Journal of Statistical Physics, Vol. 51, Nos. 5/6, 1988

Polymers on Disordered Trees, Spin Glasses, and Traveling Waves

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Received January 29, 1988

We show that the problem of a directed polymer on a tree with disorder can be reduced to the study of nonlinear equations of reaction–diffusion type. These equations admit traveling wave solutions that move at all possible speeds above a certain minimal speed. The speed of the wavefront is the free energy of the polymer problem and the minimal speed corresponds to a phase transition to a glassy phase similar to the spin-glass phase. Several properties of the polymer problem can be extracted from the correspondence with the traveling wave: probability distribution of the free energy, overlaps, etc.

KEY WORDS: Disordered system; spin glass; freezing transition; reaction–diffusion equation.

1. INTRODUCTION

One of the standard problems in the theory of disordered systems is that of directed polymers in a random medium. The model is most easily explained in a continuum notation. We consider a path (= directed polymer) $x(t)$ in $(d-1)$-dimensional space. The path has the weight

$$\exp\left[-\beta\int_0^t ds\,\{\tfrac{1}{2}\dot{x}(s)^2+V(x(s),s)\}\right] \tag{1.1}$$

Here $V(x, t)$ is the disorder. We assume that V is short-range-correlated in x and t,

$$\langle V(x,t)\,V(x',t')\rangle=\sigma^2\delta(x-x')\,\delta(t-t') \tag{1.2}$$

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One is interested in the typical fluctuations of $x(t)$. For fixed disorder V, let us denote by $\mathbb{E}(\cdot)$ the normalized average over all paths with weight (1.1). $\langle \cdot \rangle$ denotes the average over the random potential V. Since $\langle \mathbb{E}(x(t)) \rangle = 0$ by symmetry, the quantity of interest is then

$$\langle \mathbb{E}(x(t)^2) \rangle \tag{1.3}$$

for large t with $x(0) = 0$. For zero disorder, $\sigma = 0$, one has diffusive growth,

$$\langle \mathbb{E}(x(t)^2) \rangle \simeq t \tag{1.4}$$

The disorder will induce stronger fluctuations, because the walk tries to take advantage of deep potential wells even at the price of paying in entropy. Disorder roughens the walk. As a quantitative measure, one introduces the wandering or roughness exponent ζ,[1,2] through

$$\langle \mathbb{E}(x(t)^2) \rangle \simeq t^{2\zeta} \tag{1.5}$$

for large t. The goal is to understand how ζ depends on the parameters of the model, such as, e.g., the noise strength σ.

For $d = 2$ one finds $\zeta = 2/3$ for any strength of disorder. This value of ζ is the result of renormalization group calculations,[3,4] of a mode–mode coupling theory,[5] of a scaling relation,[6] and of an exactly soluble particular case.[7] For $d > 3$ one expects a division into an entropy-dominated and a disorder-dominated regime. At sufficiently small noise strength, $\zeta = \frac{1}{2}$. For a particular version of (1.1) this is proved by Imbrie and Spencer.[8] Beyond a certain critical noise strength one finds a strong coupling exponent $\zeta > \frac{1}{2}$. It is known, however, only on the basis of Monte Carlo simulations. For $d = 3$ Meakin *et al.*[9] obtain $\zeta = 0.62$. Wolf and Kértesz[10] find $\zeta = 0.66$ for $d = 3$, and $\zeta = 0.59$ for $d = 4$, and conjecture $\zeta = d/(2d - 1)$. Finally, Kardar and Zhang[11] study the zero-temperature limit, $\beta \to \infty$, of (1.1). Their results are $\zeta = 0.62$ for $d = 3$, and $\zeta = 0.64$ for $d = 4$, and they conjecture $\zeta = 2/3$ independent of dimension.

The interest in the directed polymer in a random medium comes from two sources. First of all it appears as an approximation to equilibrium systems with bond disorder. In two dimensions (1.1) is the statistical weight of an interface in the SOS approximation. Then $x(t)$ refers to the height of the interface above some reference line and $V(x, t)$ can be traced back directly to the bond disorder. Walks with weight (1.1) appear also in the high-temperature expansion of diluted ferromagnets.[12] The second source of interest is the connection to ballistic deposition as described by the Kardar–Parisi–Zhang equation,[13] equivalently by the noisy Burgers equation. In fact, Meakin *et al.* simulate a surface growing through ballistic deposition and Wolf and Kértesz study numerically the surface of a large Eden cluster.

In view of the poorly understood energy-dominated regime, it is of interest to study the directed polymer on a Cayley tree, which in some sense corresponds to a mean field limit $d \to \infty$. On the Cayley tree we lose all spatial structure. Therefore (1.5) has to be replaced by energy fluctuations, which, however, have an exponent related to the roughness exponent by a scaling relation.[11,14]

The Cayley tree problem has a fascination in itself. To our own complete surprise, there is a close conection to the traveling wave solutions of the so-called Kolmogorov–Petrovsky–Piscounov (KPP) equation[15] (also called Fisher equation), a certain nonlinear partial differential equation of diffusion–reaction type. Exploiting this connection, one can study the Cayley tree problem in great detail. In fact, it shares many properties with the random energy model of spin glasses.

Our paper is organized as follows: In Sections 2 and 3 we introduce the problem of a directed polymer on a Cayley tree with disorder (continuous walk in Section 3 and discrete walk in Section 2). We show that a suitable generating function for the partition function Z satisfies a nonlinear equation, which for the continuous walk turns out to be the KPP equation. In Section 4 we recall the relevant results on the solutions of the KPP equation and in Sections 5 and 6 we use them to describe the probability distribution of Z and to calculate the overlap in the polymer problem. In particular, we show that there is a spin-glass transition for a polymer on a disordered tree and that in the spin-glass phase the dominant configurations have overlap either 0 or 1. In Section 7 we show how the problem could be generalized to present more general overlaps and in Section 8 we return to the discrete case.

2. CAYLEY TREE WITH DISORDER

Consider a Cayley tree (cf. Fig. 1), or, more precisely, a branch of a Cayley tree. Each site (except site 0) has $K+1$ neighbors. We want to study on this tree all the self-avoiding walks of t steps starting at 0. (In Fig. 1, the path $0 \to A$ is such a walk of four steps.) On each bond (i, j) of the lattice, there is a random potential V_{ij} distributed according to a given probability distribution $\rho(V_{ij})$. Potentials at different bonds are independent. By definition, the energy $E(\omega)$ of a walk ω is the sum of the potentials of the bonds visited by the walk,

$$E(\omega) = V_{0 i_1} + V_{i_1 i_2} + \cdots V_{i_n A} \tag{2.1}$$

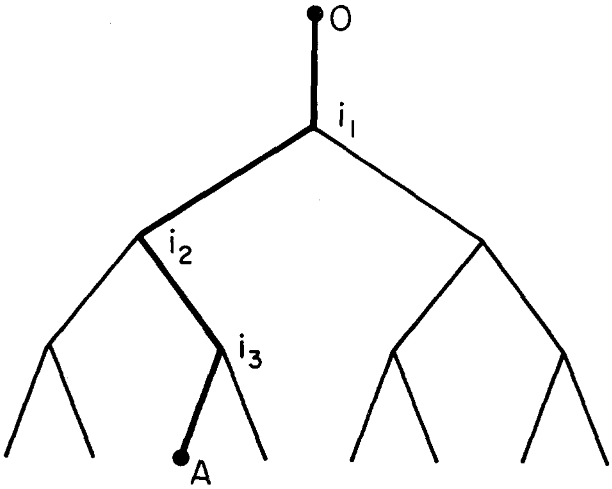

Fig. 1. Self-avoiding walk on a Cayley tree.

To describe the statistical properties of a self-avoiding walk of t steps starting at 0 on such a lattice, we have to calculate the partition function

$$Z(t) = \sum_{\omega} \exp[-\beta E(\omega)] \tag{2.2}$$

where the sum runs over the K^{t-1} walks of t steps starting 0. Here β is the inverse temperature. Because of the tree structure, the following recursion relation holds:

$$Z(t+1) = e^{-\beta V}[Z^{(1)}(t) + \cdots + Z^{(K)}(t)] \tag{2.3}$$

This recursion relation expresses the fact that each walk of $t+1$ steps starting at 0 can be decomposed into its first step on a bond of strength V and a walk of t steps in one of the K possible branches.

Since the potentials V_{ij} are random, we have to study the probability distribution $P_t(Z)$ of $Z(t)$. In Eq. (2.3), the $Z^{(i)}(t)$ are the partition functions of walks on different branches of the tree and thus they are independent random variables. Therefore, $P_t(Z)$ satisfies the recursion relation

$$P_{t+1}(Z) = \int dZ_1\, P_t(Z_1) \cdots \int dZ_k\, P_t(Z_k) \int dV\, \rho(V)\, \delta(Z - e^{-\beta V}(Z_1 + \cdots + Z_k)) \tag{2.4}$$

with initial condition

$$P_0(Z) = \delta(Z-1) \tag{2.5}$$

The recursion (2.4) with initial condition (2.5) can be used to obtain the moments of the partition function. For example, one gets

$$\langle Z(t) \rangle = K^t \mu_1^t \tag{2.6a}$$

$$\langle Z(t)^2 \rangle = K^t \mu_2^t + \frac{\mu_2(K-1)}{K\mu_1^2 - \mu_2} (K^{2t}\mu_1^{2t} - K^t \mu_2^t) \tag{2.6b}$$

$$\begin{aligned} \langle Z(t)^3 \rangle = K^t \mu_3^t &+ \frac{3K(K-1)\,\mu_3(\mu_1^2 - \mu_2)}{(K\mu_1\mu_2 - \mu_3)(K\mu_1^2 - \mu_2)} (K^{2t}\mu_1^t \mu_2^t - K^t \mu_3^t) \\ &+ \frac{(K-1)\,\mu_3[\mu_1^2 K(K-2) + \mu_2(2K-1)]}{(K^2\mu_1^3 - \mu_3)(K\mu_1^2 - \mu_2)} (K^{3t}\mu_1^{3t} - K^t \mu_3^t) \end{aligned} \tag{2.6c}$$

etc., where

$$\mu_n = \int dV\, \rho(V) e^{-n\beta V} \tag{2.7}$$

We notice that, as in spin glasses, each moment $\langle Z(t)^n \rangle$ has its own transition temperature, where the terms dominant for large t switch.

The distribution $P_t(Z)$ is determined only through the complicated integral equation (2.4), which depends on the distribution $\rho(V)$ and on the inverse temperature β. It turns out that by considering an appropriate generating function of $Z(t)$, one can obtain an integral equation independent of temperature and with a more transparent structure. Let us then define $G_t(x)$ by

$$G_t(x) = \langle \exp[-e^{-\beta x} Z(t)] \rangle \tag{2.8}$$

Using (2.4), one can show that

$$G_{t+1}(x) = \int dV\, \rho(V) [G_t(x+V)]^K \tag{2.9}$$

with the initial condition

$$G_0(x) = \exp(-e^{-\beta x}) \tag{2.10}$$

The solution $G_t(x)$ depends on temperature only through the initial condition.

It is clear from the definition (2.8) that at any time t

$$G_t(x) \to \begin{cases} 1 & \text{as} \quad x \to \infty \\ 0 & \text{as} \quad x \to -\infty \end{cases} \tag{2.11}$$

So $G_t(x)$ has the shape of a wavefront. In fact, we will see in Section 8 that in the long-time limit $G_t(x)$ is a traveling wave of the form

$$G_t(x) = w(x - ct) \tag{2.12}$$

where the velocity c is related to the initial condition, i.e., to the inverse temperature β, by

$$c = \begin{cases} \dfrac{1}{\beta} \log \left[K \int dV \ \rho(V) e^{-\beta V} \right] & \text{if} \quad \beta < \beta_c \qquad (2.13a) \\ \dfrac{1}{\beta_c} \log \left[K \int dV \, \rho(V) e^{-\beta_c V} \right] & \text{if} \quad \beta > \beta_c \qquad (2.13b) \end{cases}$$

Here β_c is defined as the inverse temperature where the velocity c of (2.13a) is minimal,

$$\frac{d}{d\beta} c(\beta_c) = 0 \tag{2.14}$$

The traveling wave comes as a sort of surprise. Its appearance and its detailed properties can be grasped more directly in a continuum approximation, to which we turn next. Then (2.9) becomes the KPP equation, a very well-studied equation.

3. CONTINUOUS TIME AND BRANCHING DIFFUSIONS

One can generalize the problem defined in Section 2 to the case of a tree branching at continuous, random rather than discrete, deterministic times. By definition of the model, the potential V on a branch of length dt is a Gaussian variable

$$\rho(V) = \frac{1}{(4\pi D \, dt)^{1/2}} \exp\left(- \frac{V^2}{4D \, dt} \right) \tag{3.1}$$

and during the time interval dt each branch has a probability $\lambda \, dt$ of branching into two branches. We could introduce branching into more than two. But since, qualitatively, our results do not depend on the branching mechanism, we stick to the simplest rule.

The recursion relation for the partition function $Z(t)$ of all walks of time span t starting at 0 is

$$Z(t+dt)=\begin{cases} e^{-\beta V}Z(t) & \text{with probability } 1-\lambda\, dt \quad (3.2a)\\ e^{-\beta V}[Z^{(1)}(t)+Z^{(2)}(t)] & \text{with probability } \lambda\, dt \quad (3.2b)\end{cases}$$

Equation (3.2a) expresses the fact that the tree has a single branch from time 0 to time dt, whereas (3.2b) represents a branching between times 0 and dt.

From (3.2) one can calculate the moments of $Z(t)$, with the result

$$\langle Z(t)\rangle = \exp\left[\left(\lambda+\frac{1}{2}\beta^2\right)t\right] \tag{3.3a}$$

$$\langle Z(t)^2\rangle = \exp[(\lambda+2\beta^2)t]+\frac{2\lambda}{\lambda-\beta^2}\{\exp[(\beta^2+2\lambda)t] - \exp[(2\beta^2+\lambda)t]\} \tag{3.3b}$$

etc. As before, the $\langle Z(t)^n\rangle$ have transition temperatures that depend on n (except for $n=1$, which has no transition).

If one defines $G_t(x)$ as in Section 2 by

$$G_t(x)=\langle \exp[-e^{-\beta x}Z(t)]\rangle \tag{3.4}$$

then one finds that for dt small

$$G_{t+dt}(x)=(1-\lambda\, dt)\int dV\,\frac{1}{(4\pi D\, dt)^{1/2}}\exp\left(-\frac{V^2}{4D\, dt}\right)G_t(x+V) + \lambda\, dt\, G_t(x)^2 \tag{3.5}$$

In the limit $dt\to 0$, (3.5) reduces to

$$\frac{\partial}{\partial t}G = D\frac{\partial^2}{\partial x^2}G+\lambda(G^2-G) \tag{3.6}$$

This is the KPP equation. By changing suitably the space and time scales, one can always set

$$D=1/2, \qquad \lambda=1 \tag{3.7}$$

As in the discrete case, the equation that governs the time evolution of G is independent of the inverse temperature β, which enters only in the initial condition,

$$G_0(x)=\exp(-e^{-\beta x}) \tag{3.8}$$

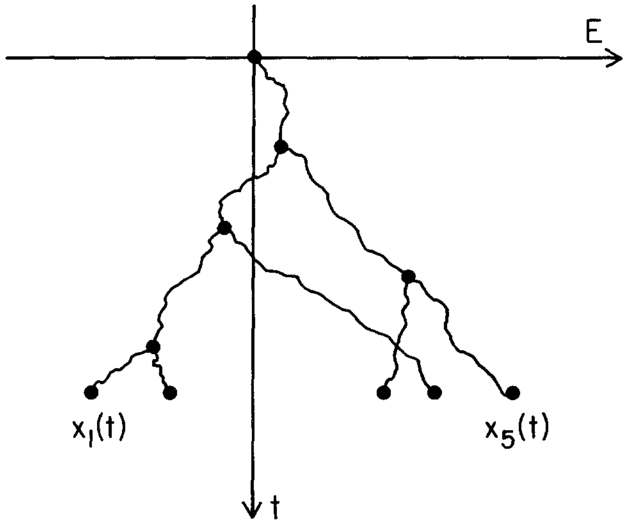

Fig. 2. Branching diffusion in energy space. Time ($\equiv$ volume) is running downward.

For our further analysis it is important to understand how the *energies* (and not only the partition function) of the walks change in time. At time t there are $n = n(t)$ walks. $n(t)$ is a random variable with mean $\langle n(t) \rangle = e^{\lambda t}$. The walks have energies $x_1(t),..., x_n(t)$. According to the rules given above, $x_1(t),..., x_n(t)$ diffuse independently of each other with diffusion coefficient D, and branch into two independently of each other with rate λ (cf. Fig. 2). At time $t = 0$, there is only one energy level and $x_1(0) = 0$. Clearly, the partition function is

$$Z(t) = \sum_{j=1}^{n} e^{-\beta x_j(t)} \tag{3.9}$$

The model introduced here is known as branching diffusions. If $x_1(t),..., x_n(t)$ are interpreted as the positions of some objects, the physical and biological origins of the model are apparent. In that context it would, however, be rather awkward to introduce a partition function.

Let us see how branching diffusions are related to the KPP equation.[16] We define $u(x, t)$ by

$$u(x. t) = \left\langle \prod_{j=1}^{n} u(x_j(t) + x) \right\rangle \tag{3.10}$$

where $u(x)$ is an arbitrary function. Then

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x, t) - u(x, t) + u(x, t)^2 \tag{3.11}$$

with $u(x, 0) = u(x)$. We see that the differential equation satisfied by $u(x, t)$ does not depend on the initial condition $u(x)$. (We adopt from now on units such that $D = \frac{1}{2}$, $\lambda = 1$.) Equation (3.6) is the particular case where $u(x) = G_0(x)$.

The proof of (3.11) is identical to the one for G_t. We only have to consider the probability of (non) splitting of the first level between times 0 and dt.

4. SOME PROPERTIES OF THE KPP EQUATION

The KPP equation (3.11) is one of the simplest nonlinear, parabolic equations that admits traveling wave solutions, i.e., solutions of the form

$$u(x, t) = w(x - ct) \tag{4.1}$$

In fact, the mechanism is easy to understand. Let us first consider solutions that are homogeneous in space. Then $u = 0$ is a stable fixed point and $u = 1$ is unstable. Therefore if $\lim_{x \to \infty} u(x) = 1$ and $\lim_{x \to -\infty} u(x) < 1$, the left-hand part of the solution drops quickly to zero. However, from the unstable fixed point to the right it can escape only through diffusion, thereby producing a wave traveling to the right. In our application $0 \leqslant u \leqslant 1$ always. Therefore also $0 \leqslant w \leqslant 1$. The wavefront satisfies the ordinary differential equation

$$\tfrac{1}{2} w''_\beta + c(\beta) w'_\beta - w_\beta(1 - w_\beta) = 0 \tag{4.2}$$

with boundary conditions $w_\beta(-\infty) = 0$, $w_\beta(\infty) = 1$, and $0 \leqslant w_\beta \leqslant 1$. For reasons that will become clear immediately, we have indexed the solutions $w = w_\beta$ by β with a corresponding speed $c(\beta)$ to be determined below. Equation (4.2) is the equation of motion for a particle in the potential $-\frac{1}{2} w^2 (1 - \frac{2}{3} w)$ with constant friction c. Solutions are admissible only for $c \geqslant \sqrt{2}$. If $c < \sqrt{2}$, the motion at $w = 1$ is an underdamped oscillation and $w \leqslant 1$ is violated. $c = \sqrt{2}$ is the minimal speed. w_β is unique up to translations. As normalization we adopt $w_\beta(0) = \frac{1}{2}$. A more detailed analysis of the motion near the fixed point $(w, w') = (1, 0)$ shows that

$$1 - w_\beta(x) \cong \begin{cases} \exp(-\beta x) & \text{for} \quad c > \sqrt{2} \\ x \exp(-\sqrt{2}\, x) & \text{for} \quad c = \sqrt{2} \end{cases} \tag{4.3}$$

as $x \to \infty$. The speed of the wavefront is related to the exponential decay of $w_\beta(x)$ by

$$c(\beta) = \frac{1}{2} \beta + \frac{1}{\beta}, \qquad \beta \leqslant \beta_c = \sqrt{2} \tag{4.4}$$

From the mechanism producing the traveling wave, it is clear that the relevant asumptotics is $x \to +\infty$.

In a beautiful piece of work, Bramson[17] studies in great detail the approach of a given initial condition $u(x)$ to a traveling wave as $t \to \infty$. We will draw heavily on his analysis. Before embarking on our own enterprise, it may be useful to explain the basic idea behind Bramson's work. He thinks of (3.11) as an (imaginary time) Schrödinger equation,

$$\frac{\partial}{\partial t} u(x, t) = \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} + [u(x, t) - 1] \right\} u(x, t) \tag{3.11$'$}$$

By the Feynman–Kac formula its solution is then written as

$$u(x, t) = \mathbb{E}_x \left(\exp \left\{ \int_0^t ds\, [u(b_s, s) - 1] \right\} u(b_t) \right) \tag{4.5}$$

Here b_t is Brownian motion and $\mathbb{E}_x$ is the expectation over all paths starting at x. Of course, because of the nonlinearity, the time-dependent potential $u(x, t) - 1$ depends itself on the solution. The crucial point and the beauty of the approach is that (4.5) allows for a sort of bootstrap strategy. A modest information on $u(x, t)$, and therefore on the potential, may be turned into a sharp information on the solution $u(x, t)$ through the use of (4.5).

Let us summarize the main results[17] of interest for our application. The initial conditions are such that u increases monotonically from $u(-\infty) = 0$ to $u(\infty) = 1$. (A more general class of initial conditions can be handled as well.) Bramson proves that, for any given initial condition $u(x)$, there exists a constant β, $0 \leqslant \beta \leqslant \sqrt{2}$, and a function $m_\beta(t)$ such that

$$\lim_{t \to \infty} \sup_x |u(x, t) - w_\beta(x - m_\beta(t))| = 0 \tag{4.6}$$

To *leading* order in t

$$m_\beta(t) = c(\beta)\, t + o(t) \tag{4.7}$$

β is determined by the asymptotic decay at $+\infty$. If

$$u(x) = 1 - e^{-\beta x} \tag{4.8}$$

for $x \to \infty$, then $c(\beta)$ and ω_β are given by (4.4) and (4.3) if $\beta \leqslant \beta_c = \sqrt{2}$. On the other hand, $c(\beta) = \sqrt{2}$ and $w_\beta = w_{\sqrt{2}}$ if $\beta > \beta_c$.

The precise time dependence of $m_\beta(t)$ is determined by the fine details of the asymptotic decay in (4.8). Bramson's major contribution is to prove logarithmic corrections and in some cases even corrections of order one. For example, for the initial condition (3.8), $u(x) = G_0(x)$, he shows that

$$m_\beta(t) = \begin{cases} c(\beta)\, t + O(1) & \text{if } \beta < \beta_c = 2^{1/2} \\ 2^{1/2} t - 2^{-3/2} \log t + O(1) & \text{if } \beta = 2^{1/2} \\ 2^{1/2} t - 3 \cdot 2^{-3/2} \log t + O(1) & \text{if } \beta > \beta_c \end{cases} \tag{4.9}$$

5. THE PROBABILITY DISTRIBUTION OF $Z(t)$

Since $G_t(x)$ defined by (3.4) satisfies the KPP equation with the initial condition (3.8), which behaves like

$$G_0(x) \cong 1 - e^{-\beta x} \tag{5.1}$$

for $x \to \infty$, we conclude that, up to order 1,

$$-\frac{1}{\beta} \langle \log Z(t) \rangle = -m_\beta(t) \tag{5.2}$$

This is because one can write

$$\begin{aligned} -\frac{1}{\beta} \langle \log Z(t) \rangle &= -\int_{-\infty}^{\infty} dx \, \langle \{ \exp(-e^{-\beta x}) - \exp[-e^{-\beta x} Z(t)] \} \rangle \\ &= -\int_{-\infty}^{\infty} dx \, [G_0(x) - G_t(x)] \end{aligned} \tag{5.3}$$

Since in the long-time limit $G_t(x)$ is a front located at the point $m_\beta(t)$, one gets (5.2). Thus, the way in which the free energy depends on β comes from the dependence of $m_\beta(t)$ on the initial condition (5.1). In particular, the free energy per unit length is given in the long-time limit by the speed $c(\beta)$ of the traveling wave,

$$\lim_{t \to \infty} -\frac{1}{t} \frac{1}{\beta} \langle \log Z(t) \rangle = -c(\beta) \tag{5.4}$$

At $\beta_c = \sqrt{2}$, there is a transition to a frozen phase. The low-temperature phase is simply reflected in the solutions of the KPP equation: they all travel with the same minimal speed as $t \to \infty$. So we see that as the temperature decreases, the free energy is given by the speed $c(\beta)$ [Eq. (4.4)] and when β reaches β_c, there is a freezing at the minimal speed $\sqrt{2}$ very similar to the freezing phenomenon in spin glasses.[18] One should also notice that, as in spin glasses, $c(\beta)$ has an analytic continuation for

$\beta > \beta_c$ given by (4.4) and that the corresponding free energy would be *lower* than the true one $c(\beta_c)$. So the phenomenon[18] that the free energy varies with temperature until it reaches its maximum and sticks there is very similar to the fact that there is a minimal speed for the traveling wave solutions of the KPP equation.

Our next goal is to understand the distribution of the free energy around its average, i.e., the distribution of $f(t)$ defined by

$$f(t) = -\frac{1}{\beta} \log Z(t) + \frac{1}{\beta} \langle \log Z(t) \rangle \tag{5.5}$$

We use the generating function

$$\langle \exp[-\nu f(t)] \rangle = \langle Z(t)^{\nu/\beta} \rangle \exp\left[-\frac{\nu}{\beta} \langle \log Z(t) \rangle \right] \tag{5.6}$$

As in ref. 19,

$$\begin{aligned} \langle Z(t)^{\nu/\beta} \rangle = & \frac{1}{\Gamma(n-\nu)} \int_{-\infty}^{\infty} dx\, \beta[\exp(-n\beta x) \exp(\nu x)] \\ & \times \langle Z(t)^n \exp\{-[\exp(-\beta x)]\, Z(t)\} \rangle \end{aligned} \tag{5.7}$$

for $n-1 < \nu/\beta < n$ and $n \geqslant 1$, and

$$\langle Z(t)^{-\nu/\beta} \rangle = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{\infty} dx\, [\exp(-\nu x)] \langle \exp\{-[\exp(-\beta x)]\, Z(t)\} \rangle \tag{5.8}$$

for $\nu > 0$. The solution of the KPP equation for large t is given by (4.4). The translation is taken care of by the subtraction in (5.5). We conclude that $\lim_{t\to\infty} f(t) = f$ in distribution and that for $n-1 < \nu/\beta < n$

$$\begin{aligned} \langle e^{-\nu f} \rangle &= \int df\, \lambda_\beta(f) e^{-\nu f} \\ &= \frac{1}{\Gamma(n-\nu)} \int_{-\infty}^{\infty} dx\, e^{\nu x} \beta e^{-n\beta x} \left(\frac{1}{\beta} e^{\beta x} \frac{d}{dx} \right)^n w_\beta(x + a(\beta)) \end{aligned} \tag{5.9}$$

whereas for $\nu > 0$

$$\langle e^{\nu f} \rangle = \int df\, \lambda_\beta(f) e^{\nu f} = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{\infty} dx\, e^{-\nu x} w_\beta(x + a(\beta)) \tag{5.10}$$

Here $a(\beta)$ is some constant.

If $\beta \geqslant \beta_c = \sqrt{2}$, then $w_\beta = w_{\sqrt{2}}$, the wavefront with minimal speed. $a(\beta)$ corresponds merely to a shift in the distribution of f. Since $\langle f \rangle = 0$, we conclude that, for $\beta \geqslant \sqrt{2}$, $\lambda_\beta(f) = \lambda_{\sqrt{2}}(f)$, i.e., the shape of the free energy distribution does not change with temperature in the low-temperature phase.

From the asymptotics of $w_\beta(x)$ for $x \to \pm\infty$ we infer the behavior of $\lambda_\beta(f)$ for $f \to \mp\infty$. Let us discuss the cases $x \to \pm\infty$ separately.

(i) $x \to \infty$ corresponding to $f \to -\infty$. For $\beta < \beta_c$ the fixed point at $(w, w') = (1, 0)$ has the eigenvalues $-\beta$, $-2/\beta$. Therefore

$$1 - w_\beta(x) = a_1 e^{-\beta x} + a_2 e^{-2\beta x} + \cdots + b_1 e^{-(2/\beta) x} + \cdots \tag{5.11}$$

In (5.9) the contributions from $e^{-\beta x}$ and its powers cancel and the leading term is $e^{-(2/\beta)x}$. Therefore (5.9) diverges for $v \geqslant 2/\beta$. For $\beta \geqslant \beta_c$,

$$1 - w_{\sqrt{2}}(x) \cong x \exp(-\sqrt{2}\, x) \tag{5.12}$$

and (5.9) diverges for $v \geqslant \sqrt{2}$.

(ii) $x \to -\infty$ corresponding to $f \to \infty$. The fixed point at $(w, w') = (0, 0)$ has one repelling direction with eigenvalue

$$\alpha = \frac{1}{2}\left\{\left[\left(\beta + \frac{2}{\beta}\right)^2 + 8\right]^{1/2} - \left(\beta + \frac{2}{\beta}\right)\right\} \tag{5.13}$$

$\langle \exp vf \rangle$ diverges for $v \geqslant \alpha$. If $\beta > \beta_c$,

$$\alpha = 2 - \sqrt{2} \tag{5.14}$$

We summarize the behavior of the distribution of the free energy $\lambda_\beta(f)$: If $\beta < \beta_c$, then

$$\lambda_\beta(f) \cong \begin{cases} \exp[(2/\beta) f] & \text{for } f \to -\infty \\ \exp(-\alpha f) & \text{for } f \to \infty \end{cases} \tag{5.15}$$

If $\beta > \beta_c$, then

$$\lambda_\beta(f) \cong \begin{cases} -f \exp(\sqrt{2}\, f) & \text{for } f \to -\infty \\ \exp[-(2 - \sqrt{2}) f] & \text{for } f \to \infty \end{cases} \tag{5.16}$$

The fluctuations in the free energy are of order one at all temperatures. At the critical point the distribution freezes.

Differentiating $\langle \exp[-\mu Z_\beta(t)] \rangle$ at $\mu = 0$, one obtains equations for the moments $\langle Z_\beta(t)^n \rangle$, which can be solved recursively [cf. (3.3)]. One finds that $\langle Z_\beta(t)^n \rangle / \langle Z_\beta(t) \rangle^n$ diverges for $\beta = (2/n)^{1/2}$, in agreement with (5.15).

At $\beta=0$, $Z_\beta(t)$ is the number n of energy levels at time t. Their distribution is

$$p_t(n) = e^{-t}(1-e^{-\lambda t})^{n-1} \tag{5.17}$$

Therefore, $y = -\log Z_0(t) + t = -\log n + t$ has the limiting distribution

$$\exp(-y)\exp[-\exp(-y)] \tag{5.18}$$

As $y \to -\infty$, the decay is faster than any exponential, consistent with the decay $\exp[(1/\beta^2)(\beta f)]$. For $y \to \infty$ the decay is e^{-y}, in agreement with $\alpha/\beta \to 1$ as $\beta \to 0$.

So much for explicit computation. Let us try to gain some further understanding by looking directly at the statistics of energy levels. The average level density at energy x is

$$e^t(2\pi t)^{-1/2}\, e^{-x^2/2t} \tag{5.19}$$

The average level density (5.19) is of order one for

$$a_0(t) = \mp [2^{1/2}t - 2^{-3/2}\log t + O(1)] \tag{5.20}$$

For $\beta \to \infty$, $-(1/\beta)\log Z_\beta(t)$ is just the lowest (ground state) energy $e_0(t)$. From (4.9) we know that $e_0(t)$ is typically located at

$$-m_{\sqrt{2}}(t) = -(2^{1/2}t - 3 \cdot 2^{-3/2}\log t) \tag{5.21}$$

The distribution of $e_0(t)$ is $w'_{\sqrt{2}}$. Comparing (5.20) with (5.21), we see that the prefactor of the logarithmic correction cannot be guessed on the basis of the average level density. One has that $e_0(t)$ is of the oder $\log t$ above $a_0(t)$. Sitting at $a_0(t)$ for most samples, one does not see any energy level at all and very rarely a large number of them. By the same method as used for the partition function, one can study the number of levels in some interval around $-m_{\sqrt{2}}(t)$. From this one concludes that above $e_0(t)$ there is a discrete set of levels, with an exponentially increasing density, however. If $\beta > \beta_c$, the partition function singles out the levels close to $e_0(t)$. Therefore $Z_\beta(t)$ is essentially a finite sum.

The random energy model (REM)[18] has an average level density identical to branching diffusions. In this case $e_0(t) \cong a_0(t)$. The statistics of levels near e_0 is Poisson with an increasing density $\exp(\sqrt{2}\,x)$. For branching diffusions we did not find a "simple" statistics of energy levels, although joint level distributions could be obtained from the solution of the KPP equation.

Also, the Sherrington–Kirkpatrick model and the GREM have the same average level density. In these cases, however, $e_0(t)$ and $a_0(t)$ differ proportional to t.

6. THE OVERLAP

In the case of spin glasses at low temperatures, phase space breaks up into many pieces separated by free energy barriers. This many-valley structure can be studied by looking at the overlap between spin configurations. For branching diffusions the overlap between $x_i(t)$ and $x_j(t)$ can be defined by

$$Q_{ij} = \text{fraction of time with } x_i(s) = x_j(s), \qquad 0 \leqslant s \leqslant t \tag{6.1}$$

Clearly, $0 \leqslant Q_{ij} \leqslant 1$. For a given tree, the probability $\tilde{Y}(q)$ of finding an overlap q is

$$\tilde{Y}(q)\, dq = \frac{1}{Z(t)^2} \sum_{i,j=1}^{n} e^{-\beta x_i(t)} e^{-\beta x_j(t)} \chi(\{q \leqslant Q_{ij} \leqslant q + dq\}) \tag{6.2}$$

The characteristic function χ restricts the average only to those levels that have an overlap in the interval $q, q + dq$.

It is clear that close to the ground-state energy $e_0(t)$ there must be energy levels that have an overlap 1 with $e_0(t)$, because they just have recently branched from $e_0(t)$. Without further insight one may expect to find near $e_0(t)$ also levels with overlap $0 < q < 1$. Surprisingly enough, this is not the case. We will show that the overlap is either zero or one, i.e., in the limit $t \to \infty$,

$$\tilde{Y}(q)\, dq = (1 - Y)\, \delta(q) + Y \delta(q - 1) \tag{6.3}$$

and that distribution for Y is identical to the one for the Sherrington–Kirkpatrick model[20] and the REM.[19]

We consider

$$Y_\beta(q) = \int_q^1 dq'\, \tilde{Y}(q') \tag{6.4}$$

To obtain the distribution of $Y_\beta(q)$ we first follow ref. 19. Then

$$\langle Y_\beta(q)^\nu \rangle = \frac{(-1)^n}{\Gamma(2\nu)\, \Gamma(n-\nu)}\, \beta \int_{-\infty}^{\infty} dx \int_0^\infty d\mu\, \mu^{n-1-\nu} \frac{\partial^n}{\partial \mu^n} \times \left\langle \exp\left[-e^{-\beta x} Z(t) - \mu e^{-2\beta x} \sum_{i,j} e^{-\beta x_j(t)} e^{-\beta x_i(t)} \chi(\{Q_{ij} \geqslant q\}) \right] \right\rangle \tag{6.5}$$

with $n \geqslant 1$ and $n-1<\nu<n$. The configuration at time qt is $\{x_j(qt)\}$. From each $x_j(qt)$ there emerges a new tree. In the double sum $\sum_{ij}$ there is no contribution from distinct trees because the overlap $Q_{ij} \geqslant q$. Therefore the average in (6.4) is given by

$$\langle \cdot \rangle = \left\langle \exp\left[-\sum_j e^{-\beta[x_j(qt)+x]} Z_j((1-q)t) \right.\right.$$
$$\left.\left. -\mu \sum_j e^{-2\beta[x_j(qt)+x]} Z_j((1-q)t)^2 \right]\right\rangle \tag{6.6}$$

Here the $Z_j((1-q)t)$ are independent copies of the partition function. They are also independent of the configuration $\{x_j(qt)\}$. Now, by (3.10)

$$\langle Y_\beta(q)^\nu \rangle = \frac{(-1)^n}{\Gamma(2\nu)\,\Gamma(n-\nu)} \beta \int_{-\infty}^{\infty} dx \int_0^\infty d\mu\, \mu^{n-1-\nu} \frac{\partial^n}{\partial \mu^n} u^{(\mu)}(x, qt) \tag{6.7}$$

where $u^{(\mu)}(x, qt)$ is the solution of the KPP equation with initial condition

$$u^{(\mu)}(x) = \langle \exp[-e^{-\beta x} Z((1-q)t) - \mu e^{-2\beta x} Z((1-q)t)^2] \rangle \tag{6.8}$$

Before continuing with the calculation of $\langle Y^\nu \rangle$, let us first study the average overlap $\langle Y_\beta(q) \rangle$. We may either take the limit $\nu \to 1$ in (6.7) or use the original definition (6.2), (6.4) and follow the reasoning given above. The net result is

$$\langle Y_\beta(q) \rangle = \lim_{t\to\infty} \beta \int_{-\infty}^{\infty} dx \frac{\partial}{\partial \mu} u^{(\mu)}(x, qt) \bigg|_{\mu=0} \tag{6.9}$$

Using (3.4) and (6.8), we conclude that $u^{(0)}(x, qt) = G_t(x)$. Therefore we have to solve only the linearized KPP equation, linearized around $G_t(x)$, with initial condition

$$\begin{aligned} \xi(x) &= [\exp(-2\beta x)] \langle Z((1-q)t)^2 \exp\{-[\exp(-\beta x)]\, Z((1-q)t)\} \rangle \\ &= \left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{1}{\beta} \frac{\partial}{\partial x} \right) G_{(1-q)t}(x) \end{aligned} \tag{6.10}$$

For large t and $q<1$,

$$G_{(1-q)t}(x) \cong w_\beta(x - m_\beta[(1-q)t])$$

Therefore the initial condition (6.10) is independent of q except for a translation, which does not change the value of the integral (6.9). In addition, as long as $q>0$, the limit (6.9) does not depend on q. Therefore $\langle Y_\beta(q) \rangle$ must be independent of q, provided $0<q<1$.

To compute the limit (6.9) it is convenient to transform to the frame moving with velocity $\dot{m}_\beta(t)$. In the moving frame the linearized KPP operator is

$$L(t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \dot{m}_\beta(t)\frac{\partial}{\partial x} - 1 + 2w_\beta \tag{6.11}$$

If $0 < q < 1$, then by (6.9), (6.10)

$$\langle Y_\beta(q) \rangle = \lim_{t\to\infty} \beta \int dx \left(\left[\exp\left(\int_0^t ds\, L(s) \right) \right] \xi \right)(x) \tag{6.12}$$

with the initial condition

$$\xi(w) = \left(\frac{1}{\beta^2}\frac{\partial^2}{\partial x^2} + \frac{1}{\beta}\frac{\partial}{\partial x} \right) w_\beta \tag{6.13}$$

The asymptotics is more clearly displayed upon the similarity transformation

$$\begin{aligned} (w'_\beta)^{-1} L(t) w'_\beta &= \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\dot{m}_\beta + \frac{w''_\beta}{w'_\beta} \right)\frac{\partial}{\partial x} + [\dot{m}_\beta - c(\beta)]\frac{w''_\beta}{w'_\beta} \\ &= \tilde{L}(t) \end{aligned} \tag{6.14}$$

Then, rewriting (6.12),

$$\begin{aligned} \langle Y_\beta(q) \rangle = \lim_{t\to\infty} \beta \int dx \int dx'\, w_\beta(x) \\ \times \exp\left[\int_0^t ds\, \tilde{L}(s) \right](x, x')[\xi(x')[\xi(x')/w'_\beta(x')] \end{aligned} \tag{6.15}$$

where $\exp[\cdot](x, x')$ denotes the fundamental solution of the linear equation $\dot{\xi} = \tilde{L}(t)\xi$.

If $\beta < \beta_c$, then $\dot{m}_\beta = c_\beta$ and $w''_\beta/w'_\beta \to -\beta$ for $x \to \infty$ and $w''_\beta/w'_\beta \to \alpha$ for $x \to -\infty$. Therefore, in (6.14) $\dot{m}_\beta + w''_\beta/w'_\beta \geqslant a > 0$ and the last term vanishes. This means that there is a net force to the right:

$$\int dx\, w'_\beta(x) \exp\left[\int_0^t ds\, \tilde{L}(s) \right](x, x')$$

is a probability distribution in x' which for large t moves with constant speed to the right. There ξ/w'_β decays exponentially. We conclude that $\langle Y_\beta(q) \rangle = 0$.

On the other hand, if $\beta > \beta_c$, then

$$\dot{m}_\beta + w''_\beta / w'_\beta = \begin{cases} 1/x - 3 \cdot 2^{-3/2}(1/t) & \text{for} \quad x \to \infty \\ \text{[illegible]}) & \text{for} \quad x \to -\infty \end{cases} \tag{6.16}$$

There is still a force to the right, although with decreasing strength. For $x \to \infty$

$$\frac{\xi}{w'_\beta} = \frac{1}{\beta}\left(\frac{1}{\beta} w''_\beta + w'_\beta\right) \Big/ w'_\beta \cong \frac{1}{\beta}\left(1 - \frac{\sqrt{2}}{\beta}\right) \tag{6.17}$$

The potential contribution $[\dot{m}_\beta - c(\beta)] w''_\beta / w'_\beta$ to $\tilde{L}(t)$ decays as $1/t$. Therefore,

$$\lim_{t \to \infty} \exp\left[\int_0^t ds\, \tilde{L}(s)\right]\left(\frac{\xi}{w'_\beta}\right)(x) = \frac{1}{\beta}\left(1 - \frac{\sqrt{2}}{\beta}\right) \tag{6.18}$$

Noting that $\int dx\, w'_\beta(x) = 1$, we conclude

$$\langle Y_\beta(q) \rangle = 1 - \frac{\sqrt{2}}{\beta} \tag{6.19}$$

For $\beta < \beta_c$ the overlap is zero with probability one. For $\beta > \beta_c$, since $Y(q)$ does not depend on q, the overlap is either zero or one (zero with probability $\sqrt{2}/\beta$ and one with probability $1 - \sqrt{2}/\beta$).

Let us return then to the task of determining the distribution of Y. We fix $\beta > \sqrt{2}$ and some q with $0 < q < 1$. The initial condition $u^{(\mu)}(x)$ of the KPP equation is given by (6.8). For large t it travels a distance $m_\beta[(1-q)t]$, which drops out in (6.7), however. Hence

$$u^{(\mu)}(x) = \int df\, \lambda(f) \exp[-e^{-\beta(x+f)} - \mu e^{-2\beta(x+f)}] \tag{6.20}$$

$\lambda(f)$ is the distribution of the free energy studied in Section 5.

For $\mu = 0$

$$u^{(0)}(x) = w_\beta(x + a(0)) \tag{6.21}$$

and the solution to the KPP equation is simply $w_\beta(x - \sqrt{2}\, t + a(0))$. For $\mu > 0$ the term in the exponent induces only a local change of $w_\beta(x)$ and the asymptotics for $x \to \infty$ is still the one of $w_\beta(x)$, i.e.,

$$1 - u^{(\mu)}(x) \cong x \exp(-\sqrt{2}\, x) \tag{6.22}$$

Bramson (ref. 17, Chapter 9) shows that also in this case the solution to the KPP equation is asymptotically of the form

$$w_\beta(x-\sqrt{2}\,t+a(\mu)) \tag{6.23}$$

$a(\mu)$ is a constant. Because the tail of the initial condition is precisely the one of $w_\beta(x)$, there are no logarithmic corrections. We insert (6.23) in (6.7) and use again $\int dx\, w'_\beta(x)=1$. Then

$$\langle Y^\nu\rangle=\frac{(-1)^n\,(\beta/\sqrt{2})}{\Gamma(2\nu)\,\Gamma(n-\nu)}\int_0^\infty d\mu\,\mu^{n-1-\nu}\frac{\partial^n}{\partial\mu^n}\sqrt{2}\,a(\mu) \tag{6.24}$$

Bramson has also determined the constant $a(\mu)$. In fact, let us go back to (4.5) using (6.23). In the space-time region where $u(x,t)=0$, paths are exponentially suppressed. To locate the front, we have to solve the diffusion equation with a linearly moving, absorbing boundary condition. The crucial point is that we need to know its location only up to order one. Bramson proves that this reasoning is indeed correct. His final result is

$$\sqrt{2}\,a(\mu)=\lim_{z\to\infty}\left[-\sqrt{2}\,z-\log v(z)\right] \tag{6.25}$$

with

$$v(z)=\lim_{t\to\infty} e^t\int dy\,[1-u(y)]$$
$$\times(2\pi t)^{-1/2}\,\{\exp[-(z+\sqrt{2}\,t-y)^2/2t]\}\{1-\exp(-2yz/t)\} \tag{6.26}$$

The term in the final curly brackets is due to the absorbing boundary conditions. The initial condition must satisfy $1-u(x)\cong x\exp(-\sqrt{2}\,x)$ for $x\to\infty$.

We insert (6.20) in (6.25). Then

$$v(z)=\int dy\,\{1-\exp[-\exp(-\beta y)-\mu\exp(-2\beta y)]\}$$
$$\times\lim_{t\to\infty}(\exp t)\int df\,\lambda(f)$$
$$\times(2\pi t)^{-1/2}\{\exp[-(z+f+\sqrt{2}\,t-y)^2/2t]\}\{1-\exp[-2(y-f)z/t]\} \tag{6.27}$$

Since $\lambda(f)\cong -f\exp(\sqrt{2}\,f)$ for $f\to-\infty$ [cf. (5.16)], the limit $t\to\infty$ is equal to const. $z \exp[-2^{1/2}(z-y)]$ with a constant independent of z and y. We conclude

$$2^{1/2}a(\mu) = -\log \int dy \, \{1 - \exp[-\exp(-\beta y) - \mu \exp(-2\beta y)]\} \exp(2^{1/2} y) \tag{6.28}$$

In combination with (6.24), this shows that Y has the same distribution[19] as found for the REM and the SK model.

The distribution of Y has a structure that is not apparent from the moments. It diverges near 1 as $(1-Y)^{-\sqrt{2}/\beta}$ and has cusp singularities at the points $1/n$, $n = 2, 3, \ldots$; see ref. 21 for details.

7. GENERAL OVERLAP

Do branching diffusions always have overlap either zero or one? The generalization of the REM[22–25] indicates that one should get any overlap desired through a simple modification: we only have to consider that the diffusion coefficient D changes slowly in time (so far we have set $D = \frac{1}{2}$). Let then $D(\tau)$, $0 \leqslant \tau \leqslant 1$, be given. If the process branches up to time t, then at time s the common diffusion coefficient is $D(s/t)$, $0 \leqslant s \leqslant t$. We also assume that $D(\tau)$ is decreasing. Other cases, e.g., time-dependent branching rate, nonmonotone $D(\tau)$, can be worked out also.

The mechanism responsible for a general overlap may be understood already from the simplest case $D(\tau) = D_1$ for $0 \leqslant \tau \leqslant \tau_0$ and $D(\tau) = D_2$ for $\tau_0 \leqslant \tau \leqslant 1$, $D_1 \geqslant D_2$. To obtain the generating function for $Z(t)$, we employ the technique developed in Section 3 iteratively. First we have to solve the KPP equation with diffusion coefficient D_2 (!) for the initial condition $u(x) = \exp[-\exp(-\beta x)]$. Let $\tilde{u}(x)$ be the solution at time $(1-\tau_0)t$. We then have to solve the KPP equation with diffusion coefficient D_1 for the initial condition $\tilde{u}(x)$. The solution at time $\tau_0 t$ is the generating function for $Z(t)$. Also, we need the velocity of the traveling wave for constant diffusion D. It is given by

$$c(\beta, D) = \begin{cases} 1/\beta + D\beta & \text{if } \beta < \beta_c = (1/D)^{1/2} \\ 2\sqrt{D} & \text{if } \beta > \beta_c \end{cases} \tag{7.1}$$

As an example, let us compute the free energy. The KPP solution first travels with speed $c(\beta, D_2)$ for a time $1-\tau_0$. Then the diffusion coefficient changes to D_1 and the front has to adjust to a new speed. Since $\beta_c(1) < \beta_c(2)$, the new speed is $c(\beta, D_1)$. We conclude that, in general, the free energy per unit length of the walk is given by

$$f(\beta) = -\int_0^1 d\tau \, c(\beta, D(\tau)) \tag{7.2}$$

For the average overlap $\langle Y(q) \rangle$, the case of interest is $\beta > (1/D_2)^{1/2}$. If $q > \tau_0$, then at $\tau_0 t$ the solution of the KPP equation needed is of the form

$$w_{\beta_c(2)}(x - 2D_2^{1/2}(q - \tau_0)t + a(\mu))$$

At $\tau_0 t$ the solution accelerates to the new speed $2D_1^{1/2}$. This has no influence on $a(\mu)$, however. Therefore

$$\langle Y(q) \rangle = 1 - (1/\beta)(1/D_2)^{1/2}, \qquad \tau_0 < q < 1 \tag{7.3}$$

On the other hand, if $q < \tau_0$, then we are back to the situation studied in Section 6. The distribution of free energies has to be taken at the inverse temperature $\beta_c(2) = (1/D_2)^{1/2}$. We conclude

$$\langle Y(q) \rangle = 1 - (1/\beta)(1/D_1)^{1/2}, \qquad 0 < q < \tau_0 \tag{7.4}$$

If $D(\tau)$ takes only two values, then the overlap is either $q = 0$, τ_0, or 1. One should notice that the case $D_1 < D_2$ would imply $\beta_c(1) > \beta_c(2)$ and lead to a rather different solution as it does in the GREM and only the overlaps $q = 0$ or 1 would be possible. In general, if $D(\tau)$ is an decreasing function of τ, one obtains

$$\langle Y(q) \rangle = 1 - (1/\beta)[1/D(q)]^{1/2} \tag{7.5}$$

for $\beta > [1/D(1)]^{1/2}$. As already noted for the GREM, if define $x(q) = 1 - \langle Y(q) \rangle$, then for $\beta > [1/D(1)]^{1/2}$

$$f(\beta) = -\int_0^1 \left[\frac{1}{x(q)} \frac{1}{\beta} + \beta x(q)\, D(q) \right] dq \tag{7.6}$$

Using the method of Section 6, one could also obtain also the full statistics of $Y(q)$ in the limit $t \to \infty$. It is precisely the intricate statistics of overlaps derived from the "superimposed" Poisson statistics with exponentially increasing density.[26]

8. BACK TO THE DISCRETE TIME PROBLEM

It is reasonable to expect that most of the properties of the KPP equation have their analogues in the discrete time problem, which was governed by Eq. (2.9),

$$G_{t+1}(x) = \int dV\, \rho(V)[G_t(x + V)]^K \tag{8.1}$$

Let us briefly indicate here the properties for the solution of this equation without giving any demonstration.

In the long-time limit, the front moves with a constant speed $c(\beta)$, which depends on the decay of the initial condition: If

$$G_0(x) = 1 - e^{-\beta x} \tag{8.2}$$

for $x \to \infty$, then the speed $c(\beta)$ of the front is given by

$$e^{\beta c(\beta)} = K \int dV\, \rho(V) e^{-\beta V}, \qquad \beta < \beta_c \tag{8.3}$$

where β_c is the value of β for which $c(\beta)$ is minimal,

$$\frac{\partial}{\partial \beta} c(\beta)\bigg|_{\beta = \beta_c} = 0 \tag{8.4}$$

For initial conditions (8.2) with $\beta > \beta_c$, the front moves with speed $c(\beta_c)$ in the long-time limit.

As in Section 5, the speed $c(\beta)$ gives the free energy per unit length in the long-time limit,

$$-\frac{1}{t}\frac{1}{\beta}\langle \log Z(t)\rangle = \begin{cases} -c(\beta) & \text{if } \beta \leqslant \beta_c \\ -c(\beta_c) & \text{if } \beta > \beta_c \end{cases} \tag{8.5}$$

The knowledge of the shape of the front determines, in principles, the shape of the probability distribution of the free energy. For example, for $f \to -\infty$, again

$$\lambda_\beta(f) = \begin{cases} e^{\psi f} & \text{if } \beta < \beta_c \\ -f e^{\psi_c f} & \text{if } \beta \geqslant \beta_c \end{cases} \tag{8.6}$$

where ψ is the solution ($\psi > \beta$) of

$$K \int dV\, \rho(V) e^{-\psi V} = \left[K \int dV\, \rho(V) e^{-\beta V} \right]^{\psi/\beta} \tag{8.7}$$

for $\beta < \beta_c$ and $\psi_c = \psi(\beta_c) = \beta_c$ for $\beta > \beta_c$. As in the continuum version, the distribution $\lambda_\beta(f)$ decays exponentially for $f \to -\infty$. For $f \to \infty$, the decay depends in a more complicated way on the distribution $\rho(V)$ and we will not discuss it here.

Finally, the overlaps can be calculated for any distribution $\rho(V)$. The result is that the overlap is 0 for $\beta < \beta_c$ and that the overlap is either 0 or 1 for $\beta > \beta_c$. For $\beta > \beta_c$, $\langle Y_\beta(q)\rangle$ is given by

$$\langle Y_\beta(q)\rangle = 1 - \gamma_m \tag{8.8}$$

where γ_m is the extremum of

$$\frac{1}{\gamma}\left[\log K + \log \int dV\,\rho(V)e^{-\gamma\beta V}\right] = \beta c(\gamma\beta) \tag{8.9}$$

where $c(\beta)$ is the solution of (8.3). Then it is clear that γ_m is given by

$$\gamma_m = \beta_c/\beta \tag{8.10}$$

and the expression (8.8) is very similar to (6.19).

9. CONCLUSIONS

In the present work we have seen that the problem of a self-avoiding walk on a disordered tree can be reduced to the study of traveling wave distribution of the free energy of the self-avoiding walk is given by the shape
of the front in the KPP equation and that the minimal speed property of the solutions of the KPP equation is closely related to a spin-glass-like transition with broken replica symmetry in the self-avoiding walk problem.

We think that it would be interesting to know whether the analogy between the mean field theory of spin glasses and the theory of traveling waves could be pushed further. One can also wonder whether the nature of the frozen phase with broken replica symmetry remains unchanged in finite dimension for the self-avoiding walk problem. This would certainly be useful to better understand the controversial subject of polymers in random media.[27]

ACKNOWLEDGMENTS

We are grateful to R. B. Griffiths, J. Langer, and J. L. Lebowitz for organizing a stimulating workshop at the Institute for Theoretical Physics, Santa Barbara. We have benefitted from discussions with P. Collet, J. P. Eckmann, and Y. Pomeau on the properties of traveling waves and with R. Peschanski on possible applications to particle physics.[28] We thank A. DeMasi and E. Presutti for showing us how to handle the linearized KPP equation. This research was supported in part by the National Science Foundation under grant PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California, Santa Barbara.

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