Polygon Evolution by Vertex Deletion

Longin Jan Latecki and Rolf Lakämper

Institut für Angewandte Mathematik, Universität Hamburg
Bundesstr. 55, 20146 Hamburg
{latecki, lakaemper}@math.uni-hamburg.de

Abstract. We propose a simple approach to evolution of polygonal curves that is specially designed to fit discrete nature of curves in digital images. It leads to simplification of shape complexity with no blurring (i.e., shape rounding) effects and no dislocation of relevant features. Moreover, in our approach the problem to determine the size of discrete steps for numerical implementations does not occur, since our evolution method leads in a natural way to a finite number of discrete evolution steps which are just the iterations of a basic procedure of vertex deletion.

Keywords: discrete curve evolution, shape simplification, shape recognition

1 Introduction

We assume that a closed polygon P is given (that does not need to be simple). In particular, any boundary curve in a digital image can be regarded as a polygon without loss of information, with possibly a large number of vertices.

The main motivation for the presented discrete curve evolution is the fact that *the boundary of a segmented object in a digital image contains misinformation but misses no information.* Clearly, there is digitization and segmentation noise on the boundary of a segmented object, that results in displacement of the boundary points. However, as long as it is possible to recognize the overall shape of the object, the shape information is contained in the given contour.

Most of the standard approaches in computer vision try to compute the original position of the displaced boundary points. This is only possible if the class of shapes to which the analyzed shape belongs is explicitly known and is sufficiently restrictive, e.g., fitting ellipses.

On the other hand, it is not necessary to recover the original position of the boundary points in order to recognize the shape. A pointwise interpretation of this fact is that there exists a subset A of the set of the boundary points B that is sufficient to represent the shape of the object. The other points in $B \setminus A$ either are redundant for the shape or had been influenced by noise. Clearly, the points in the set A may also be displaced due to noise, but nevertheless they are sufficient to recognize the shape, if the amount of displacement is such that people can still recognize the shape. For example, this is the case for the contour of the building obtained from an aerial image in Figure 1 (cf. Brunn et al. [3], Fig. 4), where it is still possible to recognize the overall shape, although the amount of displacement of boundary points is relatively large.

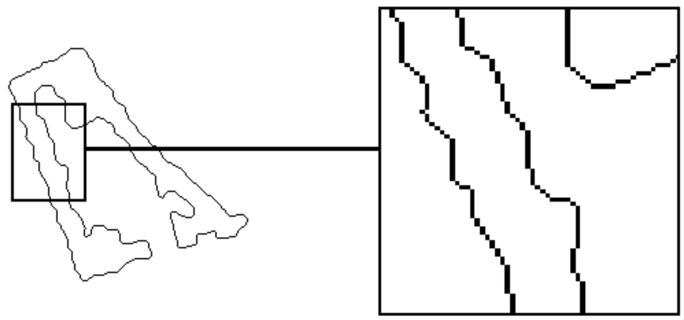

Fig. 1. It is possible to recognize the overall shape of the building, although the amount of displacement of boundary points is relatively large.

The presented discrete curve evolution allows us for a given object boundary to find a subset A of the set of the boundary points B that is sufficient to represent the shape of the object, i.e., points important for the object shape remain after the application of the discrete curve evolution. For example, compare the contour (a) with (c) in Figure 2, where the contours (b) and (c) are obtained from (a) by our discrete curve evolution. Observe also an enormous data reduction: contour (c) in Figure 2 contains only 3% of points of contour (a).

The fact that the discrete curve evolution allows us to find a subset A of the set of the boundary points B that is sufficient to represent the shape of the object is not only justified by experimental results, some of which we present in this paper, but also by the continuity theorem in [7]. This theorem states that if polygon B is sufficiently close to a polygon A, then the evolved version of polygon B will remain close to polygon A.

In *scale-space theory* a curve (or surface) Γ is embedded into a continuous family $\{\Gamma_t : t \geq 0\}$ of gradually simplified versions. The main idea of *scale-spaces* is that the original curve (or surface) $\Gamma = \Gamma_0$ should get more and more simplified and noise and small structures should vanish as parameter t increases. Thus, due to different scales (values of t), it is possible to separate small details from relevant shape properties. The ordered sequence $\{\Gamma_t : t \geq 0\}$ is referred to as *evolution* of Γ. Scale-spaces find wide application in computer vision, in particular, due to smoothing ($\Rightarrow$ noise influence is reduced) and elimination of small details ($\Rightarrow$ relevant shape features remain). Some of the main applications are quality enhancement of images, noise removal, and shape description and recognition (e.g., see Sethian [12]).

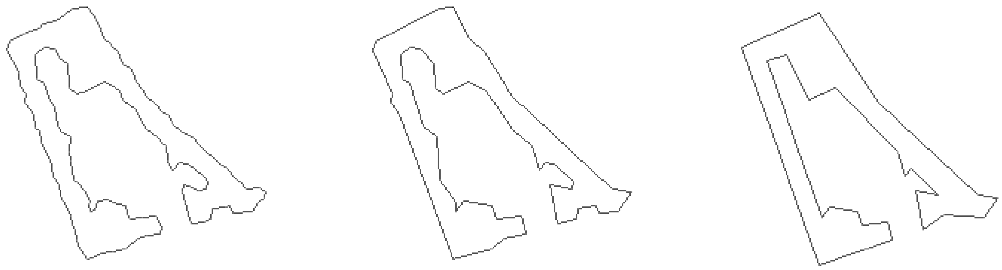

Fig. 2. (a) → (b): noise elimination. (b) → (c): extraction of relevant line segments.

The scale-space evolution is mostly based on parabolic partial differential equations. The oldest and best-studied are scale-spaces based on a linear diffusion equation (also called geometric heat equation), e.g., see Weickert [15]. The solutions of diffusion equations can be obtained by convolution of the original curve (or surface) with a Gaussian function with parameter t (Kimia and Siddiqi [5]). Hence the solutions correspond to Gaussian smoothing of the original curve (or surfaces) with support size t. This leads to a multiscale, curvature-based shape representation.

Along with the advantages of evolution based on the linear diffusion equation, there are also some serious problems (Weickert [15], p. 6):

(a) *"Gaussian smoothing does not only reduce noise, but also blurs important features such as edges and, thus, makes them harder to identify. Since Gaussian scale-space is designed to be completely uncommitted, it cannot take into account any a-priori information on structures which are worth being preserved (or even enhanced).*

(b) *Diffusion dislocates features when moving from finer to coarser scales. So features identified at a coarse scale do not give the right location and have to be traced back to the original image [16]. In practice, relating dislocated information obtained at different scales is difficult and bifurcations may give rise to instabilities. These coarse-to-fine tracking difficulties are generally denoted as the correspondence problem."*

To reduce these problems, many anisotropic and nonlinear diffusion processes have been proposed for scale-spaces (for an overview see, Weickert [15]). Also reaction-diffusion equations, which lead to reaction-diffusion scale spaces, have been considered (Kimia, et al [6]).

We propose a different approach to scale-space evolution in which both problems simply do not occur. Our departing point is a discrete nature of curves and surfaces in digital images. In opposite to standard approaches in scale-spaces, our evolution is guided neither by differential equations nor Gaussian smoothing, and it is not a discrete version of an evolution by differential equations, as it is the case in Bruckstein, et al. [2]. The main properties of the proposed evolution are (see Figure 3):

- Although it leads to noise elimination, it does not introduce any blurring effects.
- Although irrelevant features vanish during our evolution, there is no dislocation of relevant features.

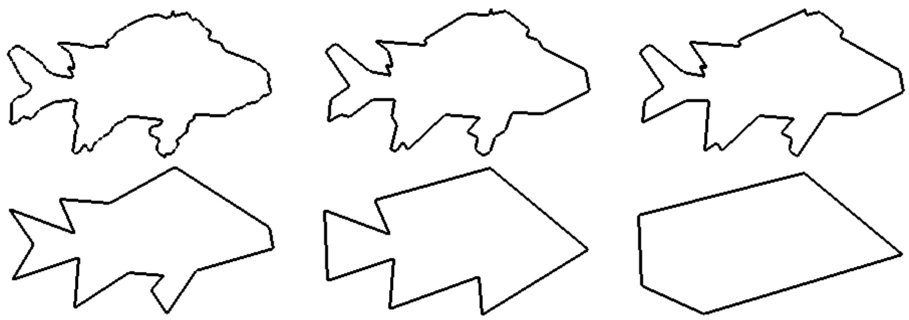

Fig. 3. A few stages of our curve evolution. The first contour is a distorted version of the contour on www-site [17].

In comparison to scale-space methods, the main differences are

1. By numerical implementations of diffusion equations, every vertex of the polygon is translated at a single evolution step, whereas in our approach the remaining vertices do not change their positions.
2. The translation vector of each point in a diffusion process is locally determined, whereas our polygonal evolution is guided by a relevance measure that is not a local property with respect to the original polygon.
3. The process of the polygonal evolution is parameter-free.

Although there exist diffusion process that are parameter-free in the sense that constant values for parameters are known that apply to large classes of curves, for most numerical implementations of parabolic differential equations several parameters are necessary and it is theoretically unknown how to relate and determine the parameters. This is due to

(c) problems with stability and computation time of discrete, numeric realizations of diffusion processes.

An example problem is to specify the discrete time steps t necessary for a stable numeric computation. Since the scale-space theories are continuous theories, i.e., scale (or time) parameter t varies over positive real numbers, the determination of discrete steps is a non-trivial problem; if the steps are too large, it can happen that too many relevant features vanish, and on the other hand, too small discrete steps lead to an inefficient computation. Additionally, a given digital curve (or surface) has some fixed grid resolution that cannot be made infinitely small, and this resolution not always satisfies the requirements for stabile numerical solutions of partial differential equations. A different but related problem is the following:

(d) *"Diffusion filters with a constant steady-state require to specify a stopping time if one wants to get nontrivial results."* (Weickert [15], p.19)

Clearly, if the stopping time (i.e., stopping parameter t) is too large, it can happen that all relevant features do not any more exist at scale t.

The proposed evolution method leads in a natural way to a finite number of discrete evolution steps which are just the iterations of a basic procedure of vertex removal. Thus, the problem to determine the size of discrete steps does not occur. This also drastically simplifies the problem of stopping time.

2 Discrete Curve Evolution

Let P be a closed polygon (that does not need to be simple). We will denote the vertices of P with $Vertices(P)$. A *discrete curve evolution* produces a sequence of polygons $P = P^0, ..., P^m$ such that $|Vertices(P^m)| \leq 3$, where $|\ .\ |$ is the cardinality function. Each vertex v in P^i is assigned a relevance measure $K(v, P^i) \in \mathbb{R}_{\geq 0}$. The relevance measure $K(v, P^i)$ that we used for our experiments is defined below. The process of the *discrete curve evolution* is very simple:
For every evolution step $i = 0, ..., m-1$, a polygon P^{i+1} is obtained after the vertices whose relevance measure is minimal have been deleted from P^i.

In order to give a precise definition of the discrete curve evolution, we first define

Definition: $K_{min}(P^i)$ to be the smallest value of the relevance measures for vertices of P^i:

$$K_{min}(P^i) = \min\{K(u, P^i) : u \in Vertices(P^i)\}$$

and the set $V_{min}(P^i)$ to contain the vertices whose relevance measure is minimal in P^i:

$$V_{min}(P^i) = \{u \in Vertices(P^i) : K(u, P^i) = K_{min}(P^i)\}$$

for $i = 0, ..., m-1$.

Definition: For a given polygon P and a relevance measure K, we call a **discrete curve evolution** a process that produces a sequence of polygons $P = P^0, ..., P^m$ such that

$$Vertices(P^{i+1}) = Vertices(P^i) \setminus V_{min}(P^i),$$

where $|Vertices(P^m)| \leq 3$.

The process of the discrete curve evolution is guaranteed to terminate, since in every evolution step, the number of vertices decreases by at least one. It is also obvious that this evolution converges to a convex polygon, since the evolution will reach a state where there are exactly three, two, one, or no vertices in P^m. Clearly, the only polygon with three vertices is a triangle. Of course, for many curves, a convex polygon with more then three vertices can be obtained in an earlier stage of the evolution. The only polygon with two vertices is a line segment. A polygon with one vertex is also trivially convex. Only when the set $Vertices(P^m)$ is empty, we obtain a degenerated polygon equal to the empty set, which is trivially convex. Thus, we obtain for every relevance measure

Proposition 1. *The discrete curve evolution converges to a convex polygon, i.e., there exists* $0 \leq i \leq m$ *such that* P^i *is convex, and if* $0 \leq i < m$*, all polygons* $P^{i+1}, ..., P^m$ *are convex.* ■

This proposition demonstrates mathematical simplicity of the relation between our evolution approach and the geometric properties of the evolved polygons. Observe that this proposition also holds for polygons that are not simple (i.e., have self-intersections). An analog theorem for evolution of continuous planar curves by diffusion equations is a deep and highly non-trivial result of differential geometry. It holds only for simple closed smooth curves evolved by the heat equation:

Theorem (Grayson [4]) *An embedded planar curve converges to a simple convex curve when evolving according to:*

$$\begin{cases} \frac{\partial C(s,t)}{\partial t} = \frac{\partial^2 C(s,t)}{\partial s^2} = \kappa(s,t)\boldsymbol{N}(s,t) \\ C(s,0) = C_0(s), \end{cases} \tag{1}$$

where $C : S^1 \times [0,T) \rightarrow \mathbb{R}^2$ is a family of smooth simple curves, s is the Euclidean arc-length, κ the Euclidean curvature, and $\boldsymbol{N}$ the inward unit normal. The diffusion equation (1) is called a *geometric heat equation* for a curve. The flow given by (1) is called the *Euclidean shortening flow*.

Polygonal analogs of the evolution by diffusion equations are presented in Bruckstein, et al. [2]. The experiments in [2] indicate that an arbitrary initial polygon converges to a convex polygon (polygonal circle). However, the proof of this fact in the Euclidean case is an open question. In [2] as well as in evolutions by numerical solutions of differential equations, each vertex of the polygon with nonzero curvature is displaced at a single evolution step, whereas in our approach some vertices are removed and the remaining vertices do not change their positions. This is an important difference which leads to several properties of our approach (described in the next section) that are favorable for many applications.

The convexity result (and some other properties of the discrete curve evolution) holds for any relevance measure. However, there are some important properties like continuity that depend on the choice of the relevance measure (see Section 3).

The key property of the evolution we used for our experiments is the order of the deletion determined by the relevance measure. Our relevance measure

$K(v, P^i)$ depends on vertex v and its two neighbor vertices u, w in P^i, i.e., $K(v, P^i) = K(v, u, w)$. It is given by the formula

$$K(v, u, w) = K(\beta, l_1, l_2) = \frac{\beta l_1 l_2}{l_1 + l_2}, \tag{2}$$

where β is the turn angle at vertex v in P^i, l_1 is the length of $\overline{vu}$, and l_2 is the length of $\overline{vw}$. (Both lengths are normalized with respect to the total length of polygon P^i.) Intuitively it reflects the shape contribution of vertex v in P^i. The main property is the following

- The higher the value of $K(v, u, w)$, the larger is the contribution of arc $\overline{vu} \cup \overline{vw}$ to the shape of polygon P^i.

Observe that this relevance measure is not a local property with respect to the polygon P, although its computation is local in P^i for every vertex v. A motivation for this measure and its properties are discussed in [8].

An algorithmic definition of the discrete curve evolution is given in [8] and live examples can be found our www-site [9]. The curve evolution in [8] differs from the one defined here if two or more vertices in P^i have the same relevance measure. The evolution in [8] removes in a single step only one vertex. If in the course of the evolution no two vertices in P^i have the same relevance measure, then the algorithmic definition in [8] and the above definitions are equivalent.

3 Properties of the Discrete Curve Evolution

We will show in this section that our discrete curve evolution has the following properties that do not depend on the choice of the relevance measure:

(P_1) It leads to a simplification of shape complexity.
(P_2) It does not introduce any blurring (i.e., shape rounding) effects and
(P_3) there is no dislocation of relevant features,

due to the fact that the remaining vertices do not change their positions. Two more important properties of our curve evolution are based on the relevance measure defined in Section 2:

(P_4) It is stable with respect to noisy deformations and noise elimination takes place in early stages of the evolution.
(P_5) It allows to find line segments in noisy images, due to the relevance order of the repeated process of linearization (e.g., Figure 2).

We begin with some examples to illustrate these properties. A few stages of the proposed curve evolution in Figure 3 illustrate the shape complexity reduction. Observe that our curve evolution does not introduce any blurring effects, which result in shape rounding for curves. (for a comparison see the curve evolution on www-site [17], based on [10]). There is no dislocation of the remaining relevant shape features, since the planar position of the remaining points of the

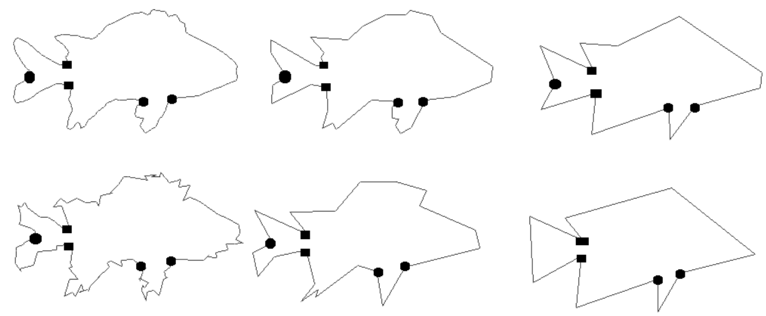

Fig. 4. Discrete curve evolution is stable with respect to distortions. The same planar position of the points marked with the same symbols demonstrates that there is no displacement of the remaining feature points.

digital polygon is unchanged. This is demonstrated by marking the corresponding points with the same symbols in Figure 4. Observe also the stability of feature points with respect to noise deformations shown in the second row in Figure 4.

By comparison of the curves (a) and (b) in Figure 2, it can be seen that our evolution method allows us first to eliminate noise influence without changing the shape of objects (P_4). If we continue to evolve the curve (b), the deletion of vertices guided by our relevance measure results in a process of repeated linearization. This way the original line segments can be recovered in noisy images, see Figure 2(c) (cf. Brunn et al. [3], Fig. 4).

Now we give a more formal justification of the above properties. The reduction of shape complexity of a polygonal curve during the evolution process (P_1) is justified by Proposition 1. Additionally, the shape complexity of a polygonal curve can be measured by the sum of the absolute values of the turn angles. Let C be a closed polygonal curve with vertices $v_0, ..., v_{n-1}$. Then the **shape complexity** of C is given by

$$SC(C) = \sum_{i=0}^{n-1} |turn(v_i)|,$$

where $turn(v_i)$ is the turn angle at vertex v_i in C. Clearly, the shape complexity of any closed convex curve is 2π and the shape complexity of a closed non-convex curve is greater than 2π.

Proposition 2. *The shape complexity $SC(C)$ of a closed polygonal curve C is monotonically decreasing in the course of the discrete evolution, i.e., if $C = C^0, ..., C^m$ with $|C^m| \leq 3$ is a sequence of simplified curves obtained by the evolution of C, then $SC(C^k) \geq SC(C^{k+1})$ for $0 \leq k \leq m-1$.*

Proof: The curves C^k and C^{k+1} differ by at least one vertex, say $v_d \in C^k \setminus C^{k+1}$. Let v_{d-1} and v_{d+1} denote the neighbor vertices of v_d in C^k, and let A be the polygonal subarc of C^k composed of the four digital line segments whose endpoints are vertices v_{d-1}, v_d, v_{d-1}. If A is a convex arc, then $SC(C^k) = SC(C^{k+1})$ (e.g, see Figure 5(a)). If A is not a convex arc, then $SC(C^k) > SC(C^{k+1})$ (e.g, see cases (b), (c), and (d) in Figure 5). ■

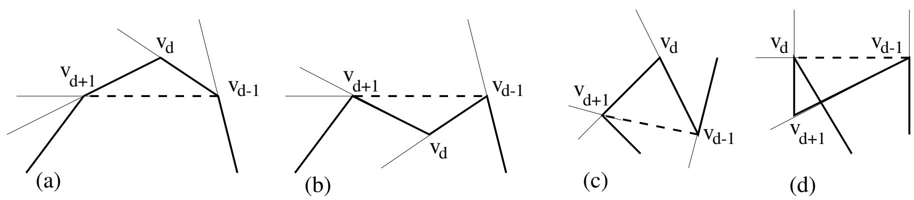

Fig. 5. The shape complexity remains the same (a) or decreases (b), (c), and (d) after a single vertex has been deleted.

The following proposition is a direct consequence of the definition of the evolution procedure:

Proposition 3. *Let $C = C^0, ..., C^m$ with $|C^m| \leq 3$ be a sequence of simplified curves obtained by the discrete evolution. For every vertex v of digital polygonal curve C that also belongs to C^k, the position of v on the plane as vertex of C is the same as the position of v as a vertex of C^k.* □

From Proposition 3, it clearly follows that there is no dislocation of the remaining features during the curve evolution. Thus, in our approach the correspondence problem of coarse-to-fine tracking difficulties does not occur. In contrary, in the course of curve evolution guided by diffusion equations, all points with non-zero curvature change their positions during the evolution. Proposition 3 also explains why our curve evolution does not introduce any blurring (i.e., rounding) effects: In a single evolution step, all vertices remain at their Euclidean positions with exception of the removed vertices. The two neighbor vertices of a removed vertex are joined by a new line segment, which does not lead to any rounding effects.

We proved that the discrete curve evolution with the relevance measure $K(v, u, w)$ is continuous (Theorem 1 in [7]): if polygon Q is close to polygon P, then the polygons obtained by their evolution are close. Continuity guarantees us the stability of the discrete curve evolution with respect to noise (P_4), which we observed in numerous experimental results.

The fact that noise elimination takes place in early stages of the evolution is justified by the relative small values of the relevance measure for vertices resulting by noise:
Mostly, if two adjacent line segments result from noise distortions, then whenever their turn angle is relatively large, their length is very small, and whenever their length is relatively large, their turn angle is very small. This implies that

if arc $\overline{vu} \cup \overline{vw}$ results from noise distortions, the value $K(v, u, w)$ of the relevance measure at vertex v will be relatively low with high probability. Hence noise elimination will take place in early stages of the evolution. This fact also contributes to the stability of our curve evolution with respect to distortions introduced by noise.

The justification of property (P_5) is based on the fact that the evolution of polygon Q corresponds to the evolution of polygon P if Q approximates P (Theorem 2 in [7]): If polygon Q is close to polygon P, then first all vertices of Q are deleted that are not close to any vertex of P, and then, whenever a vertex of P is deleted, then a vertex of Q that is close to it is deleted in the corresponding evolution step of Q. Therefore, the linear parts of the original polygon will be recovered during the discrete curve evolution.

4 Topology-Preserving Discrete Evolutions

Our discrete curve evolution yields results consistent with our visual perception even if the original polygonal curve P have self-intersections. However, it may introduce self-intersections even if the original curve were simple (e.g., see Figure 6). Now we present a simple modification that does not introduce any self-intersections for a simple polygon P.

We say that a vertex $v_i \in Vertices(P^i)$ is **blocked** in P^i if triangle $v_{i-1}v_iv_{i+1}$ contains a vertex of P^i different from v_{i-1}, v_i, v_{i+1}. We will denote the set of all blocked vertices in P^i by $Blocked(P^i)$.

Definition: For a given polygon P and a relevance measure K, the process of the discrete curve evolution in which

$$K_{min}(P^i) = \min\{K(u, P^i) : u \in Vertices(P^i) \setminus Blocked(P^i)\}$$

and

$$V_{min}(P^i) = \{u \in Vertices(P^i) \setminus Blocked(P^i) : K(u, P^i) = K_{min}(P^i)\}$$

will be called a **topology-preserving discrete curve evolution** (e.g., see Figure 6).

The question is whether this modified curve evolution will not prematurely terminate. This would be the case if $Vertices(P^i) = Blocked(P^i)$. It can be shown that this is not the case, i.e., it holds for $i = 0, ..., m-1$

$$Vertices(P^i) \setminus Blocked(P^i) \neq \emptyset.$$

5 Conclusions and Future Work

We presented a discrete approach to curve evolution that is based on the observation that in digital image processing and analysis, we deal only with digital curves that can be interpreted as polygonal curves without loss of information.

The main properties of the proposed discrete evolution approach are the following:

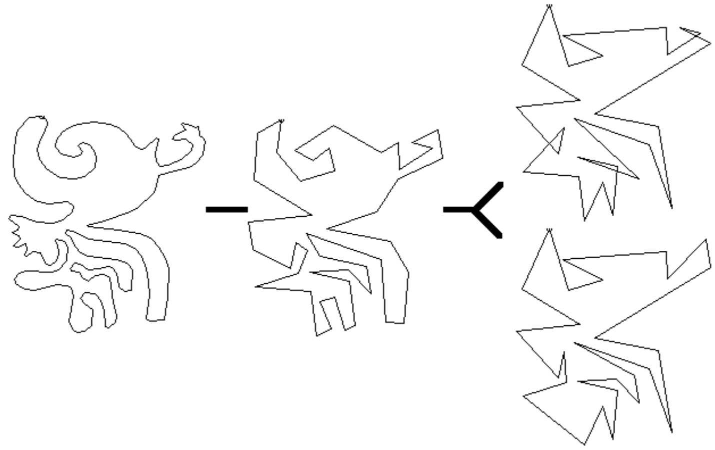

Fig. 6. The discrete curve evolution may introduce self-intersections, but after a small modification it is guaranteed to be topology-preserving.

(P_1) Analog to evolutions guided by diffusion equations, it leads to shape simplification but
(P_2) no blurring (i.e., shape rounding) effects occur and
(P_3) there is no dislocation of feature points.
(P_4) It is stable with respect to noisy deformations.
(P_5) It allows to find line segments in noisy images.

These properties are not only justified by theoretical considerations but also by numerous experimental results. Additionally, the mathematical simplicity of the proposed evolution process makes various modifications very simple, e.g., by a simple modification, a set of chosen points can be kept fixed during the evolution.

Our evolution method can be also interpreted as hierarchical approximation of the original curve by a polygonal curve whose vertices lie on the original curve. Our approximation is fine-to-coarse and it does not require any error parameters, in opposite to many standard approximations, where starting with some initial coarse approximation to a curve, whereupon line segments that do not satisfy an error criterion are split (e.g., Ramer [11]). A newer and more sophisticated split-and-merge method for polygon approximation is presented in Bengtsson and Eklundh [1], where multiscale contour approximation is obtained by varying an error parameter t, which defines a scale in a similar manner as it is the case for diffusion scale-spaces. This implies similar problems as for scale-spaces, e.g., How to determine the step size for the parameter t? Additionally, the scale-space property of shape complexity simplification does not result automatically from the approach in [1], but is enforced ([1], p. 87): *"New breakpoints, not appearing at finer scales, can occur but are then inserted also at finer levels."*

There are numerous application possibilities of our method for curve evolution in which scale-space representations play an important role, e.g., noise elimination and quality enhancement, shape decomposition into visual parts, salience measure of visual parts, and detection of critical or dominant points (Teh and Chin [13], Ueda and Suzuki [14]). The specific properties of our curve evolution yield additional application possibilities like detection of straight line segments in noisy images, which can be used for model-based shape recovery (Brunn, et al. [3]), and polygonal approximation (cf. [1]).

A paper on a discrete surface evolution that is analog to the presented polygonal evolution is in preparation.

References

1. A. Bengtsson and J.-O. Eklundh. Shape representation by mutliscale contour approximation. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 13:85–93, 1991.
2. A.M. Bruckstein, G. Shapiro, and C. Shaked. Evolutions of planer polygons. *Int. J. of of Pattern Recognition and AI*, 9:991–1014, 1995.
3. A. Brunn, U. Weidner, and W. Förstner. Model-based 2d-shape recovery. In *Proc. of 17. DAGM Conf. on Pattern Recognition (Mustererkennung)*, pages 260–268, Bielefeld, Springer-Verlag, Berlin, 1995.
4. M.A. Grayson. The heat equation shrinks embedded plane curves to round points. *Pattern Recognition*, 26:285–314, 1987.
5. B. B. Kimia and K. Siddiqi. Geometric heat eequation and nonlinear diffusion of shapes and images. *Computer Vision and Image Understanding*, 64:305–322, 1996.
6. B. B. Kimia, A. R. Tannenbaum, and S. W. Zucker. Shapes, shocks, and deformations. i: The components of shape and the reaction-diffusion space. *Int. J. Computer Vision*, 15:189–224, 1995.
7. L. J. Latecki, R.-R. Ghadially, R. Lakämper, and U. Eckhardt. Continuity of the discrete curve evolution. In *SPIE and SIAM Conf. on Vision Geometry VIII*, July 1999, to appear.
8. L. J. Latecki and R. Lakämper. Convexity rule for shape decomposition based on discrete contour evolution. *Computer Vision and Image Understanding*, 73:441–454, 1999.
9. L. J. Latecki, R. Lakämper, and U. Eckhardt. http://www.math.uni-hamburg.de/home/lakaemper/shape.
10. F. Mokhtarian and A. K. Mackworth. A theory of multiscale, curvature-based shape representation for planar curves. *IEEE Trans. PAMI*, 14:789–805, 1992.
11. U. Ramer. An iterative procedure for the polygonal approximation of plane curves. *Computer Graphics and Image Processing*, 1:244–256, 1972.
12. J.A. Sethian. *Level Set Methods*. Cambridge University Press, Cambridge, 1996.
13. C.-H. Teh and R. T. Chin. On the detection of dominant points on digital curves. *IEEE Trans. PAMI*, 11:859–872, 1989.
14. N. Ueda and S. Suzuki. Learning visual models from shape contours using multiscale convex/concave structure matching. *IEEE Trans. PAMI*, 15:337–352, 1993.
15. J. Weickert. A review of nonlinear diffusion filtering. In B. M. ter Haar Romeny, L. Florack, J. Koenderink, and M. Viergever, editors, *Scale-Space Theory in Computer Vision*, pages 3–28. Springer, Berlin, 1997.
16. A.P. Witkin. Scale-space filtering. In *Proc. IJCAI*, volume 2, pages 1019–1022, 1983.
17. www site. http://www.ee.surrey.ac.uk/research/vssp/imagedb/demo.html.