PÓLYA-SCHUR MASTER THEOREMS FOR CIRCULAR DOMAINS AND THEIR BOUNDARIES

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Abstract. We characterize all linear operators on finite or infinite-dimensional polynomial spaces that preserve the property of having the zero set inside a prescribed region $\Omega \subseteq \mathbb{C}$ for arbitrary closed circular domains Ω (i.e., images of the closed unit disk under a Möbius transformation) and their boundaries. This provides a natural framework for dealing with several long-standing fundamental problems, which we solve in a unified way. In particular, for $\Omega = \mathbb{R}$ our results settle open questions that go back to Laguerre and Pólya-Schur.

1. Introduction

Some of the main challenges in the theory of distribution of zeros of polynomials and transcendental entire functions concern the description of linear operators that preserve certain prescribed ("good") properties. Notwithstanding their fundamental character, most of these problems are in fact still open as they turn out to be surprisingly difficult in full generality. Two outstanding questions among these are the following: let $\Omega \subseteq \mathbb{C}$ be an appropriate set of interest and denote by $\pi(\Omega)$ the class of all (complex or real) univariate polynomials whose zeros lie in Ω.

Problem 1. Characterize all linear transformations $T : \pi(\Omega) \to \pi(\Omega) \cup \{0\}$.

Let π_n be the vector space (over $\mathbb{C}$ or $\mathbb{R}$) of all polynomials of degree at most n and denote by $\pi_n(\Omega)$ the subclass of $\pi(\Omega)$ consisting of polynomials of degree at most n. The finite degree analog of Problem 1 is as follows.

Problem 2. Describe all linear operators $T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\}$ for $n \in \mathbb{N}$.

The above problems were stated in precisely this general form in [10] (see also [35, pp. 182–183]) thereby encompassing essentially all similar questions or variations thereof scattered throughout the literature (cf. *loc. cit.* and references therein). As pointed out in *op. cit.*, Problems 1–2 are open for all but trivial choices of Ω, including such important cases when $\Omega = \mathbb{R}$ or Ω is a half-plane. In this paper we completely solve Problems 1–2 in arguably the most relevant cases, namely all closed circular domains (iii)–(v) and their boundaries (i)–(ii):

(i) Ω is a line,
(ii) Ω is a circle,
(iii) Ω is a closed half-plane,
(iv) Ω is a closed disk,
(v) Ω is the complement of an open disk.

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Despite their long history only relatively few results pertaining to Problems 1–2 are known. As we note in the following (very brief) survey, these deal almost exclusively with special types of linear transformations satisfying the required properties.

To prove the transcendental characterizations of linear preservers of polynomials whose zeros are located on a line or in a closed half-plane (Theorems 5 and 6) we first establish a result on uniform limits on compact sets of bivariate polynomials which are non-vanishing whenever both variables are in the upper half-plane (Theorem 12). Entire functions which are uniform limits on compact sets of sequences of univariate polynomials with only positive zeros were first described by Laguerre [23]. In the process he showed that if $Q(z)$ is a real polynomial with all negative zeros then $T(\pi(\mathbb{R})) \subseteq \pi(\mathbb{R}) \cup \{0\}$, where $T : \mathbb{R}[z] \to \mathbb{R}[z]$ is the linear operator defined by $T(z^k) = Q(k)z^k$, $k \in \mathbb{N}$. Laguerre also stated without proof the correct result for uniform limits of polynomials with all real zeros. The class of entire functions thus obtained – the so-called Laguerre-Pólya class – was subsequently characterized by Pólya [32]. A more complete investigation of sequences of such polynomials was carried out in [26]. This also led to the description of entire functions obtained as uniform limits on compact sets of sequences of univariate polynomials all whose zeros lie in a given closed half-plane [25, 28, 38] as well as the description of entire functions in two variables obtained as limits, uniformly on compact sets, of sequences of bivariate polynomials which are non-vanishing when both variables are in a given open half-plane [25].

The Laguerre-Pólya class has ever since played a significant role in the theory of entire functions [13, 25]. It was for instance a key ingredient in Pólya and Schur's (transcendental) characterization of *multiplier sequences of the first kind* [33], see Theorem 1 below. The latter are linear transformations T on $\mathbb{R}[z]$ that are diagonal in the standard monomial basis of $\mathbb{R}[z]$ and satisfy $T(\pi(\mathbb{R})) \subseteq \pi(\mathbb{R}) \cup \{0\}$. Pólya-Schur's seminal paper generated a vast literature on this topic and related subjects at the interface between analysis, operator theory and algebra but a solution to Problem 1 in the case $\Omega = \mathbb{R}$ has so far remained elusive (cf. [10]). Among the most noticeable progress in this direction we should mention Theorem 17 of [25, Chap. IX], where Levin describes a certain class of "regular" linear operators acting on the closure of the set of polynomials in one variable which have all zeros in the closed upper half-plane. However, Levin's theorem actually uses rather restrictive assumptions and seems in fact to rely on additional (albeit not explicitly stated) non-degeneracy conditions for the transformations involved. Indeed, one can easily produce counterexamples to Levin's result by considering linear operators such as the ones described in Corollary 2 (a) of this paper. In [11] Craven and Csordas established an analog of the Pólya-Schur theorem for multiplier sequences in finite degree thus solving Problem 2 for $\Omega = \mathbb{R}$ in the special case of diagonal operators. Unipotent upper triangular linear operators T on $\mathbb{R}[z]$ satisfying $T(\pi(\mathbb{R})) \subseteq \pi(\mathbb{R}) \cup \{0\}$ were described in [8]. Quite recently, in [3] the authors solved Problem 1 for $\Omega = \mathbb{R}$ and obtained multivariate extensions for a large class of linear transformations, namely all finite order linear differential operators with polynomial coefficients. Further partial progress towards a solution to Problem 1 for $\Omega = \mathbb{R}$ is preliminarily reported in [14] although the same kind of remarks as in the case of Levin's theorem apply here. Namely, the results of *op. cit.* are valid only in the presence of extra non-degeneracy or continuity assumptions for the operators under consideration. Various other special cases of Problem 1 for $\Omega = \mathbb{R}$ have been considered in [1, 5, 6, 19, 20, 21]. Finally, we should mention that to the best of our knowledge Problems 1–2 have so far been widely open in cases (ii)–(v).

To begin with, in §2.1 and §3.1–§3.2 we solve Problems 1–2 for $\Omega = \mathbb{R}$ and $\Omega = \{z \in \mathbb{C} : \mathfrak{Im}(z) \leq 0\}$ and thus obtain complete algebraic and transcendental characterizations of linear operators that preserve hyperbolicity and stability, respectively. In order to deal with Problems 1–2 for *all* closed circular domains and their boundaries we are naturally led to considering a third classification problem, namely the following more general version of Problem 2.

Problem 3. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{C}$. Describe all linear operators

$$T : \pi_n(\Omega) \setminus \pi_{n-1}(\Omega) \to \pi(\Omega) \cup \{0\}.$$

As we explain in §2.2, Problems 2 and 3 are equivalent for closed unbounded sets but for closed bounded sets the latter problem is more natural and actually turns out to be a crucial step in solving Problems 1–2 for closed discs. In §2.2 and §3.3 we fully answer Problem 3 for any closed circular domain or the boundary of such a domain and as a consequence we get complete solutions to Problems 1–2 in all cases ((i)–(v)) listed above.

On the one hand, these results accomplish the classification program originating from the works of Laguerre and Pólya-Schur that we briefly outlined in this introduction. On the other hand, they seem to have numerous applications ranging from entire function theory and operator theory to real algebraic geometry, matrix theory and combinatorics. Some of these will make the object of forthcoming publications. The paper concludes with several remarks on related open problems and potential further developments (§4).

2. Main Results

2.1. **Hyperbolicity and Stability Preservers.** To formulate the complete answers to Problems 1–2 for $\mathbb{R}$ and the half-plane $\{z \in \mathbb{C} : \mathfrak{Im}(z) \leq 0\}$ we need to introduce some notation. As in [3] – and following the commonly used terminology in e.g. the theory of partial differential equations [2] – we call a non-zero univariate polynomial with real coefficients *hyperbolic* if all its zeros are real. Such a polynomial is said to be *strictly hyperbolic* if in addition all its zeros are distinct. A univariate polynomial $f(z)$ with complex coefficients is called *stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathfrak{Im}(z) > 0$ and it is called *strictly stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathfrak{Im}(z) \geq 0$. Hence a univariate polynomial with real coefficients is stable if and only if it is hyperbolic. These classical concepts have several natural extensions to multivariate polynomials, the most general notion being as follows.

Definition 1. A polynomial $f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ is *stable* if $f(z_1, \ldots, z_n) \neq 0$ for all n-tuples $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_j) > 0$, $1 \leq j \leq n$. If in addition f has real coefficients it will be referred to as *real stable*. The sets of stable and real stable polynomials in n variables are denoted by $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{R})$, respectively.

Note that f is stable (respectively, real stable) if and only if for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$ the univariate polynomial $f(\alpha + vt) \in \mathbb{C}[t]$ is stable (respectively, hyperbolic). The connection between real stability and (Gårding) hyperbolicity for multivariate homogeneous polynomials is explained in e.g. [3, Proposition 1].

Notation 1. Henceforth it is understood that if T is a linear operator on some (real) linear subspace $V \subseteq \mathbb{R}[z_1, \dots, z_n]$ then T extends in an obvious fashion to a linear operator – denoted again by T – on the complexification $V \oplus iV$ of V.

Definition 2. A linear operator T defined on a linear subspace V of $\mathbb{C}[z_1, \dots, z_n]$ (respectively, $\mathbb{R}[z_1, \dots, z_n]$) is called *stability preserving* (respectively, *real stability preserving*) on a given subset $M \subseteq V$ if

$$T(\mathcal{H}_n(\mathbb{C}) \cap M) \subseteq \mathcal{H}_n(\mathbb{C}) \cup \{0\} \text{ (respectively, } T(\mathcal{H}_n(\mathbb{R}) \cap M) \subseteq \mathcal{H}_n(\mathbb{R}) \cup \{0\}).$$

A real stability preserver in the univariate case will alternatively be referred to as a *hyperbolicity preserver.* For $m \in \mathbb{N}$ let $\mathbb{R}_m[z] = \{f \in \mathbb{R}[z] : \deg(f) \leq m\}$ and $\mathbb{C}_m[z] = \mathbb{R}_m[z] \oplus i\mathbb{R}_m[z] = \{f \in \mathbb{C}[z] : \deg(f) \leq m\}$. If T is a stability (respectively, hyperbolicity) preserving operator on $\mathbb{C}_m[z]$ (respectively, $\mathbb{R}_m[z]$) we will also say that T *preserves stability* (respectively, *hyperbolicity*) *up to degree* m.

Pólya-Schur's characterization of multiplier sequences of the first kind that we already alluded to in the introduction is given in the following theorem [10, 25, 33].

Theorem 1 (Pólya-Schur theorem). *Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers and $T : \mathbb{R}[z] \to \mathbb{R}[z]$ be the corresponding (diagonal) linear operator given by $T(z^n) = \lambda(n)z^n$, $n \in \mathbb{N}$. Define $\Phi(z)$ to be the formal power series*

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!} z^k.$$

The following assertions are equivalent:

(i) *λ is a multiplier sequence,*
(ii) *$\Phi(z)$ defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign,*
(iii) *Either $\Phi(z)$ or $\Phi(-z)$ is an entire function that can be written as*

$$Cz^n e^{az} \prod_{k=1}^{\infty} (1 + \alpha_k z),$$

where $n \in \mathbb{N}$, $C \in \mathbb{R}$, $a, \alpha_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$,
(iv) *For all non-negative integers n the polynomial $T[(z+1)^n]$ is hyperbolic with all zeros of the same sign.*

As noted in e.g. [10, Theorem 3.3], parts (ii)-(iii) in Pólya-Schur's theorem give a "transcendental" description of multiplier sequences while part (iv) provides an "algebraic" characterization. We emphasize right away the fact that our main results actually yield algebraic and transcendental characterizations of *all* hyperbolicity and stability preservers, respectively, and are therefore natural generalizations of Theorem 1. Moreover, they also display an intimate connection between Problem 1 and its finite degree analog (Problem 2) in the case of (real) stability preservers.

Notation 2. Given a linear operator T on $\mathbb{C}[z]$ we extend it to a linear operator – denoted again by T – on the space $\mathbb{C}[z, w]$ of polynomials in the variables z, w by setting $T(z^k w^\ell) = T(z^k)w^\ell$ for all $k, \ell \in \mathbb{N}$.

Definition 3. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$ be the zeros of two hyperbolic polynomials $f, g \in \mathcal{H}_1(\mathbb{R})$. We say that these zeros *interlace* if they can be ordered so that either $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots$ or $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots$.

Note that in this case one has $|m-n| \leq 1$. By convention, the zeros of any two polynomials of degree 0 or 1 interlace.

Our first theorem characterizes linear operators preserving hyperbolicity up to some fixed degree n.

Theorem 2. *Let $n \in \mathbb{N}$ and let $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ be a linear operator. Then T preserves hyperbolicity if and only if either*

(a) *T has range of dimension at most two and is of the form*

$$T(f) = \alpha(f)P + \beta(f)Q, \quad f \in \mathbb{R}_n[z],$$

where $\alpha, \beta : \mathbb{R}_n[z] \to \mathbb{R}$ are linear functionals and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeros, or

(b) *$T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R})$, or*

(c) *$T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R})$.*

Real stable polynomials in two variables have recently been characterized by the authors [3] as the polynomials $f(z,w) \in \mathbb{R}[z,w]$ that can be expressed as

$$f(z,w) = \pm \det(zA + wB + C), \tag{2.1}$$

where A and B are positive semi-definite matrices and C is a symmetric matrix. Hence (b) and (c) in Theorem 2 can be reformulated as

$$T[(z+w)^n] = \pm \det(zA \pm wB + C),$$

where A and B are positive semi-definite matrices and C is a symmetric matrix.

We will also need to deal with the case when we allow complex coefficients.

Theorem 3. *Let $n \in \mathbb{N}$ and let $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ be a linear operator. Then $T : \pi_n(\mathbb{R}) \to \pi(\mathbb{R})$ if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}_n[z],$$

where $\alpha : \mathbb{C}_n[z] \to \mathbb{C}$ is a linear functional and $P \in \mathcal{H}_1(\mathbb{R})$, or

(b) *T has range of dimension at most two and is of the form*

$$T(f) = \eta\alpha(f)P + \eta\beta(f)Q, \quad f \in \mathbb{C}_n[z],$$

where $\eta \in \mathbb{C}$, $\alpha, \beta : \mathbb{C}_n[z] \to \mathbb{C}$ are linear functionals such that $\alpha(\mathbb{R}_n[z]) \subseteq \mathbb{R}$, $\beta(\mathbb{R}_n[z]) \subseteq \mathbb{R}$, and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeros, or

(c) *There exists $\eta \in \mathbb{C}$ such that $\eta T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R})$, or*

(d) *There exists $\eta \in \mathbb{C}$ such that $\eta T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R})$.*

The corresponding theorem for stability preservers up to some fixed degree n reads as follows.

Theorem 4. *Let $n \in \mathbb{N}$ and let $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ be a linear operator. Then T preserves stability if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}_n[z],$$

where $\alpha : \mathbb{C}_n[z] \to \mathbb{C}$ is a linear functional and $P \in \mathcal{H}_1(\mathbb{C})$, or

(b) *$T[(z+w)^n] \in \mathcal{H}_2(\mathbb{C})$.*

From Theorems 2 and 4 we deduce the following algebraic characterizations of hyperbolicity and stability preservers, respectively.

Corollary 1 (Algebraic Characterization of Hyperbolicity Preservers)**.** *A linear operator $T : \mathbb{R}[z] \to \mathbb{R}[z]$ preserves hyperbolicity if and only if either*

(a) *T has range of dimension at most two and is of the form*

$$T(f) = \alpha(f)P + \beta(f)Q, \quad f \in \mathbb{R}[z],$$

where $\alpha, \beta : \mathbb{R}[z] \to \mathbb{R}$ are linear functionals and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeros, or

(b) *$T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$, or*

(c) *$T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$.*

Corollary 2 (Algebraic Characterization of Stability Preservers)**.** *A linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ preserves stability if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}[z],$$

where $\alpha : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional and $P \in \mathcal{H}_1(\mathbb{C})$, or

(b) *$T[(z+w)^n] \in \mathcal{H}_2(\mathbb{C}) \cup \{0\}$ for all $n \in \mathbb{N}$.*

Notation 3. To any linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ we associate a formal power series in w with polynomial coefficients in z

$$G_T(z,w) = \sum_{n=0}^{\infty} \frac{(-1)^n T(z^n)}{n!} w^n \in \mathbb{C}[z][[w]].$$

Let n be a positive integer and denote by $\overline{\mathcal{H}}_n(\mathbb{C})$ and $\overline{\mathcal{H}}_n(\mathbb{R})$, respectively, the set of entire functions in n variables that are limits, uniformly on compact sets, of polynomials in $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{R})$, respectively. Hence in our notation $\overline{\mathcal{H}}_1(\mathbb{R})$ is the Laguerre-Pólya class of entire functions, sometimes denoted by $\mathcal{L}$-$\mathcal{P}$ in the literature. For a description of $\overline{\mathcal{H}}_1(\mathbb{C})$ and $\overline{\mathcal{H}}_2(\mathbb{C})$ we refer to [25, Chap. IX].

Remark 1. As noted in [3], any linear operator T on $\mathbb{C}[z]$ may be uniquely represented as a formal linear differential operator with polynomials coefficients, i.e., $T = \sum_{k=0}^{\infty} Q_k(z) \frac{d^k}{dz^k}$, where $Q_k \in \mathbb{C}[z]$, $k \geq 0$. In [3] we used the (formal) symbol of T, i.e., $F_T(z,w) := \sum_{k=0}^{\infty} Q_k(z) w^k \in \mathbb{C}[z][[w]]$. One can easily check that the "modified symbol" $G_T(z,w)$ introduced in Notation 3 satisfies $G_T(z,w)e^{zw} = F_T(z,-w)$.

Theorem 5 (Transcendental Characterization of Hyperbolicity Preservers)**.** *A linear operator $T : \mathbb{R}[z] \to \mathbb{R}[z]$ preserves hyperbolicity if and only if either*

(a) *T has range of dimension at most two and is of the form*

$$T(f) = \alpha(f)P + \beta(f)Q, \quad f \in \mathbb{R}[z],$$

where $\alpha, \beta : \mathbb{R}[z] \to \mathbb{R}$ are linear functionals and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeros, or

(b) *$G_T(z,w) \in \overline{\mathcal{H}}_2(\mathbb{R})$, or*

(c) *$G_T(z,-w) \in \overline{\mathcal{H}}_2(\mathbb{R})$.*

Theorem 6 (Transcendental Characterization of Stability Preservers)**.** *A linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ preserves stability if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}[z],$$

where $\alpha : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional and $P \in \mathcal{H}_1(\mathbb{C})$, or

(b) *$G_T(z,w) \in \overline{\mathcal{H}}_2(\mathbb{C})$.*

2.2. **Preservers of Polynomials with Zeros in a Closed Circular Domain or Its Boundary.** Recall that a Möbius transformation is a bijective conformal map of the extended complex plane, i.e., a map $\Phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by

$$\Phi(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C},\ ad-bc \neq 0. \tag{2.2}$$

The inverse of Φ is then given by

$$\Phi^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Definition 4. Let $H = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$ and $\overline{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) \geq 0\}$. An *open circular domain* is the image of H under a Möbius transformation, i.e., an open disk, the (open) complement of a closed disk or an open affine half-plane. A *closed circular domain* is the image of $\overline{H}$ under a Möbius transformation, that is, a closed disk, the (closed) complement of an open disk or a closed affine half-plane.

For technical reasons we will henceforth assume the following:

$$\begin{aligned}&\textit{If } C \textit{ is a half-plane then the corresponding Möbius transformation}\\ &\Phi : C \to H \textit{ is a translation composed with a rotation, i.e., } c = 0 \textit{ in } (2.2).\end{aligned} \tag{2.3}$$

Let us also extend Definition 1 to arbitrary sets $\Omega \subseteq \mathbb{C}$.

Definition 5. A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called Ω-*stable* if $f(\zeta_1, \ldots, \zeta_n) \neq 0$ whenever $\zeta_j \in \Omega$ for all $1 \leq j \leq n$.

Remark 2. Note that an H-stable polynomial is precisely a stable polynomial in the sense of Definition 1.

A fundamental discrepancy between $\pi_n(C_1)$ and $\pi_n(C_2)$, where C_1 is closed and unbounded and C_2 is closed and bounded, is that $\pi_n(C_1) \setminus \pi_{n-1}(C_1)$ is dense in $\pi_n(C_1)$ while $\pi_n(C_2) \setminus \pi_{n-1}(C_2)$ is not dense in $\pi_n(C_2)$ since constant non-zero polynomials do not belong to $\overline{\pi_n(C_2) \setminus \pi_{n-1}(C_2)}$. In order for a linear transformation $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ to map $\pi_n(C_1) \cup \{0\}$ into $\pi(C_1) \cup \{0\}$ it is therefore enough (by Hurwitz' theorem) for T to map $\pi_n(C_1) \setminus \pi_{n-1}(C_1)$ into $\pi(C_1) \cup \{0\}$. However, this is not the case for C_2. Indeed, take for instance C_2 to be the closed unit disk and let $T_n : \mathbb{C}_n[z] \to \mathbb{C}_{n-1}[z]$ be defined by

$$T(z^k) = (n-k)z^k + kz^{k-1}, \quad 0 \leq k \leq n.$$

An application of Theorem 7 below shows that $T : \pi_n(C_2) \setminus \pi_{n-1}(C_2) \to \pi(C_2) \cup \{0\}$ but $T(z^{n-1}) = z^{n-2}(z+n-1)$, which does not belong to $\pi(C_2)$ for $n \geq 3$. Therefore we need to solve a more general version of Problem 2, namely Problem 3 in §1. This is done in the next two theorems for all closed circular domains and their boundaries.

Notation 4. Given $\Omega \subseteq \mathbb{C}$ we denote its complement $\mathbb{C} \setminus \Omega$ by Ω', its boundary $\overline{\Omega} \setminus \mathring{\Omega}$ by $\partial\Omega$, and we let Ω^r be the interior of the complement of Ω, that is, $\Omega^r = \mathring{\Omega}'$.

Theorem 7. *Let $n \in \mathbb{N}$ and $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ be a linear operator. Let further C be an open circular domain given by $C = \Phi^{-1}(H)$, where Φ is a Möbius transformation as in (2.2). Then $T : \pi_n(C') \setminus \pi_{n-1}(C') \to \pi(C') \cup \{0\}$ if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}_n[z],$$

where $\alpha : \mathbb{C}_n[z] \to \mathbb{C}$ is a linear functional and $P \in \pi(C')$, or

(b) *The polynomial*

$$T\left[\big((az+b)(cw+d)+(aw+b)(cz+d)\big)^n\right]$$

is C-stable.

Remark 3. In the case when C' is the closed unit disk then for

$$\Phi(z)=\frac{i}{2}\frac{z+i}{z-i} \tag{2.4}$$

the polynomial in (b) of Theorem 7 reduces to $i^nT[(1+zw)^n]$.

Notation 5. Given a Möbius transformation Φ as in (2.2) and $n\in\mathbb{N}$ we define an invertible linear transformation $\phi_n:\mathbb{C}_n[z]\to\mathbb{C}_n[z]$ by $\phi_n(f)(z)=(cz+d)^nf(\Phi(z))$.

Theorem 8. *Let $n\in\mathbb{N}$ and $T:\mathbb{C}_n[z]\to\mathbb{C}[z]$ be a linear operator and let further $C=\Phi^{-1}(H)$ be an unbounded open circular domain, where Φ is a Möbius transformation as in* (2.2)*. Then $T:\pi_n(\partial C)\setminus\pi_{n-1}(\partial C)\to\pi(\partial C)\cup\{0\}$ if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f)=\alpha(f)P,\quad f\in\mathbb{C}_n[z],$$

where $\alpha:\mathbb{C}_n[z]\to\mathbb{C}$ is a linear functional and $P\in\pi(\partial C)$, or

(b) *T has range of dimension two and the linear operator given by $S=\phi_m^{-1}T\phi_n$ is a stability preserver as in (b) of Theorem 3, where $m=\max\{\deg T(f):f\in\mathbb{C}_n[z]\}$, or*

(c) *The polynomial*

$$T\left[\big((az+b)(cw+d)+(aw+b)(cz+d)\big)^n\right]$$

is both C-stable and C^r-stable, or

(d) *The polynomial*

$$T\left[\big((az+b)(cw+d)-(aw+b)(cz+d)\big)^n\right]$$

is both C-stable and C^r-stable.

Remark 4. If ∂C is the unit circle then for Φ as in (2.4) the polynomials in (c) and (d) of Theorem 8 simply become $i^nT[(1+zw)^n]$ and $T[(z-w)^n]$, respectively.

For the sake of completeness we also formulate analogs of Corollaries 1 and 2 providing algebraic characterizations in the case of closed circular domains and their boundaries.

Corollary 3 (Algebraic Characterization: Closed Circular Domain Case)**.** *Let $T:\mathbb{C}[z]\to\mathbb{C}[z]$ be a linear operator and let $C\subset\mathbb{C}$ be an open circular domain given by $C=\Phi^{-1}(H)$, where Φ is a Möbius transformation as in* (2.2)*. Then $T:\pi(C')\to\pi(C')\cup\{0\}$ if and only if either*

(a) *T has range of dimension at most one and is of the form*

$$T(f)=\alpha(f)P,\quad f\in\mathbb{C}[z],$$

where $\alpha:\mathbb{C}[z]\to\mathbb{C}$ is a linear functional and $P\in\pi(C')$, or

(b) *For all $n\in\mathbb{N}$ the polynomial*

$$T\left[\big((az+b)(cw+d)+(aw+b)(cz+d)\big)^n\right]$$

is C-stable.

Corollary 4 (Algebraic Characterization: Circle and Line Case)**.** *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator and $C = \Phi^{-1}(H)$ be an unbounded open circular domain, where Φ is a Möbius transformation as in* (2.2)*. Then $T : \pi(\partial C) \to \pi(\partial C) \cup \{0\}$ if and only if either*

(a) *T has range of dimension at most one and is of the form*
$$T(f) = \alpha(f)P, \quad f \in \mathbb{C}[z],$$
where $\alpha : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional and $P \in \pi(\partial C)$, or

(b) *T has range of dimension two and for all $n \in \mathbb{N}$ the linear operator given by $S_n = \phi_{m(n)}^{-1} T \phi_n$ is a stability preserver as in (b) of Theorem 3, where $m(n) = \max\{\deg T(f) : f \in \mathbb{C}_n[z]\}$, or*

(c) *For all $n \in \mathbb{N}$ the polynomial*
$$T\left[\left((az+b)(cw+d) + (aw+b)(cz+d)\right)^n\right]$$
is both C-stable and C^r-stable, or

(d) *For all $n \in \mathbb{N}$ the polynomial*
$$T\left[\left((az+b)(cw+d) - (aw+b)(cz+d)\right)^n\right]$$
is both C-stable and C^r-stable.

Similarly, we may characterize all linear maps that take polynomials with zeros in one closed circular domain Ω_1 to polynomials with zeros in another closed circular domain Ω_2, or the boundary of one circular domain Ω_1 to the boundary of another circular domain Ω_2. However, this only amounts to composing with linear operators of the type defined in Notation 5, namely $\phi_n : \mathbb{C}_n[z] \to \mathbb{C}_n[z]$, where $\phi_n(f)(z) = (cz+d)^n f(\Phi(z))$ and Φ is an appropriate Möbius transformation of the form (2.2).

3. Proofs of the Main Results

3.1. **Hyperbolic and Stable Polynomials.** If the zeros of two hyperbolic polynomials $f, g \in \mathcal{H}_1(\mathbb{R})$ interlace then the *Wronskian* $W[f,g] := f'g - fg'$ is either non-negative or non-positive on the whole real axis $\mathbb{R}$.

Definition 6. Given $f, g \in \mathcal{H}_1(\mathbb{R})$ we say that f and g are in *proper position*, denoted $f \ll g$, if the zeros of f and g interlace and $W[f,g] \le 0$.

For technical reasons we also say that the zeros of the polynomial 0 interlace the zeros of any (non-zero) hyperbolic polynomial and write $0 \ll f$ and $f \ll 0$. Note that if $f \ll g$ and $g \ll f$ then f and g must be constant multiples of each other, that is, $W[f,g] \equiv 0$.

The following theorem is a version of the classical Hermite-Biehler theorem [35].

Theorem 9 (Hermite-Biehler theorem)**.** *Let $h := f + ig \in \mathbb{C}[z]$, where $f, g \in \mathbb{R}[z]$. Then $h \in \mathcal{H}_1(\mathbb{C})$ if and only if $f, g \in \mathcal{H}_1(\mathbb{R})$ and $g \ll f$. Moreover, h is strictly stable if and only if f and g are strictly hyperbolic polynomials with no common zeros and $g \ll f$.*

The next theorem is often attributed to Obreschkoff [29].

Theorem 10 (Obreschkoff theorem)**.** *Let $f, g \in \mathbb{R}[z]$. Then $\alpha f + \beta g \in \mathcal{H}_1(\mathbb{R}) \cup \{0\}$ for all $\alpha, \beta \in \mathbb{R}$ if and only if either $f \ll g$, $g \ll f$, or $f = g \equiv 0$. Moreover, $\alpha f + \beta g$ is strictly hyperbolic for all $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ if and only if f and g are strictly hyperbolic polynomials with no common zeros and either $f \ll g$ or $g \ll f$.*

Remark 5. Note that if $T : \pi_n(\mathbb{R}) \to \pi(\mathbb{R})$ is an $\mathbb{R}$-linear operator then by Obreschkoff's theorem T also preserves interlacing in the following manner: if f and g are hyperbolic polynomials of degree at most n whose zeros interlace then the zeros of $T(f)$ and $T(g)$ interlace provided that $T(f)T(g) \neq 0$.

Lemma 1. *Let $n \in \mathbb{N}$. Suppose that $T : \mathbb{R}_{n+1}[z] \to \mathbb{R}[z]$ preserves hyperbolicity and that $f \in \mathbb{R}[z]$ is a strictly hyperbolic polynomial of degree n or $n+1$ for which $T(f) = 0$. Then $T(g)$ is hyperbolic for all $g \in \mathbb{R}[z]$ with $\deg g \leq n+1$.*

Let $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ be a stability preserver and suppose that $f \in \mathbb{C}[z]$ is a strictly stable polynomial of degree n for which $T(f) = 0$. Then $T(g)$ is stable for all $g \in \mathbb{C}[z]$ with $\deg g \leq n$.

Proof. Let f be a strictly hyperbolic polynomial of degree n or $n+1$ for which $T(f) = 0$ and let $g \in \mathbb{R}[z]$ be a polynomial with $\deg g \leq n+1$. From Hurwitz' theorem it follows that for $\epsilon \in \mathbb{R}$ with $|\epsilon|$ small enough the polynomial $f + \epsilon g$ is strictly hyperbolic. Since $\deg(f + \epsilon g) \leq n+1$ and T preserves hyperbolicity up to degree $n+1$ we get that $T(g) = \epsilon^{-1}T(f + \epsilon g)$ is hyperbolic.

Suppose that $f \in \mathbb{C}[z]$ is a strictly stable polynomial of degree n such that $T(f) = 0$ and that the degree of $g \in \mathbb{C}[z]$ does not exceed n. By Hurwitz' theorem $f + \epsilon g$ is strictly stable for all sufficiently small $|\epsilon|$. Since $\deg(f + \epsilon g) \leq n$ and T preserves stability up to degree n it follows that $T(g) = \epsilon^{-1}T(f + \epsilon g)$ is stable. $\square$

Lemma 2. *Suppose that $V \subseteq \mathbb{R}[z]$ is an $\mathbb{R}$-linear space whose every non-zero element is hyperbolic. Then $\dim V \leq 2$.*

Suppose that $V \subseteq \mathbb{C}[z]$ is a $\mathbb{C}$-linear space whose every non-zero element is stable. Then $\dim V \leq 1$.

Proof. We first deal with the real case. Suppose that there are three linearly independent polynomials f_1, f_2 and f_3 in V. By Obreschkoff's theorem the zeros of these polynomials mutually interlace. Wlog we may assume that $f_1 \ll f_2$ and $f_1 \gg f_3$. Consider the line segment $\ell_\theta = \theta f_3 + (1-\theta) f_2$, $0 \leq \theta \leq 1$. Since $f_1 \ll \ell_0$ and $f_1 \gg \ell_1$ by Hurwitz' theorem there is a real number η between 0 and 1 such that $f_1 \ll \ell_\eta$ and $f_1 \gg \ell_\eta$. This means that f_1 and ℓ_η are constant multiples of each other contrary to the assumption that f_1, f_2 and f_3 are linearly independent.

For the complex case let $V_R = \{p : p + iq \in V \text{ with } p, q \in \mathbb{R}[z]\}$ be the "real component" of V. By the Hermite-Biehler theorem all polynomials in V_R are hyperbolic, so by the above we have $\dim_{\mathbb{R}} V_R \leq 2$. Clearly, V is the complex span of V_R. If $\dim_{\mathbb{R}} V_R \leq 1$ we are done so we may assume that $\{p, q\}$ is a basis for V_R with $f := p + iq \in V$. By definition $W[p, q] \geq 0$ on the whole of $\mathbb{R}$ and the Wronskian is not identically zero. Assume now that g is another polynomial in V. Then

$$g = ap + bq + i(cp + dq)$$

for some $a, b, c, d \in \mathbb{R}$. We have to show that g is a (complex) constant multiple of f. Since $g \in V$ we have

$$W[ap + bq, cp + dq] = (ad - bc)W[p, q] \geq 0,$$

so that $ad - bc \geq 0$. Now by linearity we have that

$$g + (u + iv)f = (a + u)p + (b - v)q + i((c + v)p + (d + u)q) \in V$$

for all $u, v \in \mathbb{R}$ which, as above, gives

$$H(u, v) := (a + u)(d + u) - (b - v)(c + v) \geq 0$$

for all $u, v \in \mathbb{R}$. But

$$4H(u,v) = (2u + a + d)^2 + (2v + c - b)^2 - (a - d)^2 - (b + c)^2,$$

so $H(u,v) \geq 0$ for all $u, v \in \mathbb{R}$ if and only if $a = d$ and $b = -c$. This gives

$$g = ap - cq + i(cp + aq) = (a + ic)(p + iq) = (a + ic)f,$$

as was to be shown. □

Notation 6. Let $\mathcal{H}_1^-(\mathbb{C}) = \{f \in \mathbb{C}[z] : f(z) \neq 0 \text{ if } \mathfrak{Im}(z) < 0\}$. By the Hermite-Biehler theorem and Definition 6 if $f, g \in \mathbb{R}[z]$ then $f + ig \in \mathcal{H}_1^-(\mathbb{C})$ if and only if $f, g \in \mathcal{H}_1(\mathbb{R})$ and $f \ll g$. Given a linear subspace V of $\mathbb{C}[z]$ and $M \subseteq V$ we say that a linear operator T on V is *stability reversing* on M if $T(\mathcal{H}_1(\mathbb{C}) \cap M) \subseteq \mathcal{H}_1^-(\mathbb{C}) \cup \{0\}$. Note that if $S : \mathbb{C}[z] \to \mathbb{C}[z]$ is the linear involution defined by $S(f)(z) = f(-z)$ then T is stability reversing if and only if $S \circ T$ is stability preserving.

Lemma 3. *Suppose that $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ maps all hyperbolic polynomials of degree at most n to hyperbolic polynomials. Then T is either stability preserving or stability reversing or the range of T has dimension at most two. In the latter case T is given by*

$$T(f) = \alpha(f)P + \beta(f)Q, \quad f \in \mathbb{R}_n[z], \tag{3.1}$$

where P, Q are hyperbolic polynomials whose zeros interlace and α, β are real-valued linear functionals on $\mathbb{R}_n[z]$.

Proof. The lemma is obvious for $n = 0$ so we may and do assume that n is a positive integer. By Remark 5 and the Hermite-Biehler theorem we know that T maps all stable polynomials of degree n into the set $\mathcal{H}_1(\mathbb{C}) \cup \mathcal{H}_1^-(\mathbb{C}) \cup \{0\}$. We now distinguish two cases. Suppose first that there are two strictly stable polynomials f, g of degree n such that $T(f) \in \mathcal{H}_1(\mathbb{C})$ and $T(g) \in \mathcal{H}_1^-(\mathbb{C})$.

Claim. If the above conditions are satisfied then the kernel of T must contain a strictly hyperbolic polynomial of degree at least $n - 1$.

From Lemma 1 and the Claim we deduce that $T : \mathbb{R}_n[z] \to \pi(\mathbb{R}) \cup \{0\}$. Hence all non-zero polynomials in the image of T are hyperbolic, which by Lemma 2 gives that $\dim T(\mathbb{R}_n[z]) \leq 2$ and thus T must be of the form (3.1).

Proof of Claim. Suppose that f_1, f_2 are two strictly stable polynomials of degree n for which $T(f_1) \in \mathcal{H}_1(\mathbb{C})$ and $T(f_2) \in \mathcal{H}_1^-(\mathbb{C})$, respectively. By a homotopy argument, invoking again Hurwitz' theorem, there is a strictly stable polynomial h of degree n for which $T(h) \in \mathcal{H}_1(\mathbb{C}) \cap \mathcal{H}_1^-(\mathbb{C}) \cup \{0\}$. Writing h as $h = p + iq$, where p and q are strictly hyperbolic polynomials (by the Hermite-Biehler theorem) gives that $T(p)$ and $T(q)$ are constant multiples of each other. Suppose that $\deg p = n$. Then $\deg q \geq n - 1$ since the zeros of p and q interlace. If $T(q) = 0$ the claim is obviously true so suppose that $T(q) \neq 0$. Then $T(p) = \lambda T(q)$ for some $\lambda \in \mathbb{R}$. By the Obreschkoff Theorem $p - \lambda q$ is strictly hyperbolic and of degree at least $n - 1$. Clearly, $T(p - \lambda q) = 0$, which proves the Claim.

Suppose now that T maps all strictly stable polynomials of degree n into the set $\mathcal{H}_1(\mathbb{C}) \cup \{0\}$ (the case when T maps all such polynomials into $\mathcal{H}_1^-(\mathbb{C}) \cup \{0\}$ is treated similarly). Let f be a strictly stable polynomial with $\deg f < n$ and set $f_\epsilon(z) = (1 - \epsilon i z)^{n - \deg f} f(z)$. Then f_ϵ is strictly stable whenever $\epsilon > 0$ and $T(f_\epsilon) \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$ since $\deg f_\epsilon = n$. Letting $\epsilon \to 0$ we get $T(f) \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$ and since strictly stable polynomials are dense in $\mathcal{H}_1(\mathbb{C})$ it follows that T is stability preserving. □

As a final tool we will need the Grace-Walsh-Szegö coincidence theorem [15, 39, 40]. Recall that a multivariate polynomial is *multi-affine* if it has degree at most one in each variable.

Theorem 11 (Grace-Walsh-Szegö coincidence theorem)**.** *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be symmetric and multi-affine and let C be a circular domain containing the points $\zeta_1, \dots, \zeta_n$. Then there exists a point $\zeta \in C$ such that*

$$f(\zeta_1, \dots, \zeta_n) = f(\zeta, \dots, \zeta).$$

From the Grace-Walsh-Szegö coincidence theorem we immediately deduce:

Corollary 5. *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be of degree at most d in z_1 and consider the expansion of f in powers of z_1,*

$$f(z_1, \dots, z_n) = \sum_{k=0}^{d} Q_k(z_2, \dots, z_n) z_1^k, \quad Q_k \in \mathbb{C}[z_2, \dots, z_n],\, 0 \le k \le d.$$

Then f is stable if and only if the polynomial

$$\sum_{k=0}^{d} Q_k(z_2, \dots, z_n) \frac{e_k(x_1, \dots, x_d)}{\binom{d}{k}}$$

is stable in the variables $z_2, \dots, z_n, x_1, \dots, x_d$, where $e_k(x_1, \dots, x_d)$, $0 \le k \le d$, are the elementary symmetric functions in the variables $x_1, \dots, x_d$ given by $e_0 = 1$ and $e_k = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le d} x_{j_1} \cdots x_{j_k}$ for $1 \le k \le d$.

Lemma 4. *Let $T : \mathbb{C}_n[z_1] \to \mathbb{C}[z_1]$ be a linear operator such that $T[(z_1 + w)^n] \in \mathcal{H}_2(\mathbb{C})$. If $f \in \mathcal{H}_m(\mathbb{C})$ is of degree at most n in z_1 then $T(f) \in \mathcal{H}_m(\mathbb{C}) \cup \{0\}$, where T is extended to a linear operator on $\mathbb{C}[z_1, \dots, z_m]$ by setting $T(z_1^{\alpha_1} \cdots z_m^{\alpha_m}) = T(z_1^{\alpha_1}) z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ for all $\alpha \in \mathbb{N}^m$ with $\alpha_1 \le n$ (compare with Notation 2).*

Proof. Let $f \in \mathcal{H}_m(\mathbb{C})$ be of degree n in z_1. For $\epsilon > 0$ set

$$f_\epsilon(z_1, \dots, z_m) = f(z_1 + \epsilon i, z_2, \dots, z_m).$$

Fixing $\zeta_2, \dots, \zeta_m$ in the open upper half-plane we may write $f_\epsilon(z_1, \zeta_2, \dots, \zeta_m)$ as

$$f_\epsilon(z_1, \zeta_2, \dots, \zeta_m) = C(z_1 - \xi_1)(z_1 - \xi_2) \cdots (z_1 - \xi_n) = C \sum_{k=0}^{n} (-1)^k e_k(\xi_1, \dots, \xi_n) z_1^{n-k},$$

where $C \neq 0$ and $\mathfrak{Im}(\xi_j) < 0$, $1 \le j \le n$. Note that $z_1 \mapsto f_\epsilon(z_1, \zeta_2, \dots, \zeta_m)$ is indeed a polynomial of degree n in z_1. This is because

$$f_\epsilon(z_1, z_2, \dots, z_m) = z_1^n Q_n(z_2, \dots, z_m) + \text{terms of lower degree in } z_1$$

and $Q_n(z_2, \dots, z_n) := \lim_{r \to \infty} r^{-n} f(r, z_2, \dots, z_m)$ is stable by Hurwitz' theorem hence $Q_n(\zeta_2, \dots, \zeta_n) \neq 0$. Let us now write $T[(z_1 + w)^n]$ as

$$T[(z_1 + w)^n] = \sum_{k=0}^{n} \binom{n}{k} T\left(z_1^{n-k}\right) w^k \in \mathcal{H}_2(\mathbb{C}).$$

By Corollary 5 we know that

$$\sum_{k=0}^{n} T\left(z_1^{n-k}\right) e_k(w_1, \dots, w_n) \in \mathcal{H}_{n+1}(\mathbb{C}).$$

But then

$$T(f_\epsilon)(z_1,\zeta_2,\dots,\zeta_m)=C\sum_{k=0}^{n}T\left(z_1^{n-k}\right)e_k(-\xi_1,\dots,-\xi_n)\in\mathcal{H}_1(\mathbb{C}),$$

which gives $T(f_\epsilon)\in\mathcal{H}_m(\mathbb{C})$. Letting $\epsilon\to 0$ we have $T(f)\in\mathcal{H}_m(\mathbb{C})\cup\{0\}$. If $f\in\mathcal{H}_m(\mathbb{C})$ is of degree less than n in z_1 we may consider $f^\epsilon=(1-\epsilon i z_1)^{n-\deg f}f\in\mathcal{H}_m(\mathbb{C})$. Then by the above one has $T(f^\epsilon)\in\mathcal{H}_n(\mathbb{C})\cup\{0\}$ for all $\epsilon>0$. The lemma now follows from Hurwitz' theorem by letting $\epsilon\to 0$. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. If $T[(z+w)^n]\in\mathcal{H}_2(\mathbb{C})$ then by applying Lemma 4 with $m=1$ it follows that T is stability preserving.

Suppose now that T preserves stability. Assume first that there exists $w_0\in\mathbb{C}$ with $\mathfrak{Im}(w_0)>0$ such that $(z+w_0)^n$ is in the kernel of T. Since $(z+w_0)^n$ is strictly stable it follows from Lemma 1 and Lemma 2 that $\dim_{\mathbb{C}}T(\mathbb{C}_n[z])\le 1$. Hence T is given by $T(f)=\alpha(f)P$, where P is a fixed stable polynomial and $\alpha:\mathbb{C}_n[z]\to\mathbb{C}$ is a linear functional. Otherwise we may assume that $T[(z+w_0)^n]\in\mathcal{H}_1(\mathbb{C})$ for all $w_0\in\mathbb{C}$ with $\mathfrak{Im}(w_0)>0$ and conclude that $T[(z+w)^n]\in\mathcal{H}_2(\mathbb{C})$. □

The proof of Theorem 2 now follows easily.

Proof of Theorem 2. Recall Notation 6 and note that by Lemma 3 we may assume that $\dim_{\mathbb{R}}T(\mathbb{R}_n[z])>2$, so that T is either stability reversing or stability preserving. By Theorem 4 T is stability reversing or stability preserving if and only if $T[(z+w)^n]\in\mathcal{H}_2(\mathbb{C})$ or $T[(z-w)^n]\in\mathcal{H}_2(\mathbb{C})$, respectively, which proves the theorem. □

Proof of Theorem 3. By Theorem 2 it suffices to prove that if a linear operator $T:\mathbb{C}_n[z]\to\mathbb{C}[z]$ satisfies $T:\pi_n(\mathbb{R})\to\pi(\mathbb{R})$ then either

(a) There exists $\theta\in\mathbb{R}$ such that $T=e^{i\theta}\tilde{T}$, where $\tilde{T}:\mathbb{R}_n[z]\to\mathbb{R}[z]$ is a hyperbolicity preserver when restricted to $\mathbb{R}_n[z]$, or
(b) T is given by $T(f)=\alpha(f)P$, where $\alpha:\mathbb{C}_n[z]\to\mathbb{C}$ is a linear functional and P is a hyperbolic polynomial.

We prove this using induction on $n\in\mathbb{N}$. If $n=0$ there is nothing to prove so we may assume that n is a positive integer. Note that T restricts to a linear operator $T':\pi_{n-1}(\mathbb{R})\to\pi(\mathbb{R})$ and that by induction T' must be of the form (a) or (b) above. Suppose that $T'=e^{i\theta}\tilde{T}'$, where $\tilde{T}'$ is a hyperbolicity preserver up to degree $n-1$. If $T(z^n)=e^{i\theta}f_n$, where $f_n\in\mathbb{R}[z]$ (actually, $f_n\in\mathcal{H}_1(\mathbb{R})\cup\{0\}$), then T is of the form (a). Hence we may assume that $T(z^n)=e^{i\gamma}f_n$, where $0\le\gamma<2\pi$, $\gamma-\theta$ is not an integer multiple of π and f_n is a hyperbolic polynomial. Suppose that there is an integer $k<n$ such that $T(z^k)$ is not a constant multiple of f_n. Let M be the largest such k and set $R(z)=e^{-i\theta}T(z^M)$. Then

$$e^{-i\theta}T\left[z^M(1+z)^{n-M}\right]=R(z)+\left(r+e^{i(\gamma-\theta)}\right)f_n(z)$$

for some $r\in\mathbb{R}$. But this polynomial is supposed to be a complex constant multiple of a hyperbolic polynomial, which can only happen if R and thus $T\left(z^M\right)$ is a constant multiple of f_n. This contradiction means that T must be as in (b) above with $P=f_n$.

Assume now that T' is as in (b). If $T(z^n)$ is a constant multiple of P there is nothing to prove, so we may assume that $T(z^n)=e^{i\theta}f_n$, where $0\le\theta<2\pi$ and

f_n is a hyperbolic polynomial which is not a constant multiple of P. Suppose that there is an integer $k < n$ such that $\alpha(z^k)e^{-i\theta} \notin \mathbb{R}$ and let M be the largest such integer. Then

$$e^{-i\theta}T\big[z^M(1+z)^{n-M}\big] = \big(\alpha\big(z^M\big)\, e^{-i\theta} + r\big)\, P(z) + f_n(z)$$

for some $r \in \mathbb{R}$. However, the latter polynomial should be a complex constant multiple of a hyperbolic polynomial and this can happen only if $\alpha(z^M)e^{-i\theta} \in \mathbb{R}$, which contradicts the above assumption. This means that T must be as in (a). $\square$

Proof of Corollary 1. Note first that if $T : \mathbb{R}[z] \to \mathbb{R}[z]$ is as in (a), (b) or (c) of Corollary 1 then by Theorem 2 we have that T preserves hyperbolicity up to any degree $n \in \mathbb{N}$. Conversely, if $T : \mathbb{R}[z] \to \mathbb{R}[z]$ preserves hyperbolicity then for any $n \in \mathbb{N}$ the restriction $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ preservers hyperbolicity (up to degree n). The case when $\dim_{\mathbb{R}} T(\mathbb{R}[z]) \leq 2$ is clear. Suppose now that $\dim_{\mathbb{R}} T(\mathbb{R}[z]) > 2$ and that $T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R})$ for some $n \in \mathbb{N}$. Then by Lemma 4 we have that $T[(z+w)^m] \in \mathcal{H}_2(\mathbb{C}) \cup \{0\}$ for all $m \leq n$ and since the latter polynomial has real coefficients we get $T[(z+w)^m] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $m \leq n$. Similarly, if $T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R})$ then $T[(z-w)^m] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $m \leq n$. It follows that if $\dim_{\mathbb{R}} T(\mathbb{R}[z]) > 2$ then either $T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$ or $T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$. $\square$

By appropriately modifying the arguments in the proof of Corollary 1 one can easily check that Corollary 2 follows readily from Theorem 4.

3.2. **The Transcendental Characterization.** We will need the following lemma due to Szász, see [38, Lemma 3].

Lemma 5 (Szász)**.** *Let $m, n \in \mathbb{N}$ with $m \leq n$ and $f(z) = \sum_{k=m}^{n} c_k z^k \in \mathbb{C}[z]$. If $f(z) \in \mathcal{H}_1(\mathbb{C})$ and $c_m c_n \neq 0$ then for any $r \geq 0$ one has*

$$|f(z)| \leq |c_m| r^m \exp\left(r\frac{|c_{m+1}|}{|c_m|} + 3r^2\frac{|c_{m+1}|^2}{|c_m|^2} + 3r^2\frac{|c_{m+2}|}{|c_m|}\right)$$

whenever $|z| \leq r$.

For $k, n \in \mathbb{N}$ let $(n)_k = k!\binom{n}{k} = n(n-1)\cdots(n-k+1)$ if $k \leq n$ and $(n)_k = 0$ if $n < k$ denote as usual the Pochhammer symbol.

Theorem 12. *Let $F(z,w) = \sum_{k=0}^{\infty} P_k(z)w^k$ be a formal power series in w with polynomial coefficients. Then $F(z,w) \in \overline{\mathcal{H}}_2(\mathbb{C})$ if and only if $\sum_{k=0}^{n}(n)_k P_k(z)w^k \in \mathcal{H}_2(\mathbb{C}) \cup \{0\}$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $F(z,w) = \sum_{k=0}^{\infty} P_k(z)w^k \in \overline{\mathcal{H}}_2(\mathbb{C})$ has polynomial coefficients. Given $n \in \mathbb{N}$, the sequence $\{(n)_k\}_{k=0}^{n}$ is a multiplier sequence, and since it is non-negative it is a stability preserver by Theorem 3. By Corollary 2 and Lemma 4 we have that this multiplier extends to a map $\Lambda : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C}) \cup \{0\}$. Now, if $\tilde{F}_m(z,w) = \sum_{k=0}^{N_m} P_{m,k}(z)w^k$ is a sequence of polynomials in $\mathcal{H}_2(\mathbb{C})$ converging to $F(z,w)$, uniformly on compacts, then $P_{m,k}(z) \to P_k(z)$ as $m \to \infty$ uniformly on compacts for fixed $k \in \mathbb{N}$. But then $\Lambda[\tilde{F}_m(z,w)] \to \Lambda[F(z,w)]$ uniformly on compacts, which gives $\sum_{k=0}^{n}(n)_k P_k(z)w^k \in \mathcal{H}_2(\mathbb{C}) \cup \{0\}$.

Conversely, suppose that $\sum_{k=0}^{n}(n)_k P_k(z)w^k \in \mathcal{H}_2(\mathbb{C}) \cup \{0\}$ for all $n \in \mathbb{N}$ and let $F_n(z,w) = \sum_{k=0}^{n}(n)_k n^{-k} P_k(z)w^k$.

Claim. Given $r > 0$ there is a constant C_r such that

$$|F_n(z, w)| \leq C_r \text{ for } |z| \leq r, |w| \leq r \text{ and all } n \in \mathbb{N}.$$

This claim proves the theorem since $\{F_n(z, w)\}_{n=0}^{\infty}$ is then a normal family whose convergent subsequences converge to $F(z, w)$ (by the fact that $n^{-k}(n)_k \to 1$ for all $k \in \mathbb{N}$ as $n \to \infty$).

We first prove the Claim in the special case when $P_k(z) \in \mathbb{R}[z]$ for all $k \in \mathbb{N}$ and $P_K(z)$ is a non-zero constant, where K is the first index for which $P_K(z) \neq 0$.

Proof of the Claim in the special case. Let $|P_K(z)| = A$, $B_r = \max\{|P_{K+1}(z)| : |z| \leq r\}$ and $D_r = \max\{|P_{K+2}(z)| : |z| \leq r\}$. Then, if we fix $\zeta \in \mathbb{C}$ with $\mathfrak{Im}(\zeta) > 0$, we have that $F_n(\zeta, w) \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$ and by Lemma 5

$$|F_n(\zeta, w)| \leq Ar^K \exp\left(r\frac{B_r}{A} + 3r^2\frac{B_r^2}{A^2} + 3r^2\frac{D_r}{A}\right) \tag{3.2}$$

whenever $\mathfrak{Im}(\zeta) > 0$, $|\zeta| \leq r$ and $|w| \leq r$. If $\zeta \in \mathbb{C}$ is fixed with $\mathfrak{Im}(\zeta) < 0$ then $F_n(\zeta, -w) \in \mathcal{H}_1(\mathbb{C})$ (since $F_n(z, w)$ has real coefficients and $\overline{F_n(z, w)} = F_n(\overline{z}, \overline{w})$). By Lemma 5 this means that (3.2) holds also for $\mathfrak{Im}(\zeta) < 0$ and by continuity also for $\zeta \in \mathbb{R}$, which proves the Claim.

Next we assume that $\deg(P_K(z)) = d \geq 1$. An application of Theorem 4 verifies that $T = d/dz$ preserves stability, and by Lemma 4 $T = \partial/\partial z$ preserves stability in two variables. Hence $\frac{\partial F_n(z,w)}{\partial z} \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ if $F_n(z, w) \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$. To deal with this case it is therefore enough to prove that if $\left|\frac{\partial F_n(z,w)}{\partial z}\right| \leq C_r$ for $|z| \leq r, |w| \leq r$ and all $n \in \mathbb{N}$ then there is a constant D_r such that $|F_n(z, w)| \leq D_r$ for $|z| \leq r, |w| \leq r$ and all $n \in \mathbb{N}$. Clearly, $F_n(0, w) \in \mathcal{H}_1(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$. Moreover, if m is the first index such that $P_m(0) \neq 0$ then for $n \geq m$ we have

$$F_n(0, w) = (n)_m n^{-m} P_m(0) w^m + (n)_{m+1} n^{-m-1} P_{m+1}(0) w^{m+1} + \ldots \in \mathcal{H}_1(\mathbb{R}),$$

which by Lemma 5 gives

$$|F_n(0, w)| \leq |P_m(0)| r^m \exp\left(r\frac{|P_{m+1}(0)|}{|P_m(0)|} + 3r^2\frac{|P_{m+1}(0)|^2}{|P_m(0)|^2} + 3r^2\frac{|P_{m+2}(0)|}{|P_m(0)|}\right) =: E_r$$

for $|z| \leq r$. Here we have used that $(n)_k n^{-k} \leq 1$ and $(n)_{k+1} n^{-k-1}/(n)_k n^{-k} \leq 1$ for $0 \leq k \leq n$. Suppose that

$$\left|\frac{\partial F_n(z, w)}{\partial z}\right| \leq C_r$$

for $n \in \mathbb{N}$ and $|z| \leq r$, $|w| \leq r$. Then,

$$F_n(z, w) = F_n(0, w) + z\int_0^1 \frac{\partial F_n}{\partial z}(zt, w)dt,$$

so

$$|F_n(z, w)| \leq E_r + rC_r \quad \text{for } |z| \leq r, |w| \leq r \text{ and all } n \in \mathbb{N}.$$

Next we prove the above Claim in the general case. For this we need a property of multivariate stable polynomials that was established in [3, Corollary 1]:

Fact. If $h = g + if \in \mathcal{H}_n(\mathbb{C})$ then $f, g \in \mathcal{H}_n(\mathbb{R})$.

Now this means that if we let $P_k(z) = R_k(z) + iI_k(z)$ then we may write

$$F_n(z, w) = F_n^{\mathfrak{Re}}(z, w) + iF_n^{\mathfrak{Im}}(z, w)$$

with $F_n^{\mathfrak{Re}}(z,w), F_n^{\mathfrak{Im}}(z,w) \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$, where $F_n^{\mathfrak{Re}}(z,w), F_n^{\mathfrak{Im}}(z,w)$ are given by

$$F_n^{\mathfrak{Re}}(z,w) = \sum_{k=0}^{n} (n)_k n^{-k} R_k(z) w^k, \quad F_n^{\mathfrak{Im}}(z,w) = \sum_{k=0}^{n} (n)_k n^{-k} I_k(z) w^k.$$

By the above there are constants A_r and B_r such that

$$|F_n^{\mathfrak{Re}}(z,w)| \leq A_r \text{ and } |F_n^{\mathfrak{Im}}(z,w)| \leq B_r \text{ for } |z| \leq r, |w| \leq r \text{ and all } n \in \mathbb{N}.$$

Hence $|F_n(z,w)| \leq \sqrt{A_r^2 + B_r^2}$ for $|z| \leq r, |w| \leq r$ and all $n \in \mathbb{N}$, which proves the Claim in the general case. □

We can now prove the transcendental characterizations of hyperbolicity and stability preservers, respectively.

Proof of Theorems 5 and 6. We only prove Theorem 6 since the proof of Theorem 5 is almost identical. Theorem 6 follows quite easily from Theorem 12 and Corollary 2. Indeed, note that $T[(1-zw)^n] \in \mathcal{H}_2(\mathbb{C})$ if and only if $T[(z+w)^n] \in \mathcal{H}_2(\mathbb{C})$. Since

$$T[(1-zw)^n] = \sum_{k=0}^{n} (n)_k \frac{(-1)^k T(z^k)}{k!} w^k$$

for all $n \in \mathbb{N}$ we deduce the desired conclusion by comparing the above expression with the modified symbol $G_T(z,w)$ introduced in Notation 3. □

Finally, we show how Pólya-Schur's algebraic and transcendental characterizations of multiplier sequences (Theorem 1) follow from our results.

Proof of Theorem 1. We may assume that $\lambda(0) = 1$. Statements (i)–(iv) are all true if $\dim_{\mathbb{R}} T(\mathbb{R}[z]) \leq 2$ since then $\lambda(j) = 0$ for all $j \geq 2$. Hence we may assume that $\dim_{\mathbb{R}} T(\mathbb{R}[z]) > 2$. Corollary 1 implies that either

$$T[(1-zw)^n] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda(k)(zw)^k \in \mathcal{H}_2(\mathbb{R})$$

for all $n \in \mathbb{N}$ or

$$T[(1+zw)^n] = \sum_{k=0}^{n} \binom{n}{k} \lambda(k)(zw)^k \in \mathcal{H}_2(\mathbb{R})$$

for all $n \in \mathbb{N}$. We claim that if $f(z) \in \mathbb{R}[z]$ then $f(zw) \in \mathcal{H}_2(\mathbb{R})$ if and only if all the zeros of f are real and non-negative. Suppose first that $f(zw)$ is real stable. Letting $z = w = t$ we see that $f(t^2)$ is hyperbolic, which can only happen if all the zeros of f are non-negative (since $a + t^2$ is hyperbolic if and only if $a \leq 0$). On the other hand, if f has only real non-positive zeros then $f(zw)$ factors as $f(zw) = C\prod_{j=0}^{n}(zw + \alpha_j)$, where $\alpha_j \in \mathbb{R}$, $1 \leq j \leq n$. Now $zw + \alpha_j \in \mathcal{H}_2(\mathbb{R})$ if and only if $\alpha_j \leq 0$. Hence (i) ⇔ (iv) and by Theorem 5 we also have (ii) ⇔ (iv). The equivalence of (ii) and (iii) is Laguerre's classical result. □

3.3. **Closed Circular Domains and Their Boundaries.** Recall Definitions 1 and 4, Notation 4 and the linear operators ϕ_n introduced in Notation 5. In particular, an H-stable polynomial in the sense of Definition 4 is precisely a stable polynomial in the sense of Definition 1.

Lemma 6. *Let $T : \mathbb{C}_n[z] \to \mathbb{C}_m[z]$ be a linear operator and suppose m is minimal, i.e., $m = \max\{\deg T(f) : f \in \mathbb{C}_n[z]\}$. Let further $C = \Phi^{-1}(H)$ be an open circular domain, where Φ is a Möbius transformation as in (2.2), and let $S : \mathbb{C}_n[z] \to \mathbb{C}_m[z]$ be the linear operator defined by $S = \phi_m^{-1} T \phi_n$. The following are equivalent:*

(i) *$T(f)$ is C-stable or zero whenever f is of degree n and C-stable,*
(ii) *$S(f)$ is H-stable or zero whenever f is of degree n and H-stable,*
(iii) *$S(f)$ is H-stable or zero whenever f is of degree at most n and H-stable.*

The following are also equivalent:

(iv) *$T(f)$ is ∂C-stable or zero whenever f is of degree n and ∂C-stable,*
(v) *$S(f)$ is $\mathbb{R}$-stable or zero whenever f is of degree n and $\mathbb{R}$-stable,*
(vi) *$S(f)$ is $\mathbb{R}$-stable or zero whenever f is of degree at most n and $\mathbb{R}$-stable.*

Proof. Note first that the equivalences (ii)$\Leftrightarrow$ (iii) and (v)$\Leftrightarrow$ (vi) are simple consequences of the density argument used in §2.2. Let us now show that (i) $\Leftrightarrow$ (ii). This is obvious if C is an open half-plane, i.e., if $c = 0$ (cf. (2.3)). Therefore we assume that $c \neq 0$ and that the boundary of C is a circle. If C is an open disk then a/c is in the open lower half-plane, so that $-cz + a$ is H-stable. Moreover, $-d/c \in \partial C$ and thus $cz + d$ is C-stable. Hence f is H-stable if and only if $\phi_n(f)$ is C-stable so the assertion follows in this case as well.

It remains to prove that (i) $\Leftrightarrow$ (ii) in the case when C is the open complement of a (closed) disk, which we proceed to do.

(i) $\Rightarrow$ (ii). Clearly, we may assume that T is not the trivial (identically zero) operator. Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be an H-stable polynomial of degree n and suppose first that $\sum_{k=0}^{n} a_k a^k c^{n-k} \neq 0$. Then $\phi_n(p)$ is a C-stable polynomial of degree n so that by assumption $T(\phi_n(p))$ is C-stable or zero. If $S(p) \neq 0$ it follows that that we can express $S(p)$ uniquely as $S(p) = (-cz + a)^{r(p)} S_0(p)$, where $S_0(p)$ is H-stable and $r(p)$ is a non-negative integer. By a continuity argument and an application of Hurwitz' theorem we have that a factorization as above holds for any H-stable polynomial of degree n. Since the set of H-stable polynomials of degree n is dense in $\pi_n(H') \cup \{0\}$ – that is, the set of H-stable polynomials of degree at most n union the zero polynomial – we deduce that such a factorization holds for the image under S of any H-stable polynomial p of degree at most n, namely

$$S(p) = (-cz + a)^{r(p)} S_0(p),$$

where $S_0(p)$ is H-stable or zero and $r(p)$ is a non-negative integer.

Fix a basis $\{p_j(z)\}_{j=0}^{n}$ of $\mathbb{C}_n[z]$ consisting of *strictly H-stable* (that is, $\overline{H}$-stable) polynomials of degree n. We distinguish two cases:

Suppose first that $S(f) \neq 0$ for all strictly H-stable polynomials f of degree n. Since the topological space of strictly stable polynomials of degree n is (path-) connected we have by Hurwitz' theorem that $r(f)$ is constant on the set of strictly H-stable polynomials of degree n. Thus, by the minimality assumption on m and the fact that ϕ_n is invertible we must have $\deg(T(\phi_n(p_k))) = m$ for some k. It follows that $r(p_k) = 0$ hence $S(f) = S_0(f)$ for any strictly H-stable polynomial f

of degree n. Using again a standard density argument and Hurwitz' theorem we deduce that S preserves (H-)stability up to degree n.

Suppose now that $S(f) = 0$ for some strictly H-stable polynomial f of degree n and let $g \in \mathbb{C}_n[z]$. Clearly, $f + \epsilon g$ is strictly H-stable for all $\epsilon > 0$ small enough. By the above we have that

$$S(g) = \epsilon^{-1} S(f + \epsilon g) = (-cz + a)^{r(g)} S_0(g),$$

where $S_0(g)$ is H-stable or zero. It follows that $V := S(\mathbb{C}_n[z])$ is a $\mathbb{C}$-linear space such that every non-zero polynomial in V is a $(-cz + a)^r$-multiple of an H-stable polynomial. We know that $r(p_k) = 0$ for the strictly H-stable polynomial p_k above. Assume that $h \in \mathbb{C}_n[z]$ is such that $r(h) \neq 0$. Since $r(p_k) = 0$ and

$$S(h) + \delta S(p_k) = S(h + \delta p_k) = (-cz + a)^{r(h+\delta p_k)} S_0(h + \delta p_k) \in V,$$

where $S_0(h + \delta p_k)$ is H-stable or zero, we conclude that $S(h) + \delta S(p_k)$ is H-stable or zero for all $\delta \neq 0$. Letting $\delta \to 0$ we have by Hurwitz' theorem that either $S(h) = 0$ or $S(h)$ is H-stable. However, this contradicts the fact that $a/c \in H$ and $S(h)(a/c) = 0$, which follows from the assumption that $r(h) \neq 0$. Hence all non-zero polynomials in V are H-stable and thus we deduce that S preserves (H-)stability up to degree n in this case as well.

(ii) ⇒ (i). Since the set of H-stable polynomials of degree n is dense in the set of H-stable polynomials of degree at most n it follows that S preserves H-stability on the latter set (i.e., S preserves H-stability up to degree n). Let f be a C-stable polynomial of degree n. Then $\phi_n^{-1}(f)$ is H-stable. Hence so is $S(\phi_n^{-1}(f))$ and thus $T(f) = \phi_m\big(S(\phi_n^{-1}(f))\big)$ is a C-stable polynomial (note that $-d/c \in \partial C$).

The equivalence (iv) ⇔ (v) follows just as above by replacing "strictly H-stable" with "strictly hyperbolic", that is, real- and simple-rooted. □

Notation 7. Given a polynomial $f \in \mathbb{C}[z, w]$ of degree at most m in z and at most n in w and a Möbius transformation Φ as in (2.2) we let

$$\phi_{m,z}(f)(z, w) = (cz + d)^m f(\Phi(z), w), \quad \phi_{n,w}(f)(z, w) = (cw + d)^n f(z, \Phi(w)).$$

Lemma 7. *Let $f(z, w) \in \mathbb{C}[z, w]$ be of degree at most m in z and at most n in w and let $\Phi : C \to H$ be a Möbius transformation as in* (2.2)*. If either*

(a) *C is not the exterior of a disk, or*
(b) *C is the exterior of a disk and*
  (b1) *the degree in z of $\phi_{m,z}(f)(z, w)$ is m, and*
  (b2) *the degree in w of $\phi_{n,w}\phi_{m,z}(f)(z, w)$ is n,*

then f is H-stable if and only if $\phi_{n,w}\phi_{m,z}(f)$ is C-stable.

Proof. The equivalence is clear when C is a disk or a half-plane since in these cases $-cz + a$ is H-stable and $cz + d$ is C-stable (cf. (2.3)). Hence we may assume that C is the exterior of a disk. Note that if $g(z)$ is a polynomial of degree k then a/c is not a zero of $\phi_k^{-1}(g)(z)$. In particular, if $g(z)$ is a C-stable polynomial of degree k then $\phi_k^{-1}(g)(z)$ is an H-stable polynomial of degree k. Now since $-d/c \in \partial C$ we have that $cz + d$ is C-stable and therefore $\phi_{n,w}\phi_{m,z}(f)$ is C-stable if f is H-stable, which proves one of the implications.

Conversely, suppose that $G(z, w) := \phi_{n,w}\phi_{m,z}(f)(z, w) = \sum_{k=0}^n Q_k(z) w^k$ is C-stable. Then so is $\lambda^{-n} G(z, \zeta + \lambda(w - \zeta))$ whenever $\lambda \geq 1$, where ζ is the center of C'. Letting $\lambda \to \infty$ we see by Hurwitz' theorem that $(w - \zeta)^n Q_n(z)$ is C-stable.

Hence, so is $Q_n(z)$. For every $z_0 \in C$ the degree of $G(z_0, w)$ is n, from which it follows that $\phi_{n,w}^{-1}(G)(z,w) = \phi_{m,z}(f)(z,w) \neq 0$ whenever $z \in C$, $w \in H$. Similarly, if $\phi_{m,z}(f)(z,w) = \sum_{k=0}^{m} P_k(w)z^k$ then $P_m(w)$ is H-stable so that $\phi_{m,z}(f)(z,w_0)$ has degree m for every $w_0 \in H$. We conclude that $f(z,w) = \phi_{m,z}^{-1}\phi_{n,w}^{-1}(G)(z,w)$ is H-stable, as claimed. □

The following lemma is a simple consequence of Hurwitz' theorem since boundedness prevents zeros from escaping to infinity.

Lemma 8. *Let Ω_1 be a path-connected subset of $\mathbb{C}$ and let Ω_2 be a bounded subset of $\mathbb{C}$. If $T : \mathbb{C}_n[z] \to \mathbb{C}[z]$ is a linear operator such that $T : \pi_n(\Omega_1) \setminus \pi_{n-1}(\Omega_1) \to \pi(\Omega_2)$ then all polynomials in the image of $\pi_n(\Omega_1) \setminus \pi_{n-1}(\Omega_1)$ have the same degree.*

Note that in the hypothesis of Lemma 8 we do not allow the identically zero polynomial to be in the image of $\pi_n(\Omega_1) \setminus \pi_{n-1}(\Omega_1)$.

Proof of Theorem 7. Suppose that $T : \mathbb{C}_n[z] \to \mathbb{C}_m[z]$, where as before m is minimal in the sense that $m = \max\{\deg T(f) : f \in \mathbb{C}_n[z]\}$. By combining Lemma 6 with Hurwitz' theorem we see that $T : \pi_n(C') \setminus \pi_{n-1}(C') \to \pi(C') \cup \{0\}$ if and only if $\phi_m^{-1} T \phi_n : \pi_n(H') \to \pi(H') \cup \{0\}$. The case $\dim_{\mathbb{C}} T(\mathbb{C}_n[z]) \leq 1$ is clear. If $\dim_{\mathbb{C}} T(\mathbb{C}_n[z]) > 1$ then by Theorem 4 we have $\phi_m^{-1} T \phi_n : \pi_n(H') \to \pi(H') \cup \{0\}$ if and only if $f(z,w) := \phi_m^{-1} T \phi_n \left[(z+w)^n\right]$ is H-stable. Now the polynomial in (b) of Theorem 7 is precisely $\phi_{n,w}\phi_{m,z}(f)$. Moreover, by Lemma 7 (a) we may assume that C is the exterior of a disk. Therefore, in order to complete the proof it only remains to show that conditions (b1) and (b2) of Lemma 7 are satisfied.

If there were two different $\overline{C}$-stable polynomials of degree n that were mapped by T on polynomials of different degrees then by Lemma 8 there would be a $\overline{C}$-stable polynomial g of degree n in the kernel of T. However, since $\phi_n^{-1}(g)$ is strictly H-stable it would then follow from Lemmas 1 and 2 that $\phi_m^{-1} T \phi_n$ and hence also T has range of dimension at most one, which is not the case. We infer that $\deg T(h) = m$ for any $\overline{C}$-stable polynomial h of degree n.

Clearly, $\phi_{m,z}(f)(z,w) = T[\phi_{n,z}\big((z+w)^n\big)]$. Let $w_0 \in H \setminus \{-a/c\}$. The only zero of the degree n polynomial

$$p_{w_0}(z) := \phi_{n,z}\big((z+w_0)^n\big) = \big((a+w_0c)z + b + w_0d\big)^n$$

is $\Phi^{-1}(-w_0) \in \overline{C}'$, so $p_{w_0}(z)$ is $\overline{C}$-stable. By the previous paragraph we then have

$$\deg(\phi_{m,z}(f)(z,w_0)) = \deg(T[p_{w_0}(z)]) = m,$$

which verifies condition (b1) of Lemma 7.

Note next that coefficient of w^n in the polynomial defined in (b) of Theorem 7 is $T[(2acz+bc+ad)^n]$. We claim that the polynomial $(2acz+bc+ad)^n$ is $\overline{C}$-stable and of degree n. Since the image of any $\overline{C}$-stable polynomial of degree n is of degree m this would verify condition (b2) of Lemma 7. To prove the claim note first that since C is the exterior of a disk we have that $ac \neq 0$ so $\deg(2acz+bc+ad)^n = n$. Let $\zeta = -b/a$ and $\eta = -d/c$. Since $\Phi(\zeta) = 0$ and $\Phi(\eta) = \infty$ we have that $\zeta, \eta \in \partial C$, which implies – again by the assumption that C is the exterior of a disk – that the zero of $(2acz+bc+ad)^n$

$$-\frac{bc+ad}{2ac} = \frac{\zeta+\eta}{2},$$

is in $\overline{C}'$. Thus $(2acz+bc+ad)^n$ is $\overline{C}$-stable and of degree n, as required. □

Proof of Theorem 8. Let $m = \max\{\deg T(f) : f \in \mathbb{C}_n[z]\}$. By Lemma 6 we have that $T : \pi_n(\partial C) \to \pi(\partial C) \cup \{0\}$ if and only if $\phi_m^{-1} T \phi_n : \pi_n(\mathbb{R}) \to \pi(\mathbb{R}) \cup \{0\}$. Using this and Theorem 3 it is not difficult to verify the theorem in the case $\dim_{\mathbb{C}} T(\mathbb{C}_n[z]) \leq 2$. Let $f(z,w) = \phi_m^{-1} T \phi_n[(z+w)^n]$. To settle the remaining cases note first that by Theorem 3 (c) and (d) we have that $T : \pi_n(\partial C) \to \pi(\partial C) \cup \{0\}$ if and only if $f(z,w)$ or $f(z,-w)$ is a complex multiple of a real stable polynomial. Now the condition that $f(z,w)$ is a complex multiple of a real stable polynomial is equivalent to saying that $f(z,w)$ is both H-stable and H^r-stable, see [3, Proposition 3] (note that $H^r = -H$). By Theorem 7 we know that $f(z,w)$ is H-stable if and only if the polynomial in (c) of Theorem 8, that is, $\phi_{n,w}\phi_{m,z}(f)(z,w)$, is C-stable. On the other hand, since $-cz + a$ is H^r-stable and $cz + d$ is C^r-stable (since C^r is not the exterior of a disk) we also have that $f(z,w)$ is H^r-stable if and only if $\phi_{n,w}\phi_{m,z}(f)$ is C^r-stable. The proof of the fact that condition (d) in Theorem 8 is equivalent to saying that $f(z,-w)$ is a complex multiple of a real stable polynomial follows in similar fashion. This completes the proof of the theorem. □

Corollary 3 and Corollary 4 are immediate consequences of Theorem 7 and Theorem 8, respectively.

4. Open Problems

As we already noted in §1, Problems 1 and 2 have a long and distinguished history. In this paper we completely solved them for a particularly relevant type of sets, namely all closed circular domains and their boundaries. Among the most interesting remaining cases that are currently under investigation we mention:

(a) Ω is an open circular domain,
(b) Ω is a sector or a double sector,
(c) Ω is a strip,
(d) Ω is a half-line,
(e) Ω is an interval.

Let us briefly comment on the importance of the above cases.

Case (a). The classical notion of Hurwitz (or continuous-time) stability refers to univariate polynomials with all their roots in the open left half-plane. Its well-known discrete-time version – often called Schur or Schur-Cohn stability – is when all the roots of a polynomial lie in the open unit disk. Both these notions are fundamental and widely used in e.g. control theory and engineering sciences. (The authors easily found several hundreds of publications in both purely mathematical and applied areas devoted to the study of Hurwitz and Schur stability for various classes of polynomials as well as continuous or discrete-time systems.)

Case (b). Polynomials and transcendental entire functions with all their zeros confined to a (double) sector often appear as solutions to e.g. Schrödinger-type equations with polynomial potential or, more generally, any linear ordinary differential equation with polynomial coefficients and constant leading coefficient, see for instance [17, 18]. Concrete information about linear transformations preserving this class of polynomials and entire functions turns out to be very useful for asymptotic integration of linear differential equations.

Case (c). Specific examples of linear transformations preserving the class of polynomials with all their roots in the strip $|\mathfrak{Im}(z)| \leq \alpha$, $\alpha > 0$, can be found in the famous articles by Pólya [31], Lee-Yang [24] and de Bruijn [7]. A complete characterization of all such linear transformations would shed light on a great many problems in the theory of Fourier transforms and real entire functions.

Case (d). Polynomials with all their roots on either the positive or negative half-axis appeared already in Laguerre's works and in Pólya-Schur's fundamental paper [33] and have been frequently used in various contexts ever since. In addition to their description of multiplier sequences of the first kind in *op. cit.* the authors also classified multiplier sequences of the second kind, i.e., diagonal linear operators mapping polynomials with all real roots and of the same sign to polynomials with all real roots. A natural extension of these results would be to characterize all linear transformations with this property. Solving case (d) would answer this question and thus complete the program initiated by Pólya and Schur over 90 years ago.

Case (e). Numerous papers have been devoted to this case of Problems 1 and 2 and its connections with Pólya frequency functions, integral equations with totally positive or sign regular kernels, Laplace transforms, the theory of orthogonal polynomials, etc. A complete description of all linear transformations preserving the set of polynomials with all their zeros in a given interval would therefore have many interesting applications and would also answer several of the questions raised in e.g. [9, 19, 20, 21, 22, 30, 34] on these and related subjects.

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