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POLYGONS HAVE EARS

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We refer to a simple closed polygonal plane curve with a finite number of sides as a *Jordan polygon*. We assume the truth of the famous Jordan Curve Theorem only for Jordan polygons. (For elementary proofs see Appendix 2 of Chapter V of [4] or Appendix B1 of [5].) Three consecutive vertices V_1, V_2, V_3 of a Jordan polygon $P = V_1V_2V_3V_4 \cdots V_nV_1$ ($n \geqq 4$) are said to form an *ear* (regarded as the region enclosed by the triangle $V_1V_2V_3$) at the vertex V_2 if the (open) chord joining V_1 and V_3 lies entirely inside the polygon P. We say that two ears are non-overlapping if their interior regions are disjoint; otherwise they are *overlapping*. If we *remove* or cut off an ear $V_1V_2V_3$ (by drawing the chord V_1V_3) from the Jordan polygon P, then there remains the Jordan polygon $P' = V_1V_3V_4 \cdots V_nV_1$ which has one less vertex than P.

The property of Jordan polygons expressed by the following theorem seems to provide a particularly simple and conceptual bridge from the Jordan Curve Theorem for Polygons to the Triangulation Theorem for Jordan Polygons; at least simpler perhaps than that given in Appendix B2 of [5].

TWO EARS THEOREM. *Except for triangles, every Jordan polygon has at least two non-overlapping ears.*

Proof. Our proof is by induction on the number n of vertices of the Jordan polygon P. Since the proof for quadrilaterals as well as the proof for the general case ($n > 4$) can be divided into the same two cases, for the sake of brevity, we deal with quadrilaterals at the beginning of each of these two cases.

Let P denote a Jordan polygon with at least four vertices, select a vertex V of P at which the interior angle is less than 180°, and let V_- and V_+ denote the vertices of P which are adjacent to V. (Any V on any minimal triangle enclosing P will do.)

Case 1. The polygon P has an ear at V. If we remove this ear, then the remain-

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ing polygon P' is either a triangle (and hence forms another ear for P which is non-overlapping with the ear at V) or else is a Jordan polygon with more than three vertices, but with one less vertex than P, so that the induction hypothesis yields two non-overlapping ears E_1 and E_2 for P'. Since they are non-overlapping, at least one of these two ears, say E_1, is not at either of the vertices V_- and V_+. Since all ears of P' (except for any which might occur at V_- or V_+) are also ears of P, the two ears E_1 and V_-VV_+ are non-overlapping ears for P.

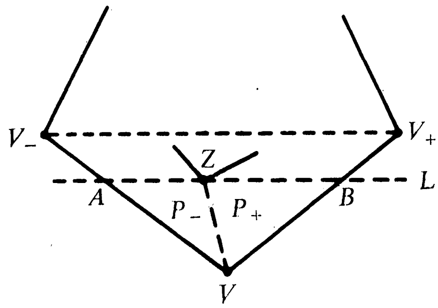

FIG. 1.

Case 2. Polygon $P = VV_+ \ldots Z \ldots V_- V$ does not have an ear at V. Chord VZ divides P into two Jordan polygons $P_- = VZ \ldots V_- V$ and $P_+ = VV_+ \ldots ZV$.

Case 2. The polygon P does not have an ear at V. (See Fig. 1.) Then the triangle V_-VV_+ must contain in its interior or on the open chord V_-V_+ at least one vertex of P. Let Z denote one such vertex with the additional property that the line L through it and parallel to V_-V_+ is as close to V as possible. Clearly such a Z must exist. If A and B denote the points of intersection of this line L with the chords VV_- and VV_+, respectively, then the open triangular region VBA cannot contain any vertex (or edge points) of P. Hence the (open) chord VZ lies entirely inside the Jordan polygon P and so divides it into two Jordan polygons $P_- = VZ \cdots V_-V$ and $P_+ = VV_+ \cdots ZV$, each with fewer vertices than P. (The polygon P_- does not contain V_+, and the polygon P_+ does not contain V_-.) If P is a quadrilateral, then VV_-Z and VV_+Z are two non-overlapping ears for P. On the other hand, if P is not a quadrilateral, then at least one of the two polygons, say P_+ is not a triangle.

Case 2a. The polygon P_- is a triangle. Then VV_-Z is an ear for P and the polygon P_+ must possess (by the induction hypothesis) two non-overlapping ears E_1 and E_2, at least one of which, say E_1, is not at either of the vertices V or Z. This ear E_1 for P_+ is then also an ear for P and obviously does not overlap with the ear VV_-Z.

Case 2b. The polygon P_- is not a triangle. The induction hypothesis now yields four mutually non-overlapping ears, two (E_1^+, E_2^+) for P_+ and two (E_1^-, E_2^-) for P_-.

At least one of the pair (E_1^-, E_2^-), say E_1^-, is not at either of the vertices V and Z; and, similarly, at least one of the pair (E_1^+, E_2^+), say E_1^+, is not at either of these vertices either. (The vertices V and Z of P are the only vertices common to P_- and P_+.) Consequently, E_1^- and E_1^+ will be (non-overlapping) ears for the original polygon P. This completes the proof of the theorem.

APPLICATION. A procedure for triangulating a Jordan polygon without introducing any new vertices is now immediately obvious: Namely, locate an ear $V_1V_2V_3$ and cut it off, then locate an ear of the remaining polygon (of one less vertex!) and cut it off, and continue this process until the remaining polygon is itself a triangle.

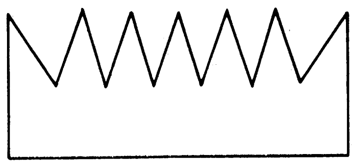

FIG. 2 — A polygon with many ears.

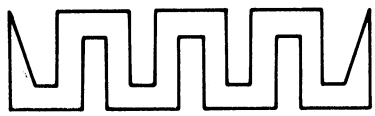

FIG. 3 — A polygon with many vertices, but only two ears.

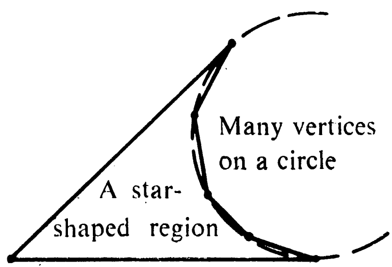

FIG. 4 — A star-shaped polygon with many vertices, but only two ears.

Note that while some Jordan polygons may have many ears (as in Fig. 2), others have only two ears no matter how many vertices they may have (as in Figs. 3 and 4). Figure 4 is also an example of a Jordan polygon with many vertices but only three interior angles less than π, and this is the minimum that is possible. Of course a Jordan polygon is convex if and only if it has an ear at every vertex.

Now consider the following two statements for Jordan polygons P with $n > 3$ vertices.

(I) *The polygon P has at least two non-overlapping ears.*

(II) *The polygon P admits a triangulation $\mathscr{T}$ with no new vertices.*

We have seen above that on the one hand (I) can be proved without assuming (II) and on the other hand (II) is an immediate consequence of (I). We now show that, conversely, (II) implies (I). Let t denote the number of triangles in $\mathscr{T}$. Then $t = t_0 + t_1 + t_2$ where t_j denotes the number of triangles in $\mathscr{T}$ which have exactly

j sides in common with P. Note that each triangle of $\mathcal{T}$ which has exactly two sides in common with P is an ear for P and no two such ears overlap. Now it can be shown that the sum Σ of the interior angles of a Jordan polygon is equal to $(n-2)\pi$. But also $\Sigma = t\pi$. Consequently, $t_0 + t_1 + t_2 = n-2$. Furthermore, it is clear that $t_1 + 2t_2 = n$. Eliminating t_1 we obtain $t_2 = t_0 + 2 \geqq 2$.

The "Shelling Theorem" on page 32 of [**1**] is closely related to our Two Ears Theorem, but our proof is entirely different and was carried out before we discovered this reference.

We close with two applications of our Two Ears Theorem.

(1) From Euler's formula $V - E + F = 1$ and the fact (just proved) that $t = n-2$, it follows that every triangulation with no new vertices contains $n-3$ non-crossing diagonals.

(2) From the existence of an ear for P and by Mathematical Induction it follows immediately that the sum Σ of the interior angles of P is equal to $[(n-1)-2]\pi + \pi = (n-2)\pi$.

For some interesting applications of triangulation of polygons see [**2**] and [**3**].

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