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Polygonizations of Point Sets in the Plane

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Abstract. We examine the different ways a set of n points in the plane can be connected to form a simple polygon. Such a connection is called a *polygonization* of the points. For some point sets the number of polygonizations is exponential in the number of points. For this reason we restrict our attention to star-shaped polygons whose kernels have nonempty interiors; these are called *nondegenerate* star-shaped polygons.

We develop an algorithm and data structure for determining the nondegenerate star-shaped polygonizations of a set of n points in the plane. We do this by first constructing an arrangement of line segments from the point set. The regions in the arrangement correspond to the kernels of the nondegenerate star-shaped polygons whose vertices are the original n points. To obtain the data structure representing this arrangement, we show how to modify data structures for arrangements of lines in the plane. This data structure can be computed in $O(n^4)$ time and space. By visiting the regions in this data structure in a carefully chosen order, we can compute the polygon associated with each region in $O(n)$ time, yielding a total computation time of $O(n^5)$ to compute a complete list of $O(n^4)$ nondegenerate star-shaped polygonizations of the set of n points.

1. Introduction

Many problems in computational geometry and pattern recognition have to do with constructing objects from sparse partial information. Given a set of points in the plane, pattern-recognition researchers have been interested in relationships between human perceptions of the patterns formed and theoretical computational representations of the point set [1], [13], [15], [16], [22]. Computational geometers, taking a slightly different viewpoint, have examined ways in which a set of points in the plane can be connected to form a simple polygon. We call such a connection a *polygonization* of the points.

The number of polygonizations of a set of n points is known to be exponential in n [9]. By placing further restrictions on the polygonizations, we hope to make the problem of generating polygonizations more tractable. One restriction is to require polygons to be *orthogonal* (all sides are parallel to the x- or y-axis). The problem of determining whether a polygonization exists in this setting, where more than two points are allowed on a side, is NP-complete [18]. If exactly two points are allowed on each edge, this existence problem is solvable in polynomial time, and the solution, if it exists, is unique [14]. In this paper we look for *star-shaped* polygonizations, and again find it necessary to place further restrictions on the problem.

A related problem is that of connecting line segments in the plane to form simple polygons. A restricted version of this problem is solved in [20], but the general problem has been shown to be NP-Complete [19].

2. Fundamentals

In this paper we assume that we have a set of n points in the plane, no three of which are collinear, and that the points are labeled with the integers from 1 to n, $n \geq 3$. Figure 1 will be used to illustrate many of the ideas presented.

A polygonization of a set of n points is designated by listing the points in the order in which they are connected. The points are listed in counterclockwise order; that is, in moving from one point to the next in the polygonization, the interior of the polygon is on the left. The polygonization is represented as a circular list of points $(i_1, i_2, \ldots, i_n)$, where i_1 is connected to i_2, i_2 to $i_3, \ldots, i_n$ to i_1, and the choice of starting point is arbitrary. For the point set of Fig. 1, $(1, 3, 2, 4, 5)$ and $(3, 2, 4, 5, 1)$ represent the same polygonization, but $(4, 1, 5, 3, 2)$ represents a different one.

A polygonization of n points defines a simple polygon P. In the terminology of Shamos [21], a polygon P is *star-shaped* if there is some point in P that can "see" all other points of P. The set of all such points is called the *kernel* of P; that is, $\ker(P) = \{x \text{ in } P \mid \forall y \text{ in } P, \overline{xy} \text{ is in } P\}$. Thus, a star-shaped polygon is one whose kernel is nonempty. Shamos [21] has proved several results about

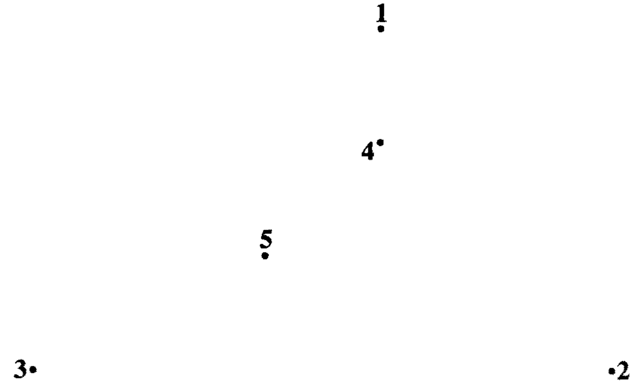

Fig. 1

star-shaped polygons. First, the kernel of any polygon P is the intersection of all the half-planes formed by the edges of P. For a given edge, the half-plane to be intersected is the one lying on the same side of that edge as the interior of P. This intersection can be computed in $O(n \log n)$ time. Moreover, the kernel of P is a convex polygon contained in P. We call a star-shaped polygon whose kernel has a nonempty interior a *nondegenerate* star-shaped polygon.

3. Star-Shaped Polygonizations of a Point Set

Next we prove two results concerning the existence and uniqueness of star-shaped polygonizations for a set of points in the plane. The techniques used in this section are due to Graham [6].

Theorem 3.1. *For a given set of $n \geq 3$ points in the plane, no three collinear, and a point r inside the convex hull of the set, there exists at least one star-shaped polygonization of the set that contains r in its kernel.*

Proof. Pick any ray emanating from r. For each point i in the point set, compute the counterclockwise angle from the chosen ray to the ray $\overrightarrow{ri}$. Sort the angles from smallest to largest and connect the points in the corresponding order, with the last being connected to the first. The rays $\overrightarrow{ri}$ divide the plane into wedges. Since r is interior to the convex hull of the set of points, the angle of each of these wedges is less than 180°, so the wedges are convex. Thus, each wedge contains the edge of the polygon that joins the points on the rays bounding it. Because each edge of the polygon is confined to a unique wedge, the polygon is simple. The fact that the wedge angles are less than 180° also ensures that r is in the interior half-plane formed by each edge, so that r is in the kernel of the polygon. Some ambiguity is possible here if two angles are equal. If so, connect the points with equal angles in either order, and r will be in the kernel of both of the resulting polygons. □

In order to eliminate the ambiguity present in Theorem 3.1, we prove the following uniqueness theorem.

Theorem 3.2. *For a set of $n \geq 3$ points in the plane, no three collinear, let r be a point interior to the convex hull of the set. Suppose that if r is collinear with any two points of the set (in this case, r is not one of the original n points), the two points lie on opposite sides of r. Then there is a unique star-shaped polygonization of the set with r in its kernel.*

Proof. Suppose P and Q are two different polygonizations of the point set, each containing r in its kernel. Let i be a point in the point set whose successor j in P is different from its successor k in Q. Without loss of generality we can assume that the counterclockwise angle θ from $\overrightarrow{ij}$ to $\overrightarrow{ik}$ satisfies $0 < \theta < \pi$, since no three points in the point set are collinear.

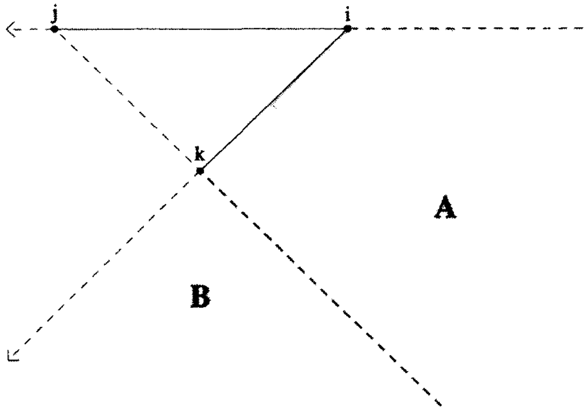

Fig. 2

Because r is in the kernel of P, it must lie to the left of $\overrightarrow{ij}$ in Fig. 2. (In general, this means that r lies in the half-plane to the left as we move out along the ray from i to j.) Similarly, because r is in the kernel of Q, it lies to the left of $\overrightarrow{ik}$. By hypothesis, r cannot lie on any of the line segments in Fig. 2 except possibly the segment $\overline{ik}$. If r lies in region A or on segment $\overline{ik}$, then in polygon Q the edge $\overline{ik}$ obscures j from the sight of r, contradicting r being in the kernel of Q. If r lies in the open region B, then the edges of polygon P adjacent to k obscure at least part of edge $\overline{ij}$ in P from the sight of r, contradicting r being in the kernel of P. Thus, r cannot be in the kernel of two distinct star-shaped polygons. □

4. Arrangements for Computing Star-Shaped Polygons

We now construct an arrangement of line segments determined by the n points of our set of points. Each region of this arrangement will be shown to be the kernal of one of the nondegenerate star-shaped polygonizations of the original n points.

Let i and j denote points of the point set. The arrangement consists of three sets of line segments:

(i) The line segments $\overline{ij}$ on the boundary of the convex hull of the set.
(ii) For each pair of points, i on the boundary of the convex hull and j in the interior of the convex hull, the line segment determined by intersecting the ray opposite to $\overrightarrow{ji}$ with the convex hull.
(iii) For each pair of points, i and j, both in the interior of the convex hull, the two line segments determined by intersecting the rays opposite $\overrightarrow{ji}$ and $\overrightarrow{ij}$ with the convex hull.

The arrangement for the point set in Fig. 1 is given in Fig. 3. In the arrangement of Fig. 3, segments $\overline{1\ 2}$, $\overline{2\ 3}$, and $\overline{3\ 1}$ form the boundary of the convex hull and

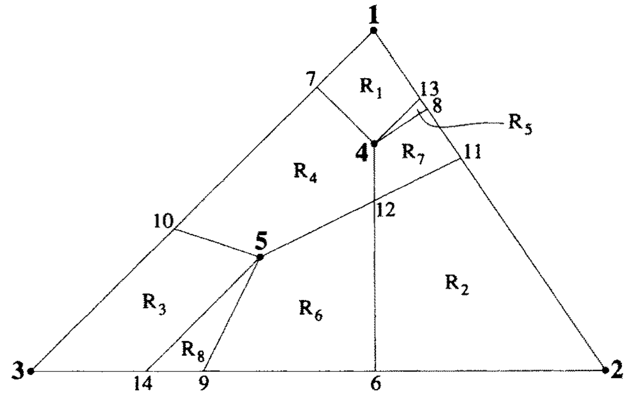

Fig. 3

are of type (i). Segments $\overline{4\ 13}$ and $\overline{5\ 14}$ are of type (iii), generated by points 4 and 5. The remaining segments are of type (ii).

Each of the original n points lying in the interior of the convex hull must have at least three segments of type (ii) incident to it, because the boundary of the convex hull must contain at least three of the original points. Thus, the arrangement of line segments divides the convex hull into (closed) regions. It follows from Theorem 3.2 that each point in the interior of one of the regions is in the kernel of a unique star-shaped polygon. The next three theorems demonstrate the relationship between the star-shaped polygonizations of a set of n planar points and the regions of the associated arrangement.

Theorem 4.1 *Let r_1 and r_2 be two points in the interior of the same region of the arrangement associated with a set of $n \geq 3$ points, no three of which are collinear. Then the unique star-shaped polygon containing r_1 in its kernel is the same as the unique star-shaped polygon containing r_2 in its kernel.*

Proof. Suppose that the polygonization associated with r_1 is different from the polygonization associated with r_2. Then there must be three points, i, j, and k, in the original point set with the property that a ray sweeping counterclockwise from $\overrightarrow{r_1k}$ intercepts the three points shown in the order kji and a ray sweeping counterclockwise from $\overrightarrow{r_2k}$ intercepts the three points in the order kij. A partial picture of the arrangement associated with the point set is shown in Fig. 4.

Notice that any of R_2, R_3, and R_4 might be empty if i, j, or k is on the convex hull, respectively. Moreover, in the complete arrangement, the regions R_1, R_2, R_3, and R_4 might possibly be subdivided further. The point r_1 must lie in region R_1, but the point r_2 must lie in one of regions R_2, R_3, or R_4. Consequently, r_1 and r_2 must lie in different regions of the arrangement. This is a contradiction. □

Theorem 4.2. *Let R_1 and R_2 be two different regions of the arrangement associated with a set of $n \geq 3$ points, no three of which are collinear. Then the unique star-shaped*

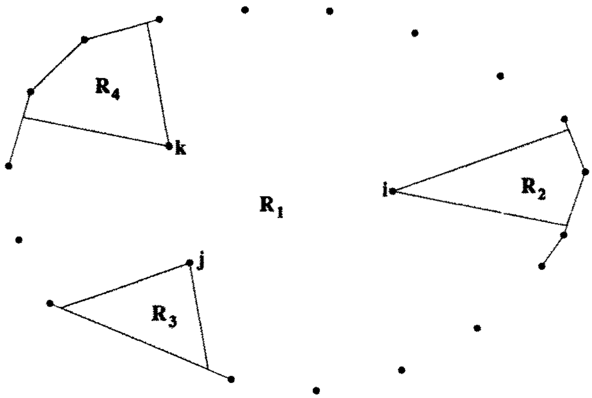

Fig. 4

polygon associated with R_1 is different from the unique star-shaped polygon associated with R_2.

Proof. Let P_1 be the polygonization associated with R_1 and P_2 be the polygonization associated with R_2. We begin by considering the case where R_1 and R_2 are adjacent regions; that is, R_1 and R_2 share a common edge L in the interior of the arrangement. L must have been generated by two points i and j where at least i is interior to the convex hull of the set of points. Let k be any third point in the set. A partial picture of the arrangement is shown in Fig. 5.

As in the proof of Theorem 4.1, any of the regions can be subdivided further in the complete arrangement. Furthermore, j or k could be on the convex hull, in which case the associated region would disappear. Points r inside the shaded regions have the property that a ray sweeping counterclockwise from $\overrightarrow{rk}$ intersects the three points shown in the order kij. Points r outside the shaded regions and inside the convex hull have the property that a ray sweeping counterclockwise

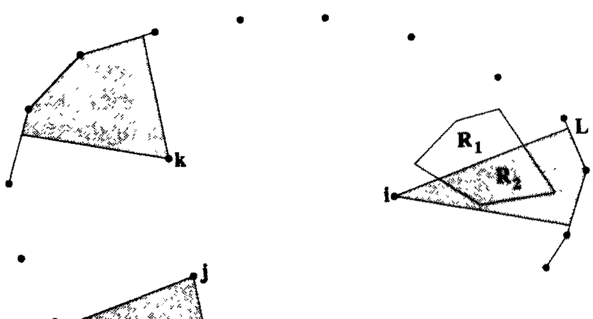

Fig. 5

from $\overrightarrow{rk}$ intersects the three points shown in the order kji. Because R_1 lies outside the shaded regions and R_2 lies inside a shaded region, they must give rise to different polygonizations.

Next suppose that R_1 and R_2 are not adjacent but that they give rise to the same polygon P. Both R_1 and R_2 lie inside the kernel of P, which is a convex set. This can happen only if at least one other region is also in the kernel of P. At least one such region must be adjacent to R_1 or R_2. We have already seen that this cannot happen, so R_1 and R_2 must have different associated polygonizations. □

It follows from Theorems 4.1 and 4.2 that each region of the arrangement has an associated star-shaped polygonization of the point set. Not all star-shaped polygonizations have associated regions, however; any degenerate star-shaped polygonization of the point set corresponds to a single point or line segment in the arrangement. For example, in the arrangement of Fig. 6, point r is the intersection of the three segments generated by the point pairs $(2,5)$, $(3,6)$, and $(1,4)$. Regions R_1–R_6 are associated with the six nondegenerate star-shaped polygonizations containing r in their kernels. The two degenerate polygonizations $(1,6,3,5,2,4)$ and $(1,4,3,6,2,5)$ have kernel r and no associated region.

In general, the more line segments that intersect in a given point in the arrangement, the more degeneracies arise. A point that is the intersection of k line segments lies in the kernels of 2^k star-shaped polygonizations. The authors are grateful to Douglas Dunham for pointing this out.

Theorem 4.3. *For a set of $n \geq 3$ points in the plane, no three of which are collinear, there is a one-to-one correspondence between the nondegenerate star-shaped polygonizations of the set and the regions determined by the arrangement of line segments associated with this set. Moreover, each region is the kernel of its associated polygon.*

Proof. Let R be a region in the arrangement, and let r be any point in the interior of R. If P is the unique star-shaped polygon containing r in its kernel,

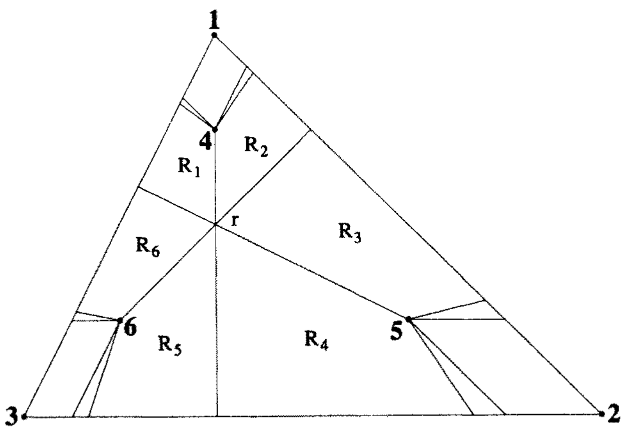

Fig. 6

then Theorem 4.1 implies that R is contained in the kernel of P. Furthermore, because R has a nonempty interior, P must be nondegenerate. By Theorem 4.2, the exterior of R does not intersect the kernel of P, and hence the kernel of P is contained in R. Thus, R and the kernel of P are equal.

Now let P be any nondegenerate star-shaped polygon on the point set. Because the kernel of P has a nonempty interior, which lies inside the convex hull of the point set, the kernel of P must have nonempty intersection with the interior of some region R of the arrangement. Then R is the region associated with P, and the kernel of P is R. □

Corollary 4.4. *Kernels of different polygonizations of a set of n points are disjoint except for the boundaries, and the union of all these kernels is the convex hull of the set.*

5. Generating Nondegenerate Star-Shaped Polygonizations

To represent the arrangement of line segments described in Section 4, we replace each line segment with the line on which it lies and use the techniques for constructing arrangements of lines in the plane developed by Chazelle *et al.* [3] and Edelsbrunner *et al.* [5]. Data structures for representing such arrangements are described in [7], [8], [11], and [17]. From n points in the plane arise $O(n^2)$ lines, and the arrangement can be constructed in $O(n^4)$ time and space. To maintain the original line-segment arrangement, we designate the edges coming from that arrangement as *real* edges and the rest as *dummy* edges [2], [4]. See Fig. 7 for the line arrangement on our set of five points.

It is now possible to traverse the regions of the line-segment arrangement by doing a depth-first search of the dual graph; by visiting the regions in this order, we always move from a region to an adjacent region, and the corresponding

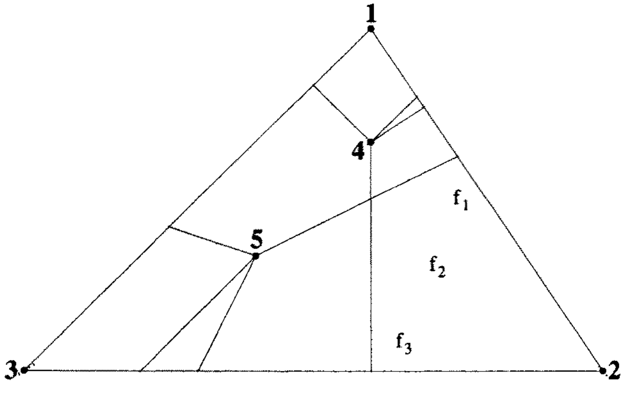

Fig. 7

polygonizations differ only by interchanging the pair of points corresponding to the real edge crossed. Thus, to print the entire list of $O(n^4)$ nondegenerate star-shaped polygonizations we do $O(n)$ work at each step to switch one pair of points and print the new polygonization, yielding a total computation time of $O(n^5)$.

Theorem 5.1. *A data structure representing the arrangement of line segments described in Section 4 for a set of $n \geq 3$ planar points, no three of which are collinear, can be constructed in $O(n^4)$ time and space. From this structure, a list of all nondegenerate star-shaped polygonizations of the n points can be generated in $O(n^5)$ time.*

A similar technique can be used to compute the nondegenerate star-shaped polygonization with minimum boundary length. The time to correct the boundary length when moving from one region to an adjacent one is $O(1)$, so the shortest length can be computed in $O(n^4)$ time. The minimum length nondegenerate star-shaped polygon is not necessarily a solution to the traveling salesman problem. For example, consider a set of points on two concentric circles, where the points are quite dense on the circles and the distance between the circles is large. Any star-shaped polygon on these points will have numerous edges between the two circles, whereas a traveling salesman tour will move back and forth between the circles only once.

6. Discussion and Open Problems

It is somewhat unappealing to represent an arrangement of *line segments* as an arrangement of *lines*. It would be much more aesthetic to simply build a data structure whose faces, edges, and vertices are those of the line-segment arrangement itself. Unfortunately, the time bound arguments given in [3] and [5] seem to be dependent upon the existence of complete lines in the arrangement. A straightforward $O(N^2)$ algorithm for constructing an arrangement of N line segments in the plane would be welcome.

A less straightforward and less appealing method for constructing a line-segment arrangement is to construct the associated line arrangement and then "clean it up" by removing dummy edges and vertices created by dummy edges. Examples exist that have $\Omega(n^4)$ nondegenerate star-shaped polygonizations, and, because $O(n)$ work must be done just to print a polygonization, a time bound of $O(n^5)$ cannot be beaten in the worst case. Thus, cleaning up the data structure may at best produce a lower constant factor. For other applications this approach may be more worthwhile.

Another method to consider is the plane-sweep. The methods described by Nievergelt and Preparata [12] can be adapted to construct a line-segment arrangement in time $O((N+S) \log N)$, where N is the number of line segments and S is the number of intersections. In our application with n points in the plane, $N = O(n^2)$ and $S = O(n^4)$, so at worst this is an $O(n^4 \log n)$ computation. For sets where S is known to be relatively small, however, this method may well be more practical than the $O(n^4)$ techniques.

Although the results in [14] on orthogonal polygonizations extend to three dimensions, the results in this paper do not. In R^3, a region of space can lie in the kernels of exponentially many polytopes on a point set, so the decomposition of the convex hull into kernels of distinct polygonizations breaks down. For example, take $n-1$ points forming a convex set in the xy-plane, and triangulate the set of points. Now lift some of the points out of the plane in the z-direction, along with the part of the xy-plane within their convex hull. This can be done so that creases develop only along the edges of the triangulation, creating a multifaceted surface. To complete the construction, place an nth point at some large distance above the xy-plane and connect it with the remaining points to form a cone-shaped polytope, where the first $n-1$ points lie around the base and last point is at the apex. Each different triangulation of the first $n-1$ points gives rise to a different polytope, and all of these polytopes have kernels which contain the apex of the cone and a neighboring region inside the cone. Because a convex set of $n-1$ points in the plane has $\Omega(4^n n^{-3/2})$ triangulations [10], there are $\Omega(4^n n^{-3/2})$ of these polytopes.

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