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Pole mass of the heavy quark: Perturbation theory and beyond

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The key quantity of the heavy quark theory is the quark mass m_Q. Since quarks are unobservable one can suggest different definitions of m_Q. One of the most popular choices is the pole quark mass routinely used in perturbative calculations and in some analyses based on heavy quark expansions. We show that no precise definition of the pole mass can be given in the full theory once nonperturbative effects are included. Any definition of this quantity suffers from an intrinsic uncertainty of order $\Lambda_{\rm QCD}/m_Q$. This fact is succinctly described by the existence of an infrared renormalon generating a factorial divergence in the high-order coefficients of the α_s series; the corresponding singularity in the Borel plane is situated at $2\pi/b$. A peculiar feature is that this renormalon is not associated with the matrix element of a local operator. The difference $\overline{\Lambda} \equiv M_{H_Q} - m_Q^{\rm pole}$ can still be defined by heavy quark effective theory, but only at the price of introducing an explicit dependence on a normalization point μ: $\overline{\Lambda}(\mu)$. Fortunately the pole mass $m_Q(0)$ *per se* does not appear in calculable observable quantities.

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I. INTRODUCTION

Significant progress has been achieved recently in the quantitative treatment of the decays of heavy flavor hadrons by employing expansions in powers of $1/m_Q$, where m_Q denotes the heavy quark mass. Exclusive transitions between two heavy flavor hadrons are conveniently dealt with by using the spin-flavor symmetry of heavy quarks [1] (see also Ref. [2]); the formalism of heavy quark effective theory (HQET) [3, 4] incorporates this symmetry concisely at the Lagrangian level. Inclusive decays, on the other hand, nonleptonic, semileptonic, or radiative transitions to any type of the final state, can be treated directly in QCD [5–10] via Wilson's operator product expansion [11].

The key parameter in most aspects of heavy quark physics is obviously "the heavy quark mass." There are no problems in defining this quantity within purely perturbative calculations. At this level one choice turns out to be particularly convenient: it is the so-called pole mass of the heavy quark, defined as the position of the pole in the quark propagator in perturbation theory. This quantity, introduced in QCD in the 1970s (see, e.g., Ref. [12]), is well defined in each *finite* order of perturbation theory; unlike many other definitions, it is introduced in a gauge invariant way. This convenient feature has made it very popular. Important results of perturbative calculations such as the total semileptonic widths (including radiative corrections) are routinely expressed in terms of the pole mass (see, e.g., [13]).

In this paper we exhibit an important drawback in the concept of a pole mass that becomes apparent as soon as one addresses leading nonperturbative corrections to order $1/m_Q$. The problem arises, of course, because the pole mass is sensitive to large distance dynamics, although this fact is not obviously seen in the standard perturbative calculations, and the corresponding treatment requires special care. It had actually been noted before that marrying full QCD with the notion of the pole mass faces subtle difficulties [14].

Our main assertions will be threefold.

(A) Infrared contributions lead to an intrinsic uncertainty in the pole mass of order $\Lambda_{\rm QCD}$, i.e., an effect of relative weight $1/m_Q$. Perturbation theory itself produces clear evidence for this nonperturbative correction to $m_Q^{\rm pole}$. The signal is the peculiar factorial growth of the high order terms in the α_s expansion corresponding to a renormalon [15, 16] residing at $2\pi/b$ in the Borel plane, where b is the first coefficient in the Gell-Mann–Low function, $b = (11N_c/3) - (2N_f/3)$. The physical reason lying behind this linear effect is just the "Coulomb" energy of the heavy quark.

A subtle point is that infrared-renormalon effects can usually be associated with the expectation values of some local operators [16, 17] of the corresponding dimension. This general pattern is not realized in the case of the pole mass for the reasons which will be clarified below.

(B) The heavy quark expansion (HQE) yields for directly observable quantities such as the total semileptonic

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width of the heavy flavor hadron decay:

$$\Gamma(B \to X_u + l + \nu_l) = \frac{G_F^2 m_b^5}{192\pi^3}|V_{ub}|^2 \left[c\frac{\langle B|\bar{b}b|B\rangle}{2M_B} + O(1/m_b^2)\right], \quad (1)$$

where for definiteness we consider B meson decay and the coefficient c is $c = 1 + O(\alpha_s)$. In the total semileptonic width nonperturbative effects of order $1/m_Q$ are absent (see Refs. [7, 9]), and the corrections start from $1/m_Q^2$ (the factor $(2M_B)^{-1}$ reflects the relativistic normalization of the state $|B\rangle$). While the pole mass is useful within purely perturbative calculations, it makes no sense to employ it in such an analysis that aims for an accuracy $O(\Lambda_{\rm QCD}/m_Q)$ or even $O(\Lambda_{\rm QCD}^2/m_Q^2)$, simply because it *cannot* be unambiguously defined at order $\Lambda_{\rm QCD}/m_Q$.

(C) There is a profound theoretical difference between the pole mass and the total semileptonic width: calculation of the latter can be formulated as an operator-product-expansion- (OPE-) based procedure, ensuring factorization of the large and small distance contributions; the pole mass, however, cannot be treated in such a way.

On the other hand, and fortunately, there is no need to use the pole mass in these inclusive transition rates. Careful analysis shows that contrary to popular opinion it is the running mass $m(\mu)$ with $\mu \gg \Lambda_{\rm QCD}$ that naturally enters. The properly defined running mass includes only the effects of momenta higher than μ; therefore if μ is chosen sufficiently high there is no infrared uncertainty in $m(\mu)$. In particular, the parameter most relevant to the inclusive heavy quark decays is $m(m_Q)$. In other problems it may turn out that the running mass normalized at a different point enters; each particular case requires its own careful analysis. It is important, however, that this normalization point never goes down to a typical hadronic scale if one wishes to avoid the corresponding infrared uncertainty in the Wilson coefficients. We stress here that the normalization point μ does *not* necessarily coincide with the off shellness of the heavy quark inside the heavy hadron.

Certainly, the definition of the running mass $m(\mu)$ depends both on the scheme and the gauge used – an obvious inconvenience. In this respect, however, the running mass does not differ, as a matter of principle, from the running gauge coupling constant $\alpha_s(\mu)$ where an explicit scheme and gauge dependence first emerges on the third-loop level. The essential difference is that for $m(\mu)$ this dependence manifests itself already at the one-loop level; the observables, however, are independent.

Correspondingly, any consistent definition of the parameter $\overline{\Lambda}$ must explicitly introduce the normalization point μ, so that $\overline{\Lambda}$ is actually running, $\overline{\Lambda}(\mu)$, although the running law is somewhat unconventional, see Eqs. (17) and (53).

The remainder of this paper will be organized as follows. In Sec. II we restate the general procedure of separating off the infrared effects within Wilson's OPE and the relation between the infrared renormalons and nonperturbative condensates; in Sec. III the perturbative series for the heavy quark mass is discussed; Sec. IV is devoted to the infrared renormalon contribution to the mass; Sec. V demonstrates that the pole mass does not appear in directly observable quantities, such as, say, the total widths; finally in Sec. VI we discuss the running of $\overline{\Lambda}$ and summarize our results.

II. SEPARATION OF THE INFRARED EFFECTS WITHIN WILSON'S OPE

In this section we remind the reader of a crucial property of the operator product expansion [17–19]: it provides us with a systematic separation of infrared and ultraviolet contributions. This brief excursion into standard OPE applications will allow us to reveal a basic difference in how infrared effects enter in the pole mass and, say, into correlation functions at large momentum transfer.

As an example let us consider the correlation function of vector currents:

$$\Pi_{\mu\nu} \equiv (q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi = i\int d^4x\, e^{iqx}\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle, \quad (2)$$

where

$$j_\mu = \bar\psi\gamma_\mu\psi$$

with ψ being a massless quark field. To avoid "external" logarithms which are irrelevant for the problem under discussion one usually deals with a modified quantity defined as

$$\tilde\Pi = -4\pi^2 Q^2 (d/dQ^2)\Pi\,, \quad Q^2 = -q^2. \quad (3)$$

The one- and two-loop graphs determining $\tilde\Pi$ in perturbation theory are depicted in Fig. 1. The well-known calculation of these diagrams yields

$$\tilde\Pi = 1 + \frac{\alpha_s(Q^2)}{\pi}. \quad (4)$$

The virtual momenta saturating the corresponding loop integrals are of order Q, and if Q^2 is chosen to be large the characteristic virtual momenta are also large. The result (4) then represents the short-distance contribution.

The fact that Eq. (4) is correct in perturbative QCD does not mean, however, that it is correct in the full theory. Indeed, in deriving Eq. (4) one integrates over all gluon momenta k, including the domain of small k where k is the momentum flowing through the gluon line in Fig. 1(b). Of course, this domain gives a relatively small contribution to the integral; nevertheless this contribution is definitely wrong since for small k the Green

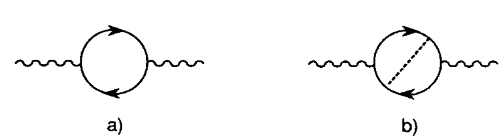

FIG. 1. Feynman diagrams for $\Pi_{\mu\nu}$ in Eq. (2). Solid lines denote the quark q, dashed lines gluons.

functions are strongly modified by nonperturbative effects and have nothing to do with the perturbative propagators one uses in obtaining Eq. (4).

The emergence of nonperturbative corrections can actually be inferred from perturbation theory *per se* if one recalls the presence of the Landau singularity in the running coupling constant in the infrared domain (for QCD). This is the essence of the concept of infrared renormalons [15, 16]. The phenomenon is quite simple and manifests itself in the behavior of the high order terms in the α_s expansion. The relevant diagrams are those where a chain of loops has been inserted into the gluon Green function (Fig. 2). This chain is equivalent to replacing the fixed coupling by the running coupling constant $\alpha_s(k^2)$ in the integrand and integrating it over with some weight function; the latter is given by the remaining propagators in the diagram. This weight function is such that the integral over d^4k converges in the ultraviolet domain. To illustrate how this works let us use, following Ref. [20], a simplified expression for $\tilde{\Pi}$:

$$\tilde{\Pi}_{\rm renorm} \sim Q^2 \int dk^2 \frac{k^2 \alpha_s(k^2)}{(k^2+Q^2)^3}. \tag{5}$$

This expression coincides with the original one in the limits $k^2 \ll Q^2$ and $k^2 \gg Q^2$ relevant to the infrared and ultraviolet renormalons, respectively. At $k^2 \gg Q^2$ the original integrand contains an extral $\ln k^2/Q^2$ which is omitted in Eq. (5).

The integral in Eq. (5) is saturated at $k^2 \sim Q^2$; to get the main contribution one substitutes $\alpha_s(Q^2)$ for $\alpha_s(k^2)$ and finds the standard two-loop result for $\tilde{\Pi}$. If, however, we are interested in high orders in $\alpha_s(Q^2)$ the tails at $k^2 \ll Q^2$ and $k^2 \gg Q^2$ become important. The contribution of these tails have the form

$$\tilde{\Pi}_{\rm IR} = Q^{-4} \int_0^{Q^2} k^2 dk^2 \alpha_s(k^2) \tag{6}$$

and

$$\tilde{\Pi}_{\rm UV} = Q^2 \int_{Q^2}^{\infty} \frac{dk^2}{k^4} \alpha_s(k^2), \tag{7}$$

where the subscripts are self-evident. Substituting the running gauge coupling

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 - [b\alpha_s(Q^2)/4\pi] \ln(Q^2/k^2)},$$

where b is the first coefficient in the Gell-Mann–Low function, and expanding in $\alpha_s(Q^2)$ we get the whole series in $\alpha_s(Q^2)$. Namely,

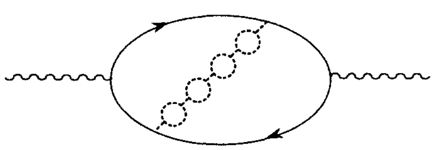

FIG. 2. "Bubbles" in the gluon Green function giving rise to the infrared renormalon in $\Pi_{\mu\nu}$.

$$\tilde{\Pi}_{\rm IR} = \alpha_s(Q^2) \sum_n \left(\frac{b\alpha_s(Q^2)}{4\pi}\right)^n \int_0^{Q^2} \frac{dk^2 k^2}{Q^4} \left(\ln \frac{Q^2}{k^2}\right)^n$$
$$= \alpha_s(Q^2) \sum_n \left(\frac{b\alpha_s(Q^2)}{4\pi}\right)^n \frac{n!}{2^{n+1}} \tag{8}$$

and

$$\tilde{\Pi}_{\rm UV} = \alpha_s(Q^2) \sum_n \left(-\frac{b\alpha_s(Q^2)}{4\pi}\right)^n \int_{Q^2}^{\infty} \frac{dk^2 Q^2}{k^4} \left(\ln \frac{k^2}{Q^2}\right)^n$$
$$= \alpha_s(Q^2) \sum_n \left(-\frac{b\alpha_s(Q^2)}{4\pi}\right)^n n!. \tag{9}$$

The first expression corresponds to the infrared renormalon while the second one refers to the ultraviolet renormalon. The factorial growth of the coefficients is explicit in both cases. There are two differences, however. The ultraviolet renormalon is represented by the sign-alternating series, in distinction to $\tilde{\Pi}_{\rm IR}$. The second difference is the n dependence of the coefficients in front of $n!(b\alpha_s/4\pi)^{n+1}$. In the infrared renormalon this coefficient is $2^{-(n+1)}$, compared to 1 in the ultraviolet renormalon. These differences result in the fact that the positions of the singularities in the Borel plane (to be denoted $\tilde{b}$) are at

$$\tilde{b}_{\rm IR} = \frac{8\pi}{b}, \quad \tilde{b}_{\rm UV} = -\frac{4\pi}{b}. \tag{10}$$

In what follows we will not consider the ultraviolet renormalon, as well as other sources of the singularities in the Borel plane, for instance, instanton–anti-instanton pairs (see, e.g., [21]) producing a singularity at $\tilde{b} = 4\pi$.

The occurrence of the factorial in $\tilde{\Pi}_{\rm IR}$ is correlated with the fact that at large n the integral is saturated not at the large scale $k^2 \sim Q^2$ but, instead, at parametrically low scale $k^2 \sim Q^2 \exp(-n)$. Therefore, when n becomes large one encounters the Landau singularity, and, in an indirect way, comes to the realization that in the infrared domain the gluon Green function cannot coincide with the perturbative expression.

The factorial growth of the coefficients points to the fact that one cannot, as a matter of principle, infinitely increase the accuracy of the perturbative approximation by including higher and higher terms in the series: starting from some order the absolute size of the corrections increases again. The best one can achieve is to truncate the series at an optimal value of n ensuring the best possible approximation. It is not difficult to check that the error introduced by this truncation is of the order of

$$\exp\left(-\frac{8\pi}{b\alpha_s(Q^2)}\right) \sim \frac{\Lambda^4_{\rm QCD}}{Q^4}. \tag{11}$$

Since the series is not Borel summable, it is in principle impossible to achieve a better accuracy within the framework of perturbation theory. All one can conclude is that there must be a nonperturbative effect of the form (11).

It is well known that this problem does not mean that one has to abandon hope for establishing theoretical control over the terms of this order of magnitude. The way out is provided by a consistent use of Wilson's OPE procedure. First one introduces a momentum scale μ such that momenta above μ can be treated perturbatively. All loop momenta are classified according to whether they exceed μ or lie below it. The integration over the infrared domain below μ is then explicitly *excluded* from the perturbative calculation and one finds

$$\tilde{\Pi}_{\rm pert} \equiv \tilde{\Pi}|_{k>\mu} = 1 + \frac{\alpha_s(Q^2)}{\pi} - \text{const} \times \frac{\alpha_s(\mu^2)\mu^4}{Q^4}. \quad (12)$$

The fact that the subtracted term in this particular case is proportional to μ^4 is not accidental, of course: it can be anticipated from Eq. (11) and it will be clarified shortly. This result automatically follows from the explicit calculation, provided that the scale μ is introduced without breaking the gauge invariance of the theory (in practice however this may turn out to be a technically highly nontrivial exercise). Excluding the domain $k < \mu$ from the perturbative calculation does not mean that we just lose this contribution. Within the Wilson OPE the contribution of this infrared domain reenters through the vacuum expectation value

$$\Delta\tilde{\Pi} = -\frac{\pi}{3Q^4}\langle\alpha_s G^2\rangle, \quad (13)$$

where $G = G^a_{\mu\nu}$ is the gluon field strength tensor, and the operator G^2 in the right-hand side is normalized at μ (i.e., by definition this vacuum expectation value includes all virtual momenta below μ). The vacuum expectation value $\langle\alpha_s G^2\rangle$ has the form

$$\langle\alpha_s G^2\rangle = \text{const} \times \Lambda^4_{\rm QCD} + \text{const} \times \alpha_s(\mu^2)\mu^4 \quad (14)$$

and in the full expression $\tilde{\Pi}_{\rm pert} + \Delta\tilde{\Pi}$ the dependence on the auxiliary parameter μ cancels. The requirement that the dependence on μ cancels in the end dictates that μ must appear in $\tilde{\Pi}_{\rm pert}$ as μ^4, since no gauge invariant operator of lower dimension exists.

Let us emphasize that the normalization point μ should be high enough to ensure the applicability of perturbative calculations above μ, i.e.,

$$\alpha_s(\mu) \ll 1.$$

On the other hand, it is desirable to have this parameter as small as possible, so that the corresponding terms will represent insignificant corrections in Eqs. (12) and (14). If this wish can be satisfied there is no need to carefully work out details of how to introduce μ explicitly. In particular, in Eq. (14) the term with $\Lambda^4_{\rm QCD}$ should be much larger than that with μ^4. In other words, the numerical values of the condensates are assumed to be much larger than their perturbative parts for some μ belonging to the window discussed above. This is what is called the practical version of OPE – powers of μ do not show up explicitly then, although, of course, the conceptual necessity of having μ should be always kept in mind (μ is certainly kept in logarithms in the practical version of OPE). It is fortunate for applications of QCD that the practical version of OPE works well in the vast majority of instances; this could not have been anticipated *a priori*. Otherwise, all calculations based on OPE would be much less useful since it would be mandatory to explicitly construct the procedure of introducing μ.

What then happens with the infrared renormalon within Wilson's procedure? If one defines the perturbative part with the infrared cutoff at the point μ, the factorial growth of the coefficients in the perturbative series in α_s stops starting from some value of n, $n \sim \ln(Q^2/\mu^2)$, since the integral "wants" to be saturated in the domain $k^2 < \mu^2$ and this domain is now simply eliminated from the perturbative sector. The price one has to pay is the introduction of a new, nonperturbative parameter, the gluon condensate. Once it is introduced, however, the perturbation theory is amended, the effects $\propto Q^{-4}$ become tractable and this accuracy is legitimate. If necessary, one goes a step further. Of course, corrections of higher order in Q^{-2} require the introduction of new higher-dimensional condensates.

Let us emphasize that the actual *uncertainty* due to the infrared renormalon, Eq. (11), should by no means be equated with the contribution from the gluon condensate. It is true that they are of the same order in the parameter $\Lambda_{\rm QCD}/Q$ and the infrared renormalon anticipates the appearance of the gluon condensate; the gluon condensate contribution, however, is much larger *numerically*, and this is the reason why the practical version of OPE is so successful. One can interpret this fact in terms of extremely strong distortions of the Green functions in the infrared domain [17]. In other words the modifications (compared to smoothly extrapolated perturbative Green functions) are not just of the order of unity but are much larger *numerically.*

The discussion above makes it clear that in principle perturbation theory already signals the emergence of a nonperturbative correction in $\tilde{\Pi}$ of order $\Lambda^4_{\rm QCD}/Q^4$. A hint is provided by the simplest diagram of Fig. 1(b). If we introduce μ and calculate the graph of Fig. 1(b) according to the Wilson procedure we discover a correction $\alpha_s\mu^4$. In combination with the general fact of strong distortion of the Green functions in the infrared domain this $\alpha_s\mu^4$ correction triggers the rest of the machinery which eventually leads to the nonperturbative gluon condensate contribution in $\tilde{\Pi}$.

The lessons one can draw from this rather standard procedure [17–19] are quite evident: although it is impossible to actually calculate nonperturbative contributions by analyzing the behavior of perturbation theory for a particular quantity, the fact that nonperturbative contributions exist and their particular form can be inferred. Moreover, one can find out what kind of nonperturbative terms are to be expected in a twofold way: by studying low order perturbative graphs within the Wilson procedure and by inspecting high orders of perturbation theory as they are generated by the infrared renormalon.

Below we will use both lines of reasoning to show that the "pole mass" of heavy quarks, if treated in the context of problems where we intend to include nonperturbative effects, contains a piece of the order of $\Lambda_{\rm QCD}$ which must be considered as an intrinsic uncertainty. Unlike the standard case, however, this piece is *not* related to any matrix element of a local gauge invariant operator. Therefore, one cannot amend the perturbation theory based on $m_Q^{\rm pole}$ in the manner it is usually done. This happens due to the fact that the notion of the "pole mass" by itself is ill defined.

III. PERTURBATIVE CORRECTIONS TO m_Q

Following the general strategy outlined above we start our analysis of the pole mass with the simplest perturbative graph of Fig. 3. Instead of a straightforward calculation of this graph[1] we follow the Wilson procedure and introduce, first of all, the normalization point μ such that the domain of virtual momenta $k < \mu$ is discarded. Thus, we are going to calculate an analogue of $\tilde{\Pi}_{\rm pert}$ in Eq. (12).

As already mentioned, separating the infrared and ultraviolet domains would in general require a rather sophisticated machinery.[2] Fortunately, the situation simplifies in this particular case because the diagram of Fig. 3 is the same as in QED. In QED we can just introduce a "photon mass" λ, which, on the one hand, preserves Ward identities associated with current conservation, and on the other, suppresses the contribution of all virtual momenta below λ. In this way, the infrared domain is automatically discarded. The photon mass λ is to be identified with the normalization point μ, cf. Eq. (12). The procedure of introducing μ suggested here cannot be extended to higher loops. It is quite satisfactory, however, for our more limited purpose, namely, to establish the presence of a correction of order $1/m_Q$ in the pole mass.

Accordingly we use the following expression for the gluon propagator:

$$D^{ab}_{\mu\nu}(k) = -\delta^{ab}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2-\lambda^2}\frac{M_0^2}{M_0^2-k^2}. \tag{15}$$

Here ξ is the gauge fixing parameter and the term $M_0^2/(M_0^2-k^2)$ ensures ultraviolet regularization (M_0 is the ultraviolet cutoff). We have introduced it to be safe in the ultraviolet regime, although it will play no role in what follows. Since we are interested in the infrared domain we further assume that

FIG. 3. One-loop diagram for the mass renormalization. Thick line denotes the heavy quark.

[1] The corresponding computation can, of course, be found in any textbook on quantum electrodynamics (QED); to this order there is no difference between QED and QCD, up to a trivial overall factor.

[2] As will be seen below, naive dimensional regularization cannot serve this purpose.

$$\frac{\lambda}{m_Q} \ll 1.$$

Explicit evaluation of the diagram of Fig. 3 yields

$$m_Q(\lambda) = m_Q^{(0)}\left[1 + \frac{\alpha_s}{\pi}\left(\ln\frac{M_0^2}{m_Q^{(0)2}} + \text{const}\right)\right] - \frac{2\pi}{3}\frac{\alpha_s}{\pi}\lambda. \tag{16}$$

Here $m_Q^{(0)}$ is the bare mass (the mass at M_0), while the quantity on the left-hand side is the *pole* mass which depends on λ. We remind the reader that the domain of virtual momenta $k < \lambda$ is absent in $m_Q(\lambda)$, and this leads to the dependence on λ.

Next, we identify λ and μ, as explained above, and express the mass at one normalization point (we will refer to this mass as the running mass) in terms of that corresponding to some "starting" normalization point μ_0:

$$m_Q(\mu) = m_Q(\mu_0) + \frac{2\pi}{3}\frac{\alpha_s}{\pi}(\mu_0 - \mu). \tag{17}$$

What is important in Eq. (17) is the occurrence of a correction of order $1/m_Q$ relative to the leading term.

The procedure outlined above is not unambiguous in both its elements – the definition of the running mass and the specific manner in which the normalization point has been introduced. In principle, one can use other prescriptions. Let us mention, for instance, the suggestion of Ref. [22] where the running mass $m_Q(\mu)$ was introduced through a certain integral over the cross section for the process $\gamma + Q \to g + Q$. The normalization point μ then enters as an upper limit in this integral (see [22] for more details; we plan to discuss this approach in a forthcoming publication). Within this procedure one gets an analogue of Eq. (17) which still contains a linear correction, albeit with a different numerical coefficient. While the numerical value of the $1/m_Q$ correction is thus scheme dependent, its presence is not.

The physical meaning of the $1/m_Q$ contribution is quite transparent. It is nothing else than the classical Coulomb self-energy of a static color source, see, e.g., Ref. [14]. The energy of the Coulomb field at a distance r_0 is given by

$$E_{\rm Coul} = \frac{2\alpha_s}{3}\frac{1}{r_0}. \tag{18}$$

Identifying r_0 with $1/\mu$ we recover Eq. (17).

The Coulomb contribution to the mass can readily be derived directly from the graph in Fig. 3. In the limit of large m_Q (compared to the gluon virtual momentum k), i.e., in the static limit, the expression for the mass shift takes the form

$$\begin{aligned}\delta m_Q &\equiv m_Q^{\rm pole} - m_Q(\mu)\\ &= -i\frac{4}{3}g_s^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k_0+i\epsilon)(k^2+i\epsilon)}\end{aligned} \tag{19}$$

with an ultraviolet cutoff in the k integration implied at μ. (The integral for δm_Q becomes linearly divergent in the static limit.) It is easily seen that the only surviving contribution in Eq. (19) comes from $1/(k_0 + i\epsilon)$ in the form of a term $-i\pi\delta(k_0)$. The fact that the static limit implies the vanishing of k_0 is quite evident by itself; the occurrence of the linear infrared effect under consideration can be traced back to this feature of the static limit.

Performing first the integration over k_0 (which reduces merely to putting $k_0 = 0$) we arrive at

$$\delta m_Q = \frac{8\pi}{3}\alpha_s \int \frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2}. \tag{20}$$

If an ultraviolet cutoff is introduced through a factor $\mu_0^2/(\mu_0^2 + \mathbf{k}^2)$ in the integrand we get

$$\delta m_Q = m_Q^{\rm pole} - m_Q(\mu_0) = \tfrac{2}{3}\alpha_s\mu_0, \tag{21}$$

cf. Eq. (17) where, to get the pole mass on the left-hand side, one should set $\mu = 0$. Were the cutoff introduced in a "hard" way we would get a different coefficient in front of the linear term, of course, but the very fact of its presence would remain intact.

The appearance of the term linear in μ in Eq. (17) tells one, according to the discussion of Sec. II, that there is a renormalon singularity in the perturbative series giving rise to a relative uncertainty in the pole mass of order $1/m_Q$. Below we will demonstrate it explicitly.

To conclude this section an important remark of a conceptual nature is in order. The calculation carried out above clearly reveals an important fact: while the OPE-like procedure routinely used in HQET resembles closely Wilson's OPE procedure [11], it has one very distinct feature. In the static limit all *energy* transfers to the heavy quark line vanish, implying that the time separations are always large. Physically that is quite transparent: the heavy quark after being placed at the origin as a static color source at $t = -\infty$ remains there at rest until $t = +\infty$. The OPE procedure in the effective low-energy theory is then based on a separation of small *spatial* from large *spatial* momenta. Therefore, below m_Q we actually deal with a three-dimensional version of OPE, which results in peculiarities that might seem strange, at first sight, to those who got used to the standard features of the four-dimensional OPE.

IV. INFRARED RENORMALON

Next one examines the impact of the high order corrections in α_s to the pole mass. We again study the chain of loops inserted into the gluon propagator, this time in the graph of Fig. 3, see Fig. 4. Summing all these "bubbles" amounts to replacing α_s in Eq. (20) by the running coupling $\alpha_s(\mathbf{k}^2)$ in the integrand:

$$m_Q^{\rm pole} - m_Q(\mu_0) = \frac{8\pi}{3}\int_{|\mathbf{k}|<\mu_0} \frac{d^3k}{(2\pi)^3}\frac{\alpha_s(\mathbf{k}^2)}{\mathbf{k}^2}. \tag{22}$$

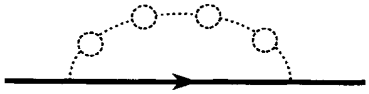

FIG. 4. Infrared renormalon in the pole mass.

The running gauge coupling is given by

$$\alpha_s(\mathbf{k}^2) = \frac{\alpha_s(\mu_0^2)}{1 - (b\alpha_s(\mu_0^2)/4\pi)\ln(\mu_0^2/\mathbf{k}^2)} = \alpha_s(\mu_0^2)\sum_{n=0}^{n=\infty}\left(\frac{b\alpha_s(\mu_0^2)}{4\pi}\ln\frac{\mu_0^2}{\mathbf{k}^2}\right)^n. \tag{23}$$

Substituting the expansion of Eq. (23) into Eq. (22) we immediately obtain the series

$$m_Q^{\rm pole} - m_Q(\mu_0) = \frac{4\alpha_s(\mu_0)}{3\pi}\mu_0\sum C_n\left(\frac{b\alpha_s(\mu_0)}{4\pi}\right)^n, \tag{24}$$

where the coefficients C_n are given by

$$C_n = \int_0^1 dx\left(\ln\frac{1}{x^2}\right)^n. \tag{25}$$

At large n these coefficients grow factorially:

$$C_n = 2^n n!\ . \tag{26}$$

In other words one can say that the position of the nearest singularity in the Borel plane [15,16] is at $\tilde{b} = 2\pi/b$. This series is not Borel-summable due to the presence of an infrared renormalon; truncating it at the optimal value of n_0,

$$n_0 \sim \frac{2\pi}{b\alpha_s(\mu_0^2)}, \tag{27}$$

one arrives at an estimate of the irreducible uncertainty in $m_Q^{\rm pole} - m_Q(\mu_0)$:

$$\Delta(m_Q^{\rm pole} - m_Q(\mu_0)) \sim \frac{8}{3b}\mu_0\exp\left(-\frac{2\pi}{b\alpha_s(\mu_0)}\right) \sim \frac{8}{3b}\Lambda_{\rm QCD}. \tag{28}$$

Thus, we see that perturbative QCD indeed does not allow one to define $m_Q^{\rm pole}$ with an accuracy better than $\Lambda_{\rm QCD}$, i.e., an infrared effect linear in $\Lambda_{\rm QCD}/m_Q$.

A simple way to explain the estimate (28) is to turn to the original integral (22):

$$m_Q^{\rm pole} - m_Q(\mu_0) = \frac{4}{3\pi}\alpha_s(\mu_0)\int_0^{\mu_0}\frac{dk}{1 - \frac{b\alpha_s(\mu_0)}{4\pi}\ln\frac{\mu_0^2}{k^2}}. \tag{29}$$

Previously we have just expanded the denominator in the powers of $\alpha_s(\mu_0)$, obtaining in this way the factorial behavior of the expansion coefficients related to the existence of the pole singularity in the integrand. Now, instead, we regularize the singularity, say, by adding $i\epsilon$ in the denominator. Then Eq. (29) acquires the imaginary part

$$\mathrm{Im}\,[m_Q^{\rm pole} - m_Q(\mu_0)] = \frac{8\pi}{3b}\Lambda_{\rm QCD}, \tag{30}$$

where $\Lambda_{\rm QCD}$ parametrizes the position of the infrared pole in the running gauge coupling. The estimate (28) above coincides with Eq. (30), up to a factor $1/\pi$ reflecting the difference between the real and imaginary parts.

[Let us parenthetically note that in Eq. (22) we substituted the soft cutoff $\mu_0^2/(\mu_0^2+\mathbf{k}^2)$ by a step function at $|\mathbf{k}|=\mu_0$. This is unimportant for the infrared renormalon where the integral is saturated at $\mathbf{k}^2 \sim \mu_0^2 e^{-2n}$. However, with the soft cutoff restored the very same integral (22) produces the ultraviolet renormalon due to the domain $\mathbf{k}^2 \sim \mu_0^2 e^{2n}$. It is not difficult to check that the corresponding factorial behavior is the same, up to a sign, $(C_n)_{\rm UV} = (-2)^n n!$. In contrast with the situation with the polarization operator $\tilde{\Pi}$ considered in Sec. II, these two singularities in the Borel plane, infrared, and ultraviolet, are symmetric with respect to the origin.]

The statement above, the impossibility of defining $m_Q^{\rm pole}$ to the accuracy better than $\Lambda_{\rm QCD}$, implies some tacit assumptions. In particular, we assumed that one should use the value of the running coupling $\alpha_s(\mathbf{k}^2)$ in the integrand in Eq. (22). This natural prescription is easily justified in QED where the Ward identity reduces the renormalization of the coupling constant to the corrections to the photon propagator. In non-Abelian theories this is not the case in covariant gauges. A general argument below illustrates the fact that one cannot get anything else.

Being interested in effects occurring at the scale μ, much below the mass of the heavy quark, one integrates out all momenta above $\sim \mu$ and arrives at an effective field theory of a nonrelativistic heavy quark with an ultraviolet cutoff μ. The parameters of QCD are then $\alpha_s(\mu)$ and $m_Q(\mu)$ (and, in principle, masses for the "light" quarks); including external interactions adds also the corresponding couplings that must be renormalized at the same scale μ as well. The dependence of these parameters defining the effective theory on the renormalization point μ follows from the requirement that physical observables do *not* depend on μ. If the renormalization procedure is such that lowering μ from a value μ_1 down to μ_2 incorporates radiative corrections due to virtual momenta $|\mathbf{k}|$ between μ_1 and μ_2, then there must be a linear dependence of $m_Q(\mu)$ on μ, necessarily of the form

$$\frac{dm_Q(\mu)}{d\mu} = \beta_m[\alpha_s(\mu)] = -\beta_m^{(1)}\alpha_s(\mu) + \cdots . \qquad (31)$$

The fact that it originates from radiative corrections is indicated by the explicit factor α_s on the right-hand side. Of course, the exact form of the function $\beta_m(\alpha_s)$ depends on the particular renormalization scheme. If one uses a scheme that coincides to one loop with the prescription of introducing a "gluon mass" μ, then, according to Eq. (16),

$$\beta_m^{(1)} = \tfrac{2}{3}. \qquad (32)$$

The renormalization-group equation (31) combined with Eq. (32) is equivalent to Eq. (22): $O(\alpha_s^2)$ terms in the function β_m (31) are neglected as subleading.

Recalling the discussion in Sec. II one is tempted to relate the infrared part in $m_Q^{\rm pole}$ in Eq. (28) to the matrix element of some local gauge invariant operator. Alas, such an attempt is doomed to fail. This is most easily seen by inspecting the local operators of the relevant dimension. The only potential candidate is

$$\bar{Q} i D_0 Q,$$

where $iD_0 = i\partial_0 + g_s A_0$ (we imply gauge invariance plus a static description of the field Q similar to that used in HQET). However, the equations of motion reduce this operator to those of dimension 5 which can generate corrections of the relative weight $\Lambda_{\rm QCD}^2/m_Q^2$ only, rather than $\Lambda_{\rm QCD}/m_Q$, provided that the definition of the quark mass is properly adjusted, so that there is no so-called residual mass [23], see below.

From the derivation given above (see Sec. III and, especially, the fact that $k_0 = 0$, as emphasized there) it is clear why the standard OPE program is inapplicable to $m_Q^{\rm pole}$. By analyzing Fig. 3 we have realized that only very small frequencies (of the order of μ_0^2/m_Q) contribute to the pole mass. In other words, even though the characteristic *spatial* distances are small in the problem at hand, the *time* separation is parametrically large. To state it in more physical terms, measuring the pole mass of the heavy quark requires a very long time, inversely proportional to the allowed uncertainty in the absolute value of the mass.

This means that the nonperturbative infrared contribution in $m_Q^{\rm pole}$ cannot be expressed in terms of a *local* condensate, but, rather, through a nonlocal expectation value. We have encountered a similar situation previously [24] in connection with the so-called temporal distribution function defined through the hadronic matrix element of the operator:

$$[D_i Q(t,\mathbf{x}=0)] e^{-i\int_0^t A_0(\tau)d\tau} [D_i Q(t=0,\mathbf{x}=0)], \qquad (33)$$

where Q is the heavy quark field and D_i is the spatial component of the covariant derivative. The pole mass formally appears in consideration of the operator (33) if one considers the matrix element of this operator over the heavy quark state in the limit $t \to \infty$. Thus, we deal here with a generalization of the standard Wilson path operator for an open path along the t direction.

V. IRRELEVANCE OF THE POLE MASS

The only systematic approach presently known that allows one to treat nonperturbative effects in QCD analytically is based on Wilson's OPE. This procedure requires, as discussed in detail in Sec. II, a careful separation of contributions from large- and short-distance dynamics. Our findings from the previous section suggest that the pole mass cannot be handled within such an approach. In this section we will demonstrate that this is indeed the case.

To be specific let us discuss the problem of charmless semileptonic decays of B mesons which has been already mentioned in Sec. I. In the parton model the total width is given by the probability of the $b \to u l \bar{\nu}_l$ transition; neglecting all corrections we have

$$\Gamma_0 = \frac{G_F^2 m_b^5}{192\pi^3}|V_{ub}|^2. \qquad (34)$$

When α_s corrections are included the result of the explicit perturbative computation is most naturally expressed in terms of the pole mass (and this is what is usually done, see, e.g., Ref. [13]):

$$\Gamma_{\rm pert} = \frac{G_F^2 (m_b^{\rm pole})^5}{192\pi^3} |V_{ub}|^2 \times \left[1 - \frac{2\alpha_s(m_b^2)}{3\pi}\left(\pi^2 - \frac{25}{4}\right) + \cdots\right]. \quad (35)$$

This formula is perfectly legitimate as long as we stay in perturbation theory. If, however, we would like to take into account nonperturbative (infrared) effects the use of Eq. (35) can be seriously misleading. Let us clarify this point.

As explained above, the perturbative series for $m_b^{\rm pole}$ diverges factorially – a fact signaling the presence of a nonperturbative contribution. A similar factorial divergence takes place in the α_s expansion in the large parentheses of Eq. (35). This perturbative series does have the renormalon singularity at $\tilde{b} = 2\pi/b$ giving rise to (uncontrollable) corrections of order $\Lambda_{\rm QCD}/m_Q$. Both effects combine, however, to cancel each other, if the running mass $m_Q(m_Q)$ is used.

To illustrate how this cancellation works let us consider one-gluon exchange graphs, Fig. 5. One must keep in mind that the gluon Green function is assumed to be dressed as in Fig. 4, so that $\alpha_s(k^2)/k^2$ must be used for the gluon propagator. This gives rise to the usual infrared renormalon.

We remind the reader that we are interested in the imaginary parts of the diagrams in Fig. 5 corresponding to the appropriate cuts (i.e., the total semileptonic width). The graphs 5(a) and 5(b) contain the effect of the radiative shift of the mass discussed in Sec. III, and, in particular, the factorial divergence, see Sec. IV. Our point is that the integrands in the Feynman integrals for diagrams 5(a) and 5(d) will completely compensate each other in the domain of virtual momenta $|k| \ll m_b$; likewise with the graphs 5(b) and 5(e). [The above assertion requires an explanation. We consistently work in the *Coulomb gauge*. In this gauge diagram 5(c) has no imaginary part and gives no contribution provided that the b quark is at rest. As for diagram 5(f) the soft gluon contribution in this graph is strongly suppressed and is irrelevant in our approximation. Here the compensation occurs between different cuts of one and the same graph.]

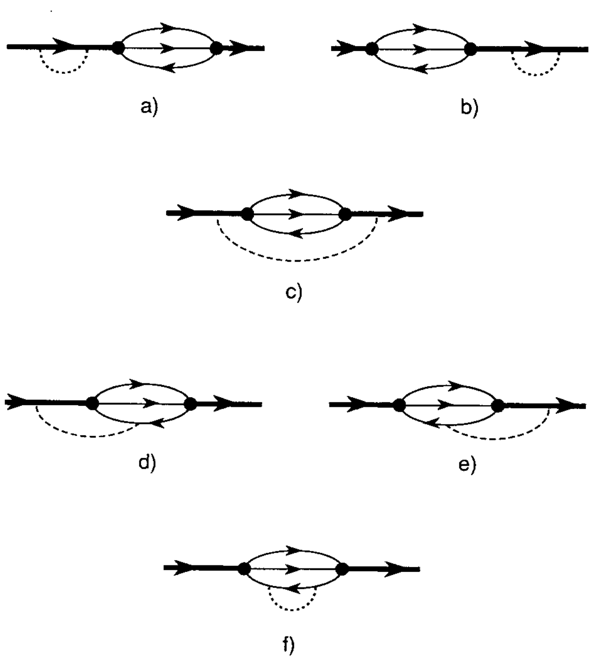

FIG. 5. Diagrams determining $O(\alpha_s)$ corrections to the total semileptonic width (taking the imaginary part is implied).

Now let us prove that the integrands in the Feynman integrals for diagrams 5(a) and 5(d) completely compensate each other in the domain of virtual momenta $|k| \ll m_b$. To this end let us freeze the virtual momentum of the gluon k and first integrate over the light fermions in the diagrams depicted in Figs. 5(a) and 5(d). In the case of Fig. 5(a) we get in this way the $b - b$ transition amplitude $U(p)$ presented by the right-hand side of the diagram of Fig. 5(a) (p stands for the b quark momentum). Pictorially this transition amplitude is denoted by a black box in Fig. 6(a). Likewise, in the case of Fig. 5(d) we obtain a two-loop induced effective vertex $b - b -$ *gluon*, to be denoted below by $V_\mu(p,k)$. This vertex is also depicted as a black box in Fig. 6(b).

The imaginary part of $U(p)$ is readily calculable and is well known; if the fermions propagating in the loops are taken to be massless,

$$\operatorname{Im} U = C \not{p}^5, \quad (36)$$

where C is a constant related to Γ_0,

$$C = \Gamma_0/(2m_b^5).$$

What is crucial for what follows is the fact that this result is infrared stable: the corresponding loop integrals are saturated by loop momenta of order of p. This implies that the limit $k \to 0$ in V_μ is smooth.

Now, it is not difficult to derive the Ward identity relating V_μ to U:

$$k_\mu V_\mu = U(p+k) - U(p). \quad (37)$$

(Here we ignore the color structure of the vertices which is not important in the approximation considered; the calculation would proceed in the very same way in QED.) The Ward identity (37) is sufficient to fix V_μ in the limit of small k:

$$V_\mu(p,0) = \frac{\partial}{\partial p_\mu} U(p). \quad (38)$$

As a matter of fact, we need only the imaginary parts of

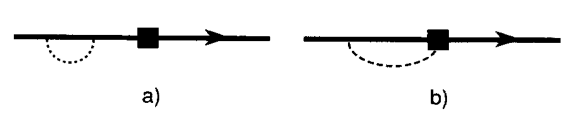

FIG. 6. Graphs 5(a) and 5(d) in the language of the effective local vertices (denoted by the black box).

both, V_μ and U; then, Eqs. (36) and (38) yield

$$\mathrm{Im}\, V_0(p,0) = 5C\gamma_0 m_b^4, \quad (39)$$

where we have accounted for the fact that one can substitute $\not{p}$ by m_b in the approximation considered.

Note that the solution (38) of the Ward identity (37) implies the absence of the infrared singularities at small k, the fact that has been already mentioned.

It is not difficult to understand that the consideration above, based on the solution of the Ward identity at small k, can be reformulated in slightly different terms as follows. The $b-b$ transition amplitude $\mathrm{Im}\, U = C\not{p}^5$ can be represented as a local effective vertex:

$$\bar{b}(i\not{\partial})^5 b.$$

The gauge invariance of the theory implies that the vertex $\mathrm{Im}\, V_\mu(p,0)$ is generated by the substitution

$$\bar{b}(i\not{\partial})^5 b \to \bar{b}(i\not{D})^5 b,$$

where D_μ is the covariant derivative. Diagrammatically these effective vertices are presented in Fig. 6. The gluon emission vertex in Fig. 6(b) is

$$5\,\not{p}^4 \not{A} \to 5(m_b^{(0)})^4 \not{A} \quad (40)$$

[see Eq. (39)].

The net effect of diagrams 5(a) and 5(b) is quite obvious: they convert the bare b quark mass into $(m_b^{\rm pole})^5$ in the total width where $m_b^{\rm pole}$ in this approximation is[3]

$$m_b^{\rm pole} = m_b^{(0)} - \Sigma \quad (41)$$

and $-\Sigma$ is given by the graph of Fig. 3. Explicitly these two graphs produce

$$\Delta\Gamma_{a+b} = \frac{G_F^2|V_{ub}|^2}{192\pi^3}\left(-5(m_b^{(0)})^4\Sigma\right). \quad (42)$$

The result for Σ is implicitly given in Eq. (16); its explicit expression is not needed for our purposes here, where we have to consider only the integrand of the integral determining Σ.

Let us now turn to diagrams 5(d) and 5(e). With the effective vertex of Eq. (40) it is straightforward to check that diagrams 5(d) and 5(e) yield

$$\Delta\Gamma_{d+e} = \frac{G_F^2|V_{ub}|^2}{192\pi^3} 5(m_b^{(0)})^4\Sigma. \quad (43)$$

We conclude that the uncertain contribution from the domain $|k| \ll m_b$ present in individual graphs and responsible for the factorial growth of the coefficients (see Sec. IV) is absent in the sum, Eqs. (42) and (43). Of course, the cancellation described above takes place only to leading order in k/m_b. The residual difference between the graphs 5(a) and 5(d) shows up at the level of integrals of the type

$$\int d^3k/|\mathbf{k}|$$

which are harmless from the point of view of the linear in $1/m_b$ effect we focus on here. Let us also note in passing that the exercise above is nothing else than a check that equations of motion at a high normalization point could have been used in OPE from the very beginning.

Thus, in applying heavy quark theory one has to avoid the pole mass altogether and to use, instead, the running mass which naturally appears in OPE; for only in this case one may hope to get consistent and well-defined expansions in powers of $1/m_Q$. The leading operator in the expansion in the problem at hand is $\bar{b}(i\not{D})^5 b$. From the consideration above it follows that the normalization point $\mu = m_b$ is the most natural choice: by adopting this normalization point we avoid any large logarithms as well as the problem of a factorial divergence of the type discussed in Sec. IV. Nonperturbative effects enter through the matrix element of this operator; they are also represented by matrix elements of other (subleading) operators, for instance, $\bar{b}(i\not{D})^3 i\sigma G b$.

Using the equations of motion one reduces the leading operator to $m_b^5\bar{b}b$, where both m_b and $\bar{b}b$ are taken at $\mu = m_b$. We then evolve $\bar{b}b$ down to a low normalization point, $\mu \ll m_b$; the net effect of this evolution is reflected in a factor of the type $c(\mu, m_b) = 1 + a_1(m_b/\mu)\alpha_s(m_b) + a_2(m_b/\mu)\alpha_s^2(m_b) + \cdots$ which is, anyway, included in the perturbative calculation. This factor contains also terms of order $(\mu/m_b)^n$ due to the exclusion of the domain below μ from the perturbative calculation. It is important that the power n starts from $n = 2$. These small terms, $(\mu/m_b)^n$, will contain the series in $\alpha_s(\mu)$, not $\alpha_s(m_b)$. The additional μ dependence occurring in this way will be canceled by that coming from appropriate local operators, e.g., $\bar{b}(i\not{D})^3 i\sigma G b$.

Once the operator $\bar{b}b$ is evolved down to a low normalization point we can use the relation

$$\bar{b}b = v_\mu \bar{b}\gamma_\mu b + \frac{1}{4m_b^2}\bar{b}i\sigma G b + \frac{1}{2m_b^2}\bar{b}[(iD)^2 - (ivD)^2]b + O(1/m_b^4) \quad (44)$$

explicitly demonstrating the absence of $1/m_Q$ corrections. Equation (44) is valid up to terms representing total derivatives.

In other words, one integrates out the momenta above the scale μ to get a generic operator product expansion for the width:

$$\hat{\Gamma} = c(\mu, m_b) m_b^5 \bar{b}b|_\mu + c_2(\mu, m_b) m_Q^3 \bar{b}i\sigma G b|_\mu + \cdots; \quad (45)$$

the matrix element is taken then by using Eq. (44).

A question which immediately comes to one's mind is as follows. What happens if we first evolve the operator $\bar{b}(i\not{D})^5 b$ down to a low normalization point and only *then* use the equations of motion. At first sight we would

[3]Strictly speaking, these graphs also give rise to the b field renormalization. In the Coulomb gauge the soft-gluon contribution to the corresponding Z factor vanishes in the linear in $1/m_b$ approximation.

get $m_b^5(\mu)$, not $m_b^5(m_b)$ in this case, and in the limit $\mu \to 0$ we would recover $(m_b^{\rm pole})^5$. The loophole in this argument is rather obvious: the operator $\bar{b}(i\not{D})^5 b$ mixes under renormalization with the operators $\bar{b}(i\not{D})^4 b$, etc., and accounting for this mixing returns us to the mass normalized at m_b.

It is instructive to dwell on this issue of mixing in more detail, the more so since it is intimately related to the notion of the residual mass, to be discussed below. Consider the operator $\bar{b} i \not{D} b|_{\mu_0}$ and evolve it down to μ. The effect of the evolution is described by the diagrams of Fig. 6(b). If only terms linear in μ are kept, the only relevant graph is the vertex renormalization; this graph is the same as that for the mass renormalization (Fig. 3), up to a sign. As a result we get

$$\bar{b} i \not{D} b|_{\mu_0} = \bar{b} i \not{D} b|_{\mu} - \frac{2\pi}{3}\frac{\alpha_s}{\pi}(\mu_0 - \mu)\bar{b}b. \tag{46}$$

Using now the equations of motion in combination with Eq. (17) we see that both the left- and the right-hand sides contain $m_b(\mu_0)$, Q.E.D. A similar relation holds, of course, for any power of $i\not{D}$.

Above we have demonstrated our assertions considering explicitly a certain class of diagrams. The result is more general, of course. It can be viewed as a statement that the inclusive widths of heavy flavors do belong to the class of observables which are given by operator product expansions in the standard understanding: physics of short and large distances [4] can be separated into operators and their coefficients, and the infrared behavior of the latter is governed, in turn, by the corresponding local operators. The existence of the $1/m_Q$ renormalon in the heavy quark mass shows that this statement is not as trivial as it might seem at first sight. However, as soon as the validity of the OPE is accepted, one necessarily arrives at the irrelevance of the pole mass.

Within the OPE-based approach one obtains the corresponding inclusive width as a series of operators of the form $\bar{b}\cdots b$ initially normalized at the scale m_b; the infrared stability of the inclusive widths ensures that the coefficient functions are finite in any perturbative order. Some of these operators contain covariant derivatives acting on the b fields; due to the equations of motions these derivatives reduce to powers of the *high* scale mass $m_b(m_b)$. Subleading operators can produce relative effects not larger than $1/m_b^2$.

Perturbative corrections actually drop out altogether from certain observable quantities. Differences in the lifetimes of different species of hadrons in the same heavy flavor family provide a prominent example: the widths of pseudoscalar mesons P_Q and baryons Λ_Q agree through order $1/m_Q$,

$$\frac{\Gamma_{\Lambda_Q} - \Gamma_{P_Q}}{\Gamma_{\Lambda_Q} + \Gamma_{P_Q}} \sim 1/m_Q^2,$$

in clear contrast to their masses,

$$\frac{M_{\Lambda_Q} - M_{P_Q}}{M_{\Lambda_Q} + M_{P_Q}} \sim 1/m_Q.$$

We plan to present a more detailed discussion in a forthcoming paper [25].

Since our conclusions obviously do not depend on the particular decay process, they apply directly and equally to radiative and nonleptonic decays.

VI. THE RUNNING OF $\bar{\Lambda}$ AND OTHER CONCLUSIONS

We have discussed in some detail the problem of the pole mass in the heavy quark theory. Because of the peculiarities of the static limit an infrared term linear in $\Lambda_{\rm QCD}/m_Q$ is generated in the pole mass, as signaled by an infrared renormalon. The presence of this nonperturbative term makes the notion of the pole mass, beyond perturbation theory, not only useless but, rather, detrimental. What is even worse, this nonperturbative term cannot be absorbed into any local condensate, unlike the usual OPE-based prescription where the infrared effects are naturally incorporated through the condensates. Thus, the pole mass should be avoided altogether in analyzing calculable observable quantities. The problem disappears provided one uses the running mass $m_Q(\mu)$ normalized at a sufficiently high point. The same situation in a different context has been noted recently in Ref. [26].

The occurrence of a new, linear infrared effect in the static limit reminds us of high-temperature QCD. In this theory an external parameter, temperature, sets the scale of the energy transfers, and if $T \gg \Lambda_{\rm QCD}$ this scale is fixed to be of order T. The integrals over the four-dimensional momenta degenerate into integrals over three-dimensional momenta. This produces new linear infrared divergences in high-temperature QCD [27]. Whether this parallel leads to nontrivial insights into our problem remains to be seen.

The fact that corrections linear in $1/m_Q$ are present in some quantities is not surprising by itself. Perhaps, the best-known example is the axial vector constant f_Q (f_D or f_B for charm and beauty, respectively). In the limit $m_Q \to \infty$ they scale with m_Q as follows [28–30]:

$$\frac{f_B}{f_D} = \left(\frac{m_c}{m_b}\right)^{1/2}\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{2/b}. \tag{47}$$

This scaling is known to be significantly modified by preasymptotic $1/m_Q$ terms if one defines the axial vector constant in the standard way via matrix elements of the axial vector current:

$$\langle 0|\bar{Q}\gamma_\mu\gamma_5 q|P_Q\rangle = i f_Q p_\mu. \tag{48}$$

However the correction is much smaller if one would de-

[4] It must be noted that it is indeed a statement of separation of large and short distances in the process, not of perturbative versus nonperturbative effects; for the Wilson coefficients in general include nonperturbative contributions generated, for example, by instantons of small size $\sim 1/\mu$ or $1/m_Q$.

fine it via the pseudoscalar current [31, 32]:

$$\langle 0|\bar{Q}i\gamma_5 q|P_Q\rangle = \tilde{f}_Q M_{P_Q}. \tag{49}$$

In the heavy quark limit the two definitions obviously coincide,

$$\frac{\tilde{f}_Q}{f_Q} \simeq \frac{M_{P_Q}}{m_Q + m_q} \tag{50}$$

and this difference is therefore indeed contained in $1/m_Q$ terms.

Calculation of f_Q cannot be directly formulated as an OPE-based procedure; therefore, the emergence of the $1/m_Q$ correction is natural. The pole mass belongs to the same class. Yet the OPE prescription is fully applicable to the calculation of the inclusive widths of heavy flavor hadrons; the infrared behavior of the corresponding Wilson coefficients is given by matrix elements of the appropriate operators. This difference between the pole mass and the inclusive widths is not accidental, of course, and could be anticipated.

A remark is in order here concerning the place the pole mass occupies in the heavy quark effective theory [3, 4]. HQET is formulated in such a way as to get rid of the heavy quark mass at all; the effective theory is left without a large parameter which might set an appropriate scale for distances. It is then only the renormalization point μ that fixes the scale of momenta for static quantities. However, the pole mass assumes that one takes the limit $\mu \to 0$ and, thus, no parameter is left at all. It is clear that in any consistent formulation of HQET it is impossible to set $\mu = 0$. One should keep μ explicitly and operate only with $m_Q(\mu)$.

It is well known that in the framework of the HQET a key role is played by the difference between the hadron and the quark mass for an asymptotically heavy quark:

$$\overline{\Lambda} = \lim_{m_Q \to \infty} (M_{H_Q} - m_Q) \ . \tag{51}$$

It is always stated that the mass m_Q entering this definition is the pole mass. We have shown, however, that this quantity is ill defined. There have been a few attempts to define it consistently in HQET. One of those has been made in Ref. [23]; namely, for pseudoscalar mesons it was suggested that

$$\overline{\Lambda}_P = \frac{\langle 0|i(v\partial)(\bar{q}i\gamma_5 h_v)|P(v)\rangle}{\langle 0|\bar{q}i\gamma_5 h_v|P(v)\rangle}. \tag{52}$$

The fields of the HQET are assumed here and standard notations are used. This definition *per se* is not better and is plagued by the same problems – the impossibility of disentangling nonperturbative effects from the perturbative contributions. Any consistent formulation has to discriminate large and small momenta rather than perturbative versus nonperturbative effects. A consistent formulation must then include the normalization point μ to ensure that momenta higher than μ do not appear in the matrix elements. If μ is introduced one must then define a "running" value of $\overline{\Lambda}(\mu)$, depending on the renormalization point μ, as

$$\overline{\Lambda}(\mu) = \lim_{m_Q \to \infty} [M_{H_Q} - m_Q(\mu)] \tag{53}$$

with μ having to exceed sufficiently typical hadronic scales. Its renormalization point dependence is given by Eq. (31); attempts to put μ to zero to arrive at the "old" $\overline{\Lambda}$, would bring about uncontrollable uncertainties of order $\Lambda_{\rm QCD}$ in $\overline{\Lambda}(0)$.

It has been suggested [33] to use the requirement that the so-called residual mass term vanishes to rigorously define the heavy quark *pole* mass $m_Q^{\rm pole}$. In Sec. V it was shown that mixing between the operators $\bar{Q}i\not{D}Q$ and $\bar{Q}Q$ arises already at the one-loop level to order μ. This means that even if the effective Lagrangian of HQET is chosen in such a way that at a certain μ the residual mass is zero, it necessarily reappears at a different value of μ. If the process is characterized by a single scale (such as m_Q in the total inclusive widths) one can certainly adjust the effective Lagrangian so that there is no residual mass. If the scale varies, however, it is mandatory to "readjust" the notion of the mechanical mass of the heavy quark. In other words by requiring the vanishing of the residual mass one defines the running quark mass. (It has been traced [34] how various observables calculated in HQET via matching with full QCD turn out to be independent of this term although it is present in intermediate stages.[5])

The fact that perturbative corrections to m_Q vanish in dimensional regularization, which is often referred to as a remedy, is an accident due to the fact that HQET does not contain a dimensional parameter at the perturbative level. References to dimensional regularization *per se* without an actual subtraction scheme are irrelevant for the effective theory which requires a clear separation of high and low momenta. For example, the modified minimal subtraction ($\overline{\rm MS}$) scheme giving a logarithmically dependent one-loop value for m_Q in the form $m_Q(\mu) = m_Q(m_Q)[1 + (2\alpha_s/\pi)\log(m_Q/\mu)]$ for an infrared cutoff $\mu \ll m_Q$ cannot achieve the momentum separation necessary for OPE.

Note that the problem of the total inclusive widths is, strictly speaking, outside pure HQET: for the large mass scale parameter, the energy release, is intrinsically involved from the very beginning, determining the characteristic space-time separations here.

Some relations are known for exclusive form factors in $b \to c$ transitions, that, in the limit $m_b, m_c \to \infty$ depend on $\overline{\Lambda}$ outside the zero recoil kinematics (see, e.g., [35]). However, they contain purely perturbative $O(\alpha_s^n)$ corrections. Their main piece reflects the hybrid renormalization [30, 36] that must be properly accounted for in constructing the effective low energy description; they are governed by the scale momenta between m_b and m_c, or, in general, above μ. Some corrections, appearing already in the effective theory itself, reflect, however, much lower scales; obviously in the heavy quark limit at $v \neq v'$ there always exists a large domain of gluon momenta $0 \lesssim |\mathbf{k}| \lesssim |\mathbf{v}|m_Q$ whose effect is not governed by $\alpha_s(m_Q)$, but rather by the running coupling at the lower scale. These corrections therefore can lead again

[5]N.U. is grateful to V.M. Braun for pointing out this constructive interpretation.

to infrared renormalon contributions. There is no reason for them not to be able to convert the "pole" $\overline{\Lambda}$ into a "running" one. It is clear that this problem deserves further theoretical studies, especially if one wants to understand the real meaning of the relations mentioned above in the presence of an infrared renormalon in the heavy quark mass. In principle a more complicated situation is conceivable – *exclusive* modes may not be described completely by the standard OPE and intrinsic uncertainties similar to that in the pole mass could have emerged.

We do not address in this paper the numerical aspects of the infrared renormalons; they are left for the future studies [25] (see also [37]). Still for orientation it is instructive to consider the estimate (28). According to this expression the uncertainty in the pole mass is

$$\delta m_Q^{\rm pole} \sim \frac{8}{3b}\Lambda_{\rm QCD}. \tag{54}$$

To determine what particular Λ (i.e., Λ corresponding to what particular subtraction scheme) enters on the right-hand side one needs to perform a two-loop calculation. In the absence of such calculations it is consistent to use the one-loop Λ which is one and the same in all schemes, $\Lambda_{\rm one\ loop} \sim 100$ to 150 MeV. Then one gets

$$\delta m_Q^{\rm pole} \sim 50\,{\rm MeV}; \tag{55}$$

certainly this estimate needs improvement via real two loop calculations.

The presence of the $1/m_Q$ renormalon in the pole mass of the heavy quark makes the attempts to extract the value of the pole mass from the absolute widths of heavy flavors [38–40] not very meaningful from a theoretical perspective; it is the masses $m_c(m_c^2)$ and $m_b(m_b^2)$ (see Ref. [38]) that can be extracted from the widths. Indeed, one can reach the required $1/m_Q$ accuracy through calculating sufficiently many perturbative corrections only if the semileptonic width of D or B mesons is expressed in terms of the high scale mass. On the other hand, one *could* determine V_{cb} from the $b \to c$ semileptonic width assuming that one is close to the small velocity (SV) limit [2]; then this width depends mainly on the difference $m_b - m_c$ meaning that the overall uncertainty in the mass cancels out in $m_b - m_c$ [38, 39, 7].

The uncertainty in the pole mass can well constitute a sizable part of the commonly accepted value for $\overline{\Lambda}$ for mesons of about 400–600 MeV, cf. Eq. (55). This is apparently due to the peculiar Coulomb enhancement of the leading radiative corrections that has been noted in a different context by Braun *et al.* [41].

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