Polyhedral divisors of Cox rings

Jarosław Wiśniewski, reporting a work with Klaus Altmann

Cox rings and MDS

- Let Z be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Cl}(Z)$ is a lattice. We define

$$\mathrm{Cox}(Z) = \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}(D))$$

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  with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$.
- Assume $\mathrm{Cox}(Z)$ finitely generated and call Z a Mori Dream Space (or MDS). The $\mathrm{Cl}(Z)$-grading of $\mathrm{Cox}(Z)$ yields action of torus $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathrm{Cl}(Z))}$ on the affine variety $\mathrm{Spec}(\mathrm{Cox}(Z))$.

polyhedral group

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- If $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone, then we denote by $\mathrm{Pol}(N_{\mathbb{Q}}, \sigma)$ the Grothendieck group of the semigroup

$$\mathrm{Pol}^+(N_{\mathbb{Q}}, \sigma) := \{\Delta \subseteq N_{\mathbb{Q}} \mid \Delta = \sigma + [\text{compact polytope}]\}$$

with respect to Minkowski addition. Via $a \mapsto a + \sigma$, the latter contains $N_{\mathbb{Q}}$. Moreover, $\mathrm{tail}(\Delta) := \sigma$ is called the tail cone of the elements of $\mathrm{Pol}(N_{\mathbb{Q}}, \sigma)$.

polyhedral divisors

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- Moreover, it is called *semiample* if the evaluations $\mathcal{D}(u) := \sum_i \min\langle \Delta_i, u\rangle D_i$ are semiample for $u \in \sigma^\vee \cap M$

p-divisors $\longrightarrow$ varieties

- For $u \in \sigma^\vee$ we get $\min\langle \Delta_i, u\rangle > -\infty$ and therefore $\mathcal{D}$ defines a function $\sigma^\vee \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ denoted by the same name. By abuse, $\mathcal{D} : M_{\mathbb{Q}} \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ however $\Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))) = 0$ makes sense for $u \notin \sigma^\vee$.

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- Semiample polyhedral divisors are called *p-divisors*. We get a structure of finitely generated $\mathbb{C}$-algebra on $\bigoplus_{u \in M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$ and define

$$X := X(\mathcal{D}) := \operatorname{Spec} \bigoplus_{u \in M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$$

which is an affine variety with T action.

p-divisors $\leftrightarrow$ varieties

- The variety $X(\mathcal{D})$ does not change if $\mathcal{D}$ is pulled back via a birational modification $Y' \to Y$ or if $\mathcal{D}$ is altered by a polyhedral *principal* divisor from Y. P-divisors that differ by (a chains of) those operations are called equivalent.

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- **Task:** Find description of $\mathrm{Cox}(\bullet)$ in terms of $\mathcal{D}_{\mathrm{Cox}}$.
- **Motivation:** Zariski decomposition, base point loci loci and multiplicities.

warnings

- Equivalent divisors define the same map $N_{\mathbb{Q}} \to \operatorname{CaDiv}_{\mathbb{Q}}(Y)/\operatorname{PDiv}(Y) \neq \operatorname{Pic}_{\mathbb{Q}}(Y)$ which we can extend to $N_{\mathbb{Q}} \to \operatorname{Pic}_{\mathbb{Q}}(Y)$.

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- The effective grading of $\mathrm{Cox}(\bullet)$ is in the cone of effective divisors $\mathrm{Eff} \subset \mathrm{Cl}_{\mathbb{Q}}$ while the values of the expected p-divisor is in the cone of semi-ample divisors $\mathrm{Nef} \subset \mathrm{Pic}_{\mathbb{Q}}$ hence $\mathcal{D}_{\mathrm{Cox}}$ defines a map

$$\mathrm{Cl}_{\mathbb{Q}} \supset \mathrm{Eff} \to \mathrm{Nef} \subset \mathrm{Pic}_{\mathbb{Q}}$$

Zariski decomposition

- Let D be an effective $\mathbb{Q}$ divisor on a surface S. Then $D = P + N$ where $P \in \mathrm{Nef}(S)$ and N is effective, if non-empty then supported on a contractible divisor transversal to P. Moreover for $n \geq 0$ we have

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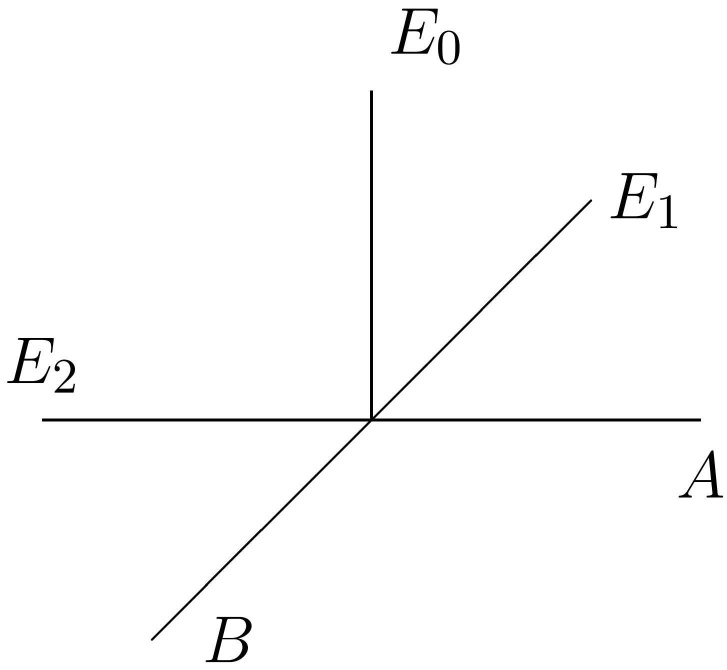
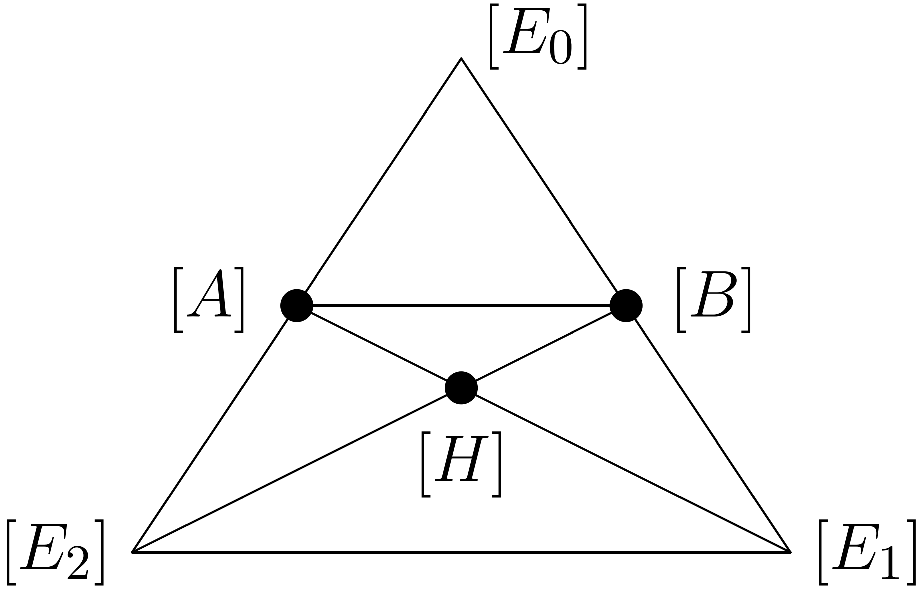
$$\mathcal{D}_{\mathrm{Cox}} : \mathrm{Cl}_{\mathbb{Q}}(S) \supset \mathrm{Eff}(S) \to \mathrm{Nef}(S) \subset \mathrm{Pic}_{\mathbb{Q}}(S)$$

is a piecewise linear retraction of cones.

$\mathbb{P}^2$ blown-up in two points





$$\mathcal{D} = \mathrm{id}_{\mathrm{Cl}(S)} + \sum_i \left(\overline{0[E_i]} + \mathrm{Nef}(S) \right)$$

MDS and SQM's

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MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel]
Z_i share the same Cox ring and, by strict transform, we identify $\mathrm{Div}(Z_i)$ and $\mathrm{Cl}(Z_i)$ with $\mathrm{Div}(Z)$ and $\mathrm{Cl}(Z)$, respectively; same holds for effective and movable cones

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However, the cones $\mathrm{Nef}(Z_i)$ are different, that is $\mathrm{int}\,\mathrm{Nef}(Z_i) \cap \mathrm{int}\,\mathrm{Nef}(Z_j) = \emptyset$ if $Z_i \neq Z_j$ and we have decomposition

$$\mathrm{Mov}(Z) = \bigcup_i \mathrm{Nef}(Z_i)$$

Chow limit

- GIT quotients and their morphisms form a projective system. Take its limit, normalize it and get *Chow limit* Y which dominates all Z_i via birational morphisms:

$$\operatorname{Spec}\operatorname{Cox}(Z) \dashrightarrow Y \xrightarrow{\psi_i} Z_i$$

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- Note that the maps

$$\psi_i^*,\ \psi_j^* : \operatorname{Pic}_{\mathbb{Q}}(Z_i) = \operatorname{Pic}_{\mathbb{Q}}(Z_j) \longrightarrow \operatorname{Pic}_{\mathbb{Q}}(Y)$$

are different if $i \neq j$, but

$$\psi_i^*(\operatorname{Nef}(Z_i)),\ \psi_j^*(\operatorname{Nef}(Z_j)) \subset \operatorname{Nef}(Y)$$

exceptional divisors on Y

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- An irreducible divisor $E \subset Y$ is an exceptional divisor of the second kind if it is a strict transform to Y of a (divisorial) component of an exceptional locus of a birational morphism (divisorial contraction) of a Z_i; i.e. E is a non-movable divisor on Z.

stable multiplicities

- For a birational $\psi : Y \to Z$ and a big divisor B on Z

$$\mathrm{mult}_E^{\mathrm{st}}(\psi^*[B]) := \inf_{D \in |B|_{\mathbb{Q}}} \mathrm{mult}_E(\psi^* D)$$

  where $D \in |B|_{\mathbb{Q}}$ means that D is an (effective) $\mathbb{Q}$-divisor with $mD \in |mB|$ for $m \gg 0$.

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  where $D \in |B|_{\mathbb{Q}}$ means that D is an (effective) $\mathbb{Q}$-divisor with $mD \in |mB|$ for $m \gg 0$.
- Eventually, for an MDS Z the stable multiplicity function $\mathrm{mult}_E^{\mathrm{st}} := \mathrm{mult}_E^{\mathrm{st}} \circ \psi^*$ can be extended to a concave, fanwise linear function on $\mathrm{Eff}(Z) \subseteq \mathrm{Cl}(Z)_{\mathbb{Q}}$. [Ein, Lazarsfeld, Mustaţă, Nakamaye, Popa]

the theorem

Altmann, ——: In the above situation

$$\mathcal{D}_{\mathrm{Cox}} = \psi_i^* + \sum_{E \subset Y} \Delta_E^i \otimes E$$

where E are exceptional divisors described above and

$$\Delta_E^i := \{C \in \mathrm{Cl}^*(Z_i)_{\mathbb{Q}} \mid \langle C, [B] \rangle \geq -\operatorname{mult}_E^{\mathrm{st}} \psi_i^* B\}$$

In short: for a big B on Z the class $\mathcal{D}_{\mathrm{Cox}}([B])$ is the stable base-point free part of $\psi^*(B)$.

$\mathbb{P}^3$ blown-up in two points

As a map of cones $\mathcal{D}_{\mathrm{Cox}} : \mathrm{Eff}(Z) \to \mathrm{Nef}(Y)$ it is a composition of piecewise retraction of $\mathrm{Eff}(Z)$ to $\mathrm{Mov}(Z)$ with ψ_i^* on each $\mathrm{Nef}(Z_i)$.