0). Therefore, $\operatorname{Bs}|mH| = \emptyset$ for $m \gg 0$. Let $\sigma\colon V \to X$ be a birational morphism onto a normal projective variety X defined by the free linear system $|mH|$ for $m \gg 0$. Then

$H \sim \sigma^* A$ for an ample divisor A on X. Since $(\mu_* E)H^{n-1} = RH^{n-1} = 0$, $\mu_* E$ and R are σ-exceptional. In particular, X has only canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. By considering the Stein factorization of the composite $\sigma \circ f \colon V \to X$, we have an endomorphism f_X of X such that $f_X \circ \sigma = \sigma \circ f$. Then, f_X is étale in codimension one and $f_X^* A \sim qA$. □

The following gives a sufficient condition for a normal projective variety admitting polarized endomorphisms to be Q-abelian:

Theorem 3.3. *Let $f \colon X \to X$ be a non-isomorphic polarized endomorphism of a normal projective variety X. Assume that f is étale in codimension one, and that for any point $P \in \operatorname{Sing} X$, there is an analytic open neighborhood $\mathcal{U}$ of P such that $\pi_1(\mathcal{U}_{\mathrm{reg}})$ is finite. Then X is a Q-abelian variety.*

Proof. Let A be an ample divisor on X such that $f^* A \sim qA$. For a positive integer k, let $\theta_k \colon V_k \to X$ be the Galois closure of f^k, and let τ_k, θ_k, g_k, and h_k be as in Lemma 2.4. We set A_k to be the ample divisor $\tau_k^* A$. Then $g_k^* A_k \sim A_{k+1}$ and $h_k^* A_k \sim qA_{k+1}$.

For a point $P \in \operatorname{Sing} X$, let $\mathcal{U} \subset X$ be an analytic open neighborhood such that $\pi_1(\mathcal{U}_{\mathrm{reg}})$ is finite. For a point $Q \in \theta_k^{-1}(P)$, let $\mathcal{V}$ be the connected component of $\theta_k^{-1}(\mathcal{U})$ containing Q. Then

$$\Pi(\mathcal{U}; k) := \pi_1(\mathcal{V} \setminus \theta_k^{-1}(\operatorname{Sing} X)) = \pi_1(\mathcal{V}_{\mathrm{reg}})$$

is a normal subgroup of $\pi_1(\mathcal{U}_{\mathrm{reg}})$, and it is independent for the choice of $Q \in \theta_k^{-1}(P)$, since θ_k is Galois. Since $\operatorname{Sing} X$ is compact, there is a positive integer k_0 such that $\sharp\Pi(\mathcal{U}; k) = \sharp\Pi(\mathcal{U}; k+1)$ for any $k \geq k_0$ and for any such open neighborhood $\mathcal{U}$ of any point $P \in \operatorname{Sing} X$. Then g_k, $h_k \colon V_{k+1} \to V_k$ are both étale for $k \geq k_0$. In particular, $g_k^{-1}(\operatorname{Sing} V_k) = h_k^{-1}(\operatorname{Sing} V_k) = \operatorname{Sing} V_{k+1}$, and the mapping degrees of $g_k \colon \operatorname{Sing} V_{k+1} \to \operatorname{Sing} V_k$ and $h_k \colon \operatorname{Sing} V_{k+1} \to \operatorname{Sing} V_k$ are $\deg g_k$ and $\deg h_k$, respectively. For $d = \dim \operatorname{Sing} V_k < \dim V_k = n$, we have the equality

$$(\deg g_k)(\operatorname{Sing} V_k)A_k^d = (\operatorname{Sing} V_{k+1})A_{k+1}^d = q^{-d}(\deg h_k)(\operatorname{Sing} V_k)A_k^d$$

of intersection numbers. Thus, $\operatorname{Sing} V_k = \emptyset$, since $q^{-d} \deg h_k = q^{n-d} \deg g_k$ and A_k is ample.

Then g_k, $h_k \colon V_{k+1} \to V_k$ are étale morphisms between smooth projective varieties with $\deg h_k = (\deg f)(\deg g_k)$. Hence, we have

$$c_1(V_k)A_k^{n-1} = c_1(V_k)^2 A_k^{n-2} = c_2(V_k)A_k^{n-2} = 0$$

by a similar calculation of intersection numbers as above. Then $c_1(V_k)$ is numerically trivial by the hard Lefschetz theorem. Moreover, the vanishing of c_2 implies that an étale covering of V_k is an abelian variety by [54] (cf. [3]). Therefore, X is a Q-abelian variety. $\square$

Remark. A result of Campana [10], Corollary 6.3 gives another proof of Theorem 3.3 in the case where $K_X \sim_{\mathbb{Q}} 0$ and X has only quotient singularities.

The following conjecture is proved in [14], Theorem 4.2, in case X is smooth:

Conjecture 3.4. *A non-uniruled normal projective variety admitting a non-isomorphic polarized endomorphism is Q-abelian.*

For the case of normal varieties, we have the following partial answer:

Proposition 3.5. Conjecture 3.4 *is true if* $\dim X \leq 3$ *or if* X *has only quotient singularities.*

Proof. By Theorem 3.3, it is enough to show that any singular point has a connected analytic open neighborhood $\mathcal{U}$ such that $\pi_1(\mathcal{U}_{\mathrm{reg}})$ is finite. If X has only quotient singularities, then this is true. We know that X has only canonical singularities by Theorem 3.2. If $\dim X \leq 2$, then X has only quotient singularities. If $\dim X = 3$, then the finiteness of $\pi_1(\mathcal{U}_{\mathrm{reg}})$ is proved in [51], Theorem 3.6. Thus, we are done. $\square$

Even though Conjecture 3.4 is not solved yet, we have the following:

Proposition 3.6. *Let X be a normal projective variety with a non-isomorphic polarized endomorphism f. If X is not uniruled, then there exist an abelian variety A, a weak Calabi–Yau variety S, a finite morphism $\tau\colon A \times S \to X$ and polarized endomorphisms $f_A\colon A \to A$, $f_S\colon S \to S$ such that*

(1) *τ is étale in codimension one, and*
(2) *$\tau \circ (f_A \times f_S) = f \circ \tau$.*

Proof. We know that X has only canonical singularities and $K_X \sim_{\mathbb{Q}} 0$ by Theorem 3.2. Let $X' \to X$ be the global index-one cover, i.e., the minimal cyclic covering satisfying $K_{X'} \sim 0$. By the uniqueness of the global index-one cover, there is an endomorphism $f'\colon X' \to X'$ compatible with f. If $q^{\circ}(X) = 0$, then X' is a weak Calabi–Yau variety, and the assertion holds. Hence, we may assume that $q^{\circ}(X) > 0$.

Let $\widetilde{X} = (X')^{\sim} \to X'$ be the Albanese closure of X' in codimension one (cf. Lemma 2.6). By the uniqueness of Albanese closure, $\widetilde{X}$ admits an endomorphism $\tilde{f}$ compatible with f and f'. For the Albanese map $\alpha\colon \widetilde{X} \to A := \mathrm{Alb}(\widetilde{X})$, by [30], Theorem 8.3, there is an étale covering $\theta\colon T \to A$ such that

$$\widetilde{X} \times_A T \simeq S \times T$$

over T for a fibre S of α. This S is weak Calabi–Yau by the definition of q°. Taking a further étale covering, we may assume that $T \simeq A$ and $\theta\colon T \to A$ is just the multiplication by a positive integer m for certain group structure of A. Let $f'_A\colon A \to A$ be the induced endomorphism of A satisfying $\alpha \circ \tilde{f} = f'_A \circ \alpha$. By [48], Lemma 4.9, there is an endomorphism f_A of A such that $\theta \circ f_A = f'_A \circ \theta$. Let W be the fiber product of $\alpha\colon \widetilde{X} \to A$ and $\theta\colon A \to A$. Then $\tilde{f} \times f_A$ induces an endomorphism f_W of W which is compatible with f. In particular, f_W is polarized by the pullback of an ample divisor on X. We have an endomorphism $f_S\colon S \to S$ such that $f_W = f_S \times f_A$ by Lemma 2.7. Here, f_S and f_A are polarized by Lemma 2.8. Thus, we are done. □

Theorem 1.1 for non-uniruled X is proved by Theorem 3.2 and Propositions 3.5, 3.6.

4. The proof of Theorems 1.1 and 1.2

The following result gives a descent property of polarized endomorphisms by maximal rationally connected fibrations, which is proved in [46], Section 4.3.

Lemma 4.1. *Let $f\colon X \to X$ be a quasi-polarized endomorphism of a normal projective variety X. Suppose that X is uniruled but not rationally connected. Then there exist a birational morphism $\sigma\colon W \to X$, an equi-dimensional surjective morphism $p\colon W \to Y$, and quasi-polarized endomorphisms $f_W\colon W \to W$, $f_Y\colon Y \to Y$ such that*

(1) *W and Y are normal projective varieties,*
(2) *Y is not uniruled,*
(3) *a general fiber of p is rationally connected,*
(4) *$\sigma \circ f_W = f \circ \sigma$, and $p \circ f_W = f_Y \circ p$.*

Here, if f is polarized, then f_Y is also polarized and $\deg f_Y = (\deg f)^{\dim Y/\dim X}$.

The dominant rational map $p \circ \sigma^{-1}\colon X \cdots\!\to Y$ is the *special MRC fibration* defined in [46], Section 4.3. The variety W is characterized as the normalization of the graph of $p \circ \sigma^{-1}$. If f is polarized, then so is f_W since it is induced from $f \times f_Y$.

Remark 4.2. The same assertion as Lemma 4.1 for polarized endomorphisms is stated in [55], Proposition 2.2.4. However, the argument there is valid only when the maximal rationally connected fibration is flat, which is not a priori available. The study of "intersection sheaves" in [46] renders the flatness requirement redundant, and consequently the expected assertion is proved in [46], Section 4.3.

Remark 4.3. In Lemma 4.1, there is a countable dense subset $\mathcal{Y} \subset Y$ such that for every $y \in \mathcal{Y}$, the fiber $W_y = p^{-1}(y)$ satisfies $f^k(W_y) = W_y$ for some $k = k(y) > 0$, and that f^k induces a quasi-polarized endomorphism of W_y (cf. [14], Theorem 5.1).

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Apply Lemma 4.1. Then Y has only canonical singularities and $K_Y \sim_{\mathbb{Q}} 0$ by Theorem 3.2. Moreover, by Proposition 3.6, there exist a finite morphism $A \times S \to Y$, which is étale in codimension one, from the direct product $A \times S$ for an abelian variety A and a weak Calabi–Yau variety S, and polarized endomorphisms f_A, f_S compatible with f_Y. Let Z be the normalization of the fiber product of $W \to Y$ and $A \times S \to Y$. Then the first projection $Z \to W$ is étale in codimension one, since π is equi-dimensional. Moreover, Z is irreducible since a general fiber of π is connected. Thus, the endomorphisms f_W and $f_A \times f_S$ induce a polarized endomorphism f_Z of Z. Let $Z \to V \to X$ be the Stein factorization of $Z \to W \to X$. Then $V \to X$ is étale in codimension one and V admits a polarized endomorphism f_V compatible with f and f_Z. Let $\pi\colon V \cdots\to A \times S$ be the induced rational map from the birational map $Z \to V$ and the second projection $Z \to A \times S$. Then Z is just the normalization of the graph of π. Thus, all the required things are proved. $\square$

Proposition 4.4. *Let X be an n-dimensional normal projective variety admitting a non-isomorphic polarized endomorphism. If* Conjecture 3.4 *is true, then $\pi_1(X)$ contains a finite index subgroup which is a finitely generated abelian group of rank at most $2q^\natural(X)$. In particular, this holds if $q^\natural(X) \geq n - 3$.*

Proof. In Lemma 4.1, Y is Q-abelian by Conjecture 3.4. Note that $q^\natural(X) = q^\circ(Y)$ by definition. Thus, if $q^\natural(X) \geq n - 3$, then Y is Q-abelian by Proposition 3.5. Therefore, there is a Galois covering $A \to Y$ from an abelian variety which is étale in codimension one over Y. Thus, $\pi_1(A) \to \pi_1(Y_{\mathrm{reg}})$ is an injection and its image is a finite index subgroup. In particular, $\pi_1(Y)$ has a finite index finitely generated abelian subgroup of rank at most $2 \dim A = 2 \dim Y = 2q^\natural(X)$.

Let $W \to Y$ be the morphism in Lemma 4.1. Let $\widetilde{W} \to W$ and $\widetilde{Y} \to Y$ be birational morphisms from smooth projective varieties such that the induced rational map $\widetilde{W} \to \widetilde{Y}$ is holomorphic and smooth over the complement of a normal crossing divisor on $\widetilde{Y}$. Then $\pi_1(\widetilde{W}) \simeq \pi_1(\widetilde{Y})$ since a general fiber is rationally connected, by [35], Theorem 5.2 (cf. [48], Lemma 5.3). Here, $\pi_1(\widetilde{Y}) \simeq \pi_1(Y)$ by [52] since Y has only quotient singularities. For the birational morphism $\widetilde{W} \to X$, we have a surjection $\pi_1(\widetilde{W}) \to \pi_1(X)$. Thus, there is a surjection $\pi_1(Y) \to \pi_1(X)$, and the assertion holds. $\square$

Lemma 4.5. *Let Z be the normalization of the graph of $V \cdots\to A \times S$ in* Theorem 1.1 *and let $\varpi\colon Z \to A \times S$ be the induced equi-dimensional morphism. Suppose that* $\dim S = 0$. *Then ϖ is flat, and any fiber of ϖ is irreducible and reduced. If* $\dim Z = \dim A + 1$, *then ϖ is a $\mathbb{P}^1$-bundle.*

Proof. Let $V \to X$, $Z \to V$, $Z \to A$, and $A \to Y$ be as in the proof of Theorem 1.1, where S is a point. Let Z_1 be the fiber product of $\varpi\colon Z \to A$ and $f_A\colon A \to A$. Then the other endomorphism f_Z induces a commutative diagram

$$(*)\qquad \begin{array}{ccccc} Z & \xrightarrow{\psi} & Z_1 & \xrightarrow{p_1} & Z \\ {\scriptstyle\varpi}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle\varpi}\downarrow \\ A & = & A & \xrightarrow{f_A} & A, \end{array}$$

where p_1 and p_2 denote the first and second projections, and $f_Z = p_1 \circ \psi$.

Step 1. We shall show that a non-empty Zariski closed subset $\Sigma \subset A$ satisfying $f_A^{-1}(\Sigma) \subset \Sigma$ is A:

We have a positive integer l such that $f_A^{-l}(\Sigma) = f_A^{-l-1}(\Sigma)$ by Noetherian condition. Hence, $f_A^{-1}(\Sigma) = \Sigma$. Replacing f with a power f^k, we may assume that f_A^{-1} preserves any irreducible component of Σ. Thus, we may assume that Σ is irreducible. Let f_Σ be the polarized endomorphism of Σ induced from f_A. Then $\deg f_A = q^{\dim A}$ and $\deg f_\Sigma = q^{\dim \Sigma}$ for some $q > 1$ by Lemma 2.1. On the other hand, $\deg f_\Sigma = \deg f_A$ since f_A is étale. Thus, $\dim \Sigma = \dim A$, i.e., $\Sigma = A$.

Step 2. We shall prove that any fiber of ϖ is irreducible:

Let Σ be the set of points $y \in A$ such that $\varpi^{-1}(y)$ is reducible. Then $\varpi^{-1}(y')$ is reducible for any $y' \in f_A^{-1}(y)$, since ψ in the diagram (*) is surjective. Thus, $f_A^{-1}(\Sigma) \subset \Sigma$, and hence $\Sigma = \emptyset$ by *Step* 1.

Step 3. We shall prove that ϖ is flat:

Let L be an ample divisor on Z such that $f_Z^* L \sim qL$. Since $\mathcal{O}_{Z_1}$ is a direct summand of $\psi_* \mathcal{O}_Z$, we infer that $\varpi_* \mathcal{O}_Z(f_Z^* L) = \varpi_* \mathcal{O}_Z(qL)$ contains

$$p_{2*}\mathcal{O}_{Z_1}(p_1^* L) \simeq f_A^* \left(\varpi_* \mathcal{O}_Z(L)\right)$$

as a direct summand. In particular, if $\varpi_* \mathcal{O}_Z(qL)$ is locally free at a point $y \in A$, then so is $\varpi_* \mathcal{O}_Z(L)$ at $f_A(y)$. Let U be the set of points $y \in A$ such that ϖ is flat along $\varpi^{-1}(y)$. Then U is a Zariski open dense subset. The argument above says that $f_A(U) \subset U$, since $y \in U$ if and only if $\varpi_* \mathcal{O}_Z(mL)$ is free at y for $m \gg 0$. Thus, for the complement Σ of U in A, we have $f_A^{-1}(\Sigma) \subset \Sigma$. Then $\Sigma = \emptyset$ by *Step* 1, and hence ϖ is flat.

Step 4. We shall prove that any fiber of ϖ is reduced:

Let Σ be the set of points $y \in A$ such that the fiber $F_y := \varpi^{-1}(y)$ is non-reduced. Then $L^d F_{y,\mathrm{red}} < L^d F_y$ for $d = \dim Z - \dim A$. For a point $y' \in f_A^{-1}(y)$, let δ be the degree of the finite morphism $f_Z\colon F_{y',\mathrm{red}} \to F_{y,\mathrm{red}}$. Then

$$q^d L^d F_{y',\mathrm{red}} = (f_Z^* L)^d F_{y',\mathrm{red}} = L^d (f_{Z*} F_{y',\mathrm{red}}) = \delta L^d F_{y,\mathrm{red}}$$

implies that $\delta = q^d$ and

$$L^d F_{y',\mathrm{red}} = L^d F_{y,\mathrm{red}} < L^d F_y = L^d F_{y'}.$$

Thus, $f_A^{-1}(\Sigma) \subset \Sigma$, and hence $\Sigma = \emptyset$ by *Step* 1.

Step 5. The remaining case: $\dim Z/A = 1$.

Since a general fiber of ϖ is rationally connected, we infer that any fiber of ϖ is $\mathbb{P}^1$ by *Step* 2–4. In particular, ϖ is smooth and is a holomorphic $\mathbb{P}^1$-bundle. $\square$

Next, we shall prove Theorem 1.2.

Proof of Theorem 1.2. Most things are derived from Theorems 3.2, 3.3, Lemma 4.1, and Proposition 4.4. Note that $q^\natural(X) = q^\circ(Y) \leq \dim Y$ for Y in Lemma 4.1. If $q^\circ(Y) \geq \dim Y - 3$, then Y is Q-abelian by Proposition 3.5.

(1): If $\dim X \leq 3$ and $q^\natural(X) = 0$, then Y is a point by Proposition 3.5. Thus, X is rationally connected, since $W \to Y$ is birational to a maximal rationally connected fibration of a nonsingular model of W.

(2): If $q^\natural(X) = q^\circ(Y) = n$, then $W \simeq Y$ and Y is Q-abelian. Thus, X is not uniruled and is Q-abelian by Theorem 3.2.

(3) and (4): Suppose that $q^\natural(X) = q^\circ(Y) \geq n - 3$. Then Y is Q-abelian. Let $V \to X$, $Z \to V$, $\varpi\colon Z \to A$, and $A \to Y$ be as in the proof of Theorem 1.1, where S is a point. Then $\varpi\colon Z \to A$ is a flat morphism whose fibers are all irreducible and reduced rationally connected varieties by Lemma 4.5. Then the assertion (3) follows.

Suppose that $q^\natural(X) = n - 1$. Then $Z \to A$ is a $\mathbb{P}^1$-bundle by Lemma 4.5. Hence, if $V \simeq Z$, then this is the case of (4a). Assume that $Z \to V$ is not isomorphic. Let $E \subset Z$ be the exceptional locus. Then $f_Z^{-1}(E) = E$ and $f_A^{-1}(\varpi(E)) = \varpi(E)$. Thus, $\varpi(E) = A$ by *Step* 1 in the proof of Lemma 4.5. Let $\Sigma \subset A$ be the set of points $y \in A$ such that $\varpi^{-1}(y) \subset E$. Then $f_A^{-1}(\Sigma) \subset \Sigma$. Hence, $\Sigma = \emptyset$ by *Step* 1 in the proof of Lemma 4.5. Therefore, $\varpi|_E\colon E \to A$ is a finite surjective morphism. It is enough to show that E is a section of ϖ. Let Γ be a fiber of $Z \to V$ and $M \to \varpi(\Gamma)$ a resolution of singularity. Then $Z_M := Z \times_A M \to M$ is a $\mathbb{P}^1$-bundle. In order to show E to be a section of ϖ, it is enough to show that $E \times_A M$ is a section of $Z_M \to M$. An irreducible component B of $E \times_A M$, which is a prime divisor of Z_M, is contracted to a point by $Z_M \to Z \to V$. Thus, $Z_M \simeq \mathbb{P}_M(\mathcal{E}_M)$ for a locally free sheaf $\mathcal{E}_M$ of rank two on M, and there is a surjection $\mathcal{E}_M \to \mathcal{L}_M$ to an invertible sheaf $\mathcal{L}_M$ such that the section corresponding to $\mathcal{E}_M \to \mathcal{L}_M$ is B. Since B is contracted to a point, the kernel $\mathcal{M}$ of $\mathcal{E}_M \to \mathcal{L}_M$ is the maximal destabilizing sheaf of $\mathcal{E}_M$ for any ample divisor on M. Therefore, $E \times_A M$ has no other irreducible component. Hence, $E \times_A M = B$ is a section of $Z_M \to M$. Therefore, E is a section of ϖ. Thus, (4) has been proved. $\square$

5. Endomorphisms of projective spaces

In this section, we shall prove Theorem 1.4. Let f be an endomorphism of $\mathbb{P}^n$ with $\deg f > 1$ for $n \geq 2$. Then $f^*H \sim qH$ for the hyperplane section H and $\deg f = q^n$ for a positive integer $q > 1$.

Lemma 5.1 (cf. [15], Proposition 4.2; [12], §3)**.** *Let $V \subset \mathbb{P}^n$ be a hypersurface such that $f^{-1}(V) = V$. Then $\deg V \leq n+1$ and every irreducible component of V is uniruled. If $\deg(V) = n+1$, then $f \colon \mathbb{P}^n \setminus V \to \mathbb{P}^n \setminus V$ is étale. If V is irreducible and $\deg(V) = n+1$, then $V \cap L$ has non-nodal singularities for a general plane $L \subset \mathbb{P}^n$.*

Proof. Replacing f with a positive power f^k, we may assume that $f^{-1}(V_i) = V_i$ for every irreducible component V_i of V. Thus $f^*V_i = qV_i$. Hence, the ramification formula for f is written as

$$K_{\mathbb{P}^n} = f^*(K_{\mathbb{P}^n}) + (q-1)V + \Delta$$

for an effective divisor Δ not containing any irreducible component of V. Comparing the degrees, we have

$$(q-1)\deg(V) \leq (1-q)\deg K_{\mathbb{P}^n} = (q-1)(n+1).$$

Thus, $\deg(V) \leq n+1$. If $\deg(V) = n+1$, then $\Delta = 0$; hence f is étale outside V.

Suppose that V is irreducible. Let $\nu \colon \widetilde{V} \to V$ be the normalization. Then $K_{\widetilde{V}} = \nu^*(K_V) - C$ for the conductor C. Thus,

$$K_{\widetilde{V}} \sim (\deg(V) - (n+1))\nu^*H - C.$$

If $\deg(V) < n+1$ or V is non-normal, then $K_{\widetilde{V}}$ is not pseudo-effective, and hence V is uniruled by [40]. Assume that $\deg(V) = n+1$. Then $V \cap L$ is neither smooth nor nodal for a general plane $L \subset \mathbb{P}^n$ by Theorem 5.2 below, since the degree $(\deg f)^k$ of the étale covering $f^k \colon \mathbb{P}^n \setminus V \to \mathbb{P}^n \setminus V$ is not bounded. □

In the proof of Lemma 5.1, we use the following result related to a conjecture of Zariski.

Theorem 5.2 ([21], [13], [27])**.** *Let $V \subset \mathbb{P}^n$ be a hypersurface of degree d for $n \geq 2$. If V has only normal crossing singularities in codimension one, then the fundamental group $\pi_1(\mathbb{P}^n \setminus V)$ is abelian. In particular, if V is irreducible, then $\pi_1(\mathbb{P}^n \setminus V) \simeq \mathbb{Z}/d\mathbb{Z}$.*

Proof. The assertion for $n = 2$ is just [13], Théorèm 1. In case $n \geq 3$, let $L \subset \mathbb{P}^n$ be a general plane. Then $V \cap L$ is a nodal curve. Here, $\pi_1(L \setminus V) \simeq \pi_1(\mathbb{P}^n \setminus V)$ by [27], Corollarie (0.1.2) (cf. [13], §1). Thus, we are done. □

The following result gives a property of ramification divisors of endomorphisms, which is originally proved in the case of curves in [47].

Lemma 5.3. *Let $f\colon X \to X$ be a finite surjective endomorphism of a normal algebraic variety X with $\deg f = d > 1$. Let D and Δ be effective divisors such that the ramification divisor R_f of f is expressed as $f^*D - D + \Delta$, i.e., $K_X + D = f^*(K_X + D) + \Delta$. Then D and Δ have no common irreducible components, and $f^{-1}(D) = D$. Moreover, there is a positive integer k such that $(f^k)^{-1}\Gamma = \Gamma$ for any irreducible component Γ of D. In particular,*

$$D = \sum\nolimits_{1 \le a \,|\, d^k} D_a$$

for effective divisors D_a such that

(1) *$(f^k)^*D_a = aD_a$ for any a,*
(2) *D_a is reduced or zero for $a > 1$,*
(3) *D_a and $D - D_a$ have no common irreducible components for any a.*

Proof. Let Γ be an irreducible component of D and Θ an irreducible component of $f^{-1}(\Gamma)$. We set $a := \mathrm{mult}_\Theta(f^*\Gamma)$. Since $\mathrm{mult}_\Theta(R_f) = a - 1$, we have

$$(**) \qquad \mathrm{mult}_\Theta(D) - 1 = a(\mathrm{mult}_\Gamma(D) - 1) + \mathrm{mult}_\Theta(\Delta) \ge a(\mathrm{mult}_\Gamma(D) - 1).$$

In particular, $\mathrm{mult}_\Theta(D) \ge 1$. Let $\mathcal{S}(D)$ be the set of irreducible components of D. Then we have the inclusion $\mathcal{S}(f^{-1}D) \subset \mathcal{S}(D)$ of finite sets. Since the map

$$\mathcal{S}(f^{-1}D) \ni \Theta \mapsto f(\Theta) \in \mathcal{S}(D)$$

is surjective, we have $\sharp\mathcal{S}(f^{-1}D) = \sharp\mathcal{S}(D)$. Thus, $\mathcal{S}(f^{-1}D) = \mathcal{S}(D)$, $f^{-1}(D) = D$, and $\Gamma \mapsto f^{-1}(\Gamma)$ gives a permutation of $\mathcal{S}(D)$. In particular, $f^*\Gamma = a\Theta$. Hence $a \,|\, d$. Let k be a positive integer such that f^{-k} induces the identity on $\mathcal{S}(D)$. Then $(f^k)^*\Gamma = a_k\Gamma$ for an integer a_k, where $a \,|\, a_k \,|\, d^k$. Since $K_X + D = (f^k)^*(K_X + D) + \Delta_k$ for

$$\Delta_k = (f^{k-1})^*\Delta + \cdots + f^*\Delta + \Delta,$$

one of the following two cases occurs by the same inequality for f^k as (**):

(i) $a_k = 1$ and $\mathrm{mult}_\Gamma(\Delta_k) = 0$.
(ii) $\mathrm{mult}_\Gamma(D) = 1$ and $\mathrm{mult}_\Gamma(\Delta_k) = 0$.

In both cases, we have $\Gamma \not\subset \mathrm{Supp}\,\Delta_k$. Hence, D and Δ have no common irreducible components. In Case (i), $f^k\colon X \to X$ is not branched along Γ, and $f^k_*\Gamma = d^k\Gamma$. In Case (ii), $f^k_*\Gamma = d^k/a_k\Gamma$. We set $D_a = \sum_{a_k = a}\Gamma$ for $a > 1$ and $D_1 = D - \sum_{a>1} D_a$. Then all the required conditions are satisfied. □

Proposition 5.4. *Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-isomorphic surjective endomorphism, and V a hypersurface. Assume that V is not normal and $f^{-1}(V_i) = V_i$ for any irreducible component V_i of V. Then V has only normal crossing singularities in codimension one,* *and $f^{-1}(\Sigma) = \Sigma$ for the non-normal locus Σ of V. In particular, $V \cap L$ is nodal for a general plane $L \subset \mathbb{P}^n$.*

Proof. The ramification divisor R_f is expressed as $(q-1)V + \Delta$ for $q = (\deg f)^{1/n}$ and an effective divisor Δ as before. For the normalization $\widetilde{V_i} \to V_i$, we have the normalization map $\nu\colon \widetilde{V} = \bigsqcup \widetilde{V_i} \to V$ of V and an effective divisor $C = \nu^* K_V - K_{\widetilde{V}}$ called the conductor of V. There is an endomorphism $h\colon \widetilde{V} \to \widetilde{V}$ such that $\nu \circ h = f \circ \nu$ and $h^{-1}(\widetilde{V_i}) = \widetilde{V_i}$ for any i. Moreover, we have the same formula

$$K_{\widetilde{V}} + C = h^*(K_{\widetilde{V}} + C) + \nu^*(\Delta|_V)$$

as in the proof of Lemma 5.1. Applying Lemma 5.3 to every connected component $\widetilde{V_i}$ of $\widetilde{V}$, we infer that $h^{-1}(C) = C$ and that C and $\nu^*(\Delta|_V)$ have no common irreducible components. In particular, $f^{-1}(\Sigma) = \Sigma$ for $\Sigma = \nu(C)$. Since h is also a polarized endomorphism with $\deg h = q^{n-1}$, we infer by Lemma 2.1 that if $(h^k)^{-1}(\Gamma) = \Gamma$ for a prime divisor Γ and a positive integer k, then $(h^k)^*\Gamma = q^k\Gamma$. Hence, C is reduced and $(h^k)^*C = q^kC$ for some $k > 0$ by Lemma 5.3. If a plane curve has a reduced conductor over a singular point, then the singularity is nodal. Hence, V has only normal crossing singularities in codimension one. $\square$

By Lemma 5.1 and Proposition 5.4, we have:

Corollary 5.5. *Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-isomorphic surjective endomorphism. If V is an irreducible hypersurface of degree $n+1$, then $f^{-1}(V) \neq V$.*

Proposition 5.6. *Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-isomorphic surjective endomorphism and let V be a union of hyperplanes such that $f^{-1}(V) = V$. Then V is normal crossing.*

Proof. If $n = 2$, then it follows from Proposition 5.4. Let $V = \sum V_i$ be the irreducible decomposition. We may assume that $f^{-1}(V_i) = V_i$ by replacing f with a power f^k. We set $V_i' = V_i \cap V_1$ for $i \geq 2$. Then V_i' is a hyperplane of $V_1 \simeq \mathbb{P}^{n-1}$, and $V_i' \neq V_j'$ for $i \neq j$ by Proposition 5.4. Now $f_1 = f|_{V_1}\colon V_1 \to V_1$ is a non-isomorphic surjective endomorphism with $f_1^{-1}(V_i') = V_i'$ for $2 \leq i \leq n$. By induction on n, we may assume that $\sum_{i\geq 2} V_i'$ is normal crossing. Thus, $V = V_1 + \sum_{i\geq 2} V_i$ is also normal crossing along V_1. Therefore, V is normal crossing. $\square$

Theorem 5.7. *Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-isomorphic surjective endomorphism for $n \geq 2$. If $f^{-1}(V) = V$ for a smooth hypersurface V, then V is a hyperplane.*

Proof. Assume that $d := \deg(V) > 1$. Then the following results are known:

- $d \leq 2$ by [4].
- If $n \geq 4$, then $d \neq 2$ by [49], Proposition 8.

- $(n, d) \in \{(2, 2)\}$ by [12].
- $n > 2$ by [23].

Thus, the assertion is proved by the results above. However, we shall give another proof in the case where $d = 2$ and $n \geq 2$, applying results in [38] and [49], Proposition 8.

Let $\tau\colon Y \to \mathbb{P}^n$ be the double cover totally branched over V, associated with the relation $V \sim 2H$ for a hyperplane H. Then $K_Y = \tau^*(K_{\mathbb{P}^n} + H) = -n\tau^*H$ and Y is a smooth quadric hypersurface in $\mathbb{P}^{n+1}$. Let X be the normalization of the fibre product $\mathbb{P}^n \times_{\mathbb{P}^n} Y$ of $f\colon \mathbb{P}^n \to \mathbb{P}^n$ and $\tau\colon Y \to \mathbb{P}^n$.

Suppose that $q = (\deg f)^{1/n}$ is odd. Then X is also the double covering totally branched along V. Hence, the second projection produces an endomorphism $f_Y\colon Y \to Y$ of degree q^n. Then $n = 2$ by [49], Proposition 8. Therefore, Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in which the inverse image $D = \tau^{-1}(V)$ corresponds to the diagonal locus. Here, $f_Y^{-1}(D) = D$. By replacing f with f^2, we may assume that f_Y preserves each projection $Y \to \mathbb{P}^1$. Then $f_Y = h \times h$ for an endomorphism h of $\mathbb{P}^1$. However, $f_Y^{-1}(D)$ contains D as a proper subset since $\deg h = q > 1$. This is a contradiction.

Suppose next that q is even. Then X is a disjoint union of two copies of $\mathbb{P}^n$. Thus, we have a factorization $\mathbb{P}^n \to Y \to \mathbb{P}^n$ of f. Then $Y \simeq \mathbb{P}^n$ by Lazarsfeld's theorem [38], absurd!

Therefore, the case $d = 2$ does not occur, and V is a hyperplane. $\square$

Lemma 5.8. *Let $L \subset \mathbb{P}^n$ a linear subspace of codimension $m + 1$ for $m \geq 1$ and $\pi\colon \mathbb{P}^n \cdots\to \mathbb{P}^m$ the projection from L. Then a surjective endomorphism $f\colon \mathbb{P}^n \to \mathbb{P}^n$ satisfying $f^{-1}(L) = L$ defines a surjective endomorphism $h\colon \mathbb{P}^m \to \mathbb{P}^m$ satisfying the following conditions*:

(1) *$\deg h = (\deg f)^{m/n}$.*
(2) *If $f^{-1}(\pi^{-1}(B_1)) = \pi^{-1}(B_2)$ for two subvarieties B_1, $B_2 \subset \mathbb{P}^m$, then $h^{-1}(B_1) = B_2$.*

Proof. Let $(\mathtt{X}_0, \ldots, \mathtt{X}_n)$ be a homogeneous coordinate of $\mathbb{P}^n$ such that $L = \{\mathtt{X}_0 = \cdots = \mathtt{X}_m = 0\}$. Then π is written as

$$(\mathtt{X}_0 : \mathtt{X}_1 : \cdots : \mathtt{X}_n) \mapsto (\mathtt{X}_0 : \mathtt{X}_1 : \cdots : \mathtt{X}_m).$$

The endomorphism f is also expressed as

$$f^*\mathtt{X}_i = F_i(\mathtt{X}_0, \mathtt{X}_1, \ldots, \mathtt{X}_n)$$

for a homogeneous polynomial F_i of degree q for any $0 \leq i \leq n$, where $\deg f = q^n$. By assumption, $\bigcap_{i=0}^{m}\{F_i = 0\} = L$. Thus, we can define an endomorphism $h\colon \mathbb{P}^m \to \mathbb{P}^m$ by

$$h^*\mathtt{X}_i = F_i(\mathtt{X}_0, \mathtt{X}_1, \ldots, \mathtt{X}_m, 0, \ldots, 0)$$

for $0 \le i \le m$. Here, $\deg h = q^m = (\deg f)^{m/n}$. For a point $P = (a_0 : a_1 : \cdots : a_m) \in \mathbb{P}^m$, let F_P be the linear subspace $\pi^{-1}(P)$ and set

$$P_0 := (a_0 : a_1 : \cdots : a_m : 0 : \cdots : 0) \in \mathbb{P}^n.$$

Then $F_P \supset L$, $F_P \ni P_0$, and $\pi \circ f(P_0) = h(P)$. If $P \in B_2$, then $P_0 \in F_P \subset \pi^{-1}(B_2) = f^{-1}(\pi^{-1}(B_1))$, $f(P_0) \in f(F_P) \subset \pi^{-1}(B_1)$, and hence $h(P) \in B_1$. Conversely, if $h(P) \in B_1$, then $f(P_0) \in F_{h(P)} \subset \pi^{-1}(B_1)$, $P_0 \in f^{-1}(\pi^{-1}(B_1)) = \pi^{-1}(B_2)$, and hence $P \in B_2$. Therefore, $h^{-1}(B_1) = B_2$. □

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4.

(1) and (2) are consequences of Lemma 5.1, Proposition 5.4, Corollary 5.5 and Theorem 5.7 above. The assertion (3) is proved as Proposition 5.6. The assertion (4) follows from (1) and (2). Thus, it remains to prove (5). Here, we may assume that V is irreducible by replacing f with a power f^k. Then $\deg(V) \le 3$ by Corollary 5.5. It remains to consider the cases of $\deg(V) = 2$ and $\deg(V) = 3$.

Case $\deg(V) = 3$: If V is not normal, then V is a rational surface by [50], Theorem 1.1, or [1], Theorem 1.5. Thus, we assume that V is normal. Then V is either a rational Gorenstein del Pezzo surface or a cubic cone over an elliptic curve by [25]. Therefore, we have to consider the latter case, where the cubic cone V is obtained from a relatively minimal elliptic ruled surface by contracting the unique negative section. Here, $f^{-1}(P) = P$ for the vertex P of the cone, since (V, P) is not a germ of quotient singularity. Let $\pi \colon \mathbb{P}^3 \cdots\to \mathbb{P}^2$ be the projection from P. Then $C = \pi(V)$ is a smooth cubic curve. By Lemma 5.8, there is a non-isomorphic surjective endomorphism $h \colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $h^{-1}(C) = C$. This contradicts Theorem 5.7.

Case $\deg(V) = 2$: Then V is a singular quadric cone by Theorem 5.7. Thus, $V \simeq \{\mathtt{Z}_0^2 + \mathtt{Z}_1^2 + \mathtt{Z}_2^2 = 0\} \subset \mathbb{P}^3$ for a homogeneous coordinate $(\mathtt{Z}_0, \mathtt{Z}_1, \mathtt{Z}_2, \mathtt{Z}_3)$ of $\mathbb{P}^3$. Let $\tau \colon Y \to \mathbb{P}^3$ be the double cover branched along V. Then Y is isomorphic to a singular quadric cone $\{\mathtt{X}_0^2 + \mathtt{X}_1^2 + \mathtt{X}_2^2 + \mathtt{X}_3^2 = 0\} \subset \mathbb{P}^4$ for a homogeneous coordinate $(\mathtt{X}_0, \mathtt{X}_1, \mathtt{X}_2, \mathtt{X}_3, \mathtt{X}_4)$ of $\mathbb{P}^4$, where τ is given by

$$(\mathtt{X}_0 : \mathtt{X}_1 : \mathtt{X}_2 : \mathtt{X}_3 : \mathtt{X}_4) \mapsto (\mathtt{X}_0 : \mathtt{X}_1 : \mathtt{X}_2 : \mathtt{X}_4).$$

For the vertices $P_V = (0:0:0:1) \in \mathbb{P}^3$ and $P_Y = (0:0:0:0:1) \in \mathbb{P}^4$ of the cones V and Y, respectively, we have $\tau^{-1}(P_V) = \{P_Y\}$. Now, P_Y is a unique singular point of Y, and (Y, P_Y) is not a $\mathbb{Q}$-factorial singularity, since it has a small resolution, whose exceptional locus is $\mathbb{P}^1$ with the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We shall construct a surjective endomorphism $f_Y \colon Y \to Y$ such that $\tau \circ f_Y = f \circ \tau$. Let Z be the normalization of the fiber product $\mathbb{P}^3 \times_{\mathbb{P}^3} Y$ of $f \colon \mathbb{P}^3 \to \mathbb{P}^3$ and $\tau \colon Y \to \mathbb{P}^3$. If

$\deg(f)$ is odd, then Z is irreducible and is isomorphic to Y; thus the natural composition $f_Y : Y \xrightarrow{\simeq} Z \to Y$ is an expected endomorphism. If $\deg(f)$ is even, then Z is a disjoint union of two copies of $\mathbb{P}^3$ and $f = \tau \circ \theta$ for a surjective morphism $\theta : \mathbb{P}^3 \to Y$; thus the composition $f_Y = \theta \circ \tau$ is an expected endomorphism.

For the endomorphism f_Y, we have $f_Y^{-1}(P_Y) = \{P_Y\}$; otherwise P_Y is dominated by a non-singular point of Y, which implies that (Y, P_Y) is $\mathbb{Q}$-factorial. Hence, $f^{-1}(P_V) = P_V$ for $\tau^{-1}(P_V) = P_Y$. Now applying Lemma 5.8, we have an non-isomorphic surjective endomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h^{-1}(C) = C$ for the smooth conic $C = \{\mathsf{Z}_0^2 + \mathsf{Z}_1^2 + \mathsf{Z}_2^2 = 0\} \subset \mathbb{P}^2$. This contradicts Theorem 5.7.

The proof of Theorem 1.4 is completed. $\square$

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Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502, Japan
E-mail address: `nakayama@kurims.kyoto-u.ac.jp`

Department of Mathematics
National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore
E-mail address: `matzdq@nus.edu.sg`