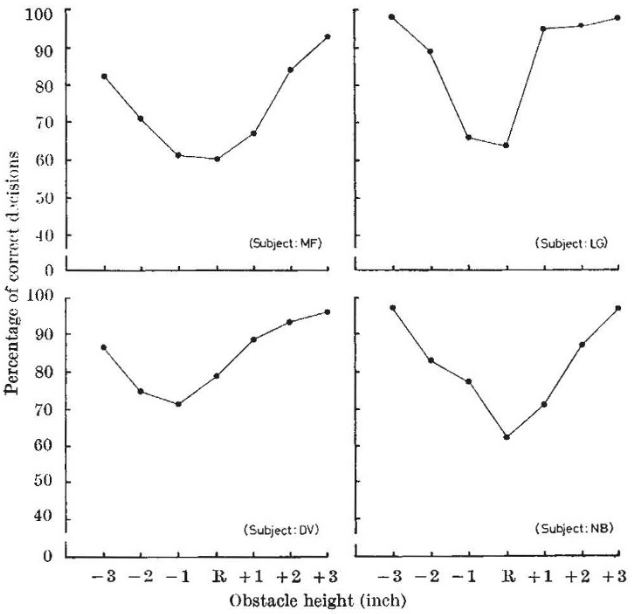

Fig. 1.

There is no doubt that in this laboratory situation the blindfolded subjects are monitoring both the long cane and the sonic aid as an effective combined travel system to ensure efficient mobility. Some pilot studies in real street situations have indicated that this combination of mobility aids does greatly reduce the incidence of collisions with overhanging branches and bushes which protrude onto the pavement. The findings also make it clear that the sonic aid mounted on the head can be used with great accuracy to make decisions about the positions of objects in near space while the user is moving. Obviously, conclusions derived from experiments involving blindfolded subjects in a laboratory situation cannot be confidently generalized to blind people in real life situations. Further experiments are now under way, with blind people, to establish the usefulness of the combined mobility system in street situations.

I thank St Dunstan's for financial support and Dr J. A. Leonard and Mr C. Espin for assistance.

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Received January 21; revised April 21, 1969.

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Polynomial Algebra: an Application of the Fast Fourier Transform

Much use is made in combinatorial problems of generating functions in the form of polynomials and infinite power series, these being obtained by the manipulation of other algebraic expressions. In order to save time and improve accuracy in the evaluation of the coefficients, one can, of course, make use of computer programs for doing algebra[1,2]. But it is often easier to use the following method which relies only on arithmetical operations available in all programming languages.

Suppose that the polynomial required is of degree at most $n-1$, so that it is known to be of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1}$. Let $x_s = \omega^s$, where $\omega = \exp(2\pi i/n)$. By straightforward arithmetical operations using complex numbers, we can compute $a_s^* = f(x_s) = \sum_{r=0}^{n-1} a_r\omega^{rs}$ $(s = 0, 1, \ldots, n-1)$. This is called the (mod n) discrete Fourier transform[3,4] of the sequence $(a_0, a_1, \ldots, a_{n-1})$, although sometimes it is divided by $\sqrt{n}$. By the formula for the inverse Fourier transform we have

$$a_r = \frac{1}{n}\sum_{s=0}^{n-1} a_s^*\omega^{-rs}$$

(The use of $\sqrt{n}$ leads to greater symmetry here: compare the use of $\sqrt{2\pi}$ in ordinary Fourier transforms.) Thus we can compute all the coefficients in the required polynomial. If they are known to be integers they can be obtained accurately, if necessary by making use of multiple-precision arithmetic. Moreover, the calculations can be performed by making use of the Fast Fourier Transform[5] provided that n is a suitable integer. Because n can always be increased we can certainly choose it to be of one of the forms required by the Fast Fourier Transform techniques, such as a power of 2 or the product of small relatively prime integers.

The method can be extended to polynomials in several variables merely by replacing n by a vector $\mathbf{n} = (n_1, n_2, \ldots)$, r and s by vectors $\mathbf{r}$ and $\mathbf{s}$, and the discrete Fourier transform by the multivariate form

$$c_{\mathbf{s}}^* = \sum_{\mathbf{r}} a_{\mathbf{r}}\boldsymbol{\omega}^{\mathbf{rs}}\ (r_1, s_1, = 0, 1, \ldots, n_1 - 1, \text{etc.})$$

where $\boldsymbol{\omega}^{\mathbf{rs}}$ is the "scalar indicial" way of writing $\omega_1^{r_1s_1}\omega_2^{r_2s_2}\ldots$ and where $\omega_1 = \exp(2\pi i/n_1)$, and so on. The inverse formula[4] is

$$a_{\mathbf{r}} = \frac{1}{n_1 n_2 \ldots}\sum_{\mathbf{s}} a_{\mathbf{s}}^*\,\boldsymbol{\omega}^{-\mathbf{rs}}$$

The method described here is buried in some papers on statistics[6], and is not mentioned in the reviews of those papers. In the interests of information retrieval it seems useful to publish the present explicit statement of the method on its own, especially because the Fast Fourier Transform has now become familiar.

As a simple example we take $f(x) = (1+x)^{n-1}$ and obtain

$$\binom{n-1}{r} = \frac{2^{n-1}}{n}\sum_{s=0}^{n-1}(-1)^s\left(\cos\frac{\pi s}{n}\right)^{n-1}\cos\frac{\pi s(1+2r)}{n}$$

from which the binomial coefficients for large n can be rapidly computed with considerable proportional accuracy by taking, say, $2\sqrt{n}$ terms at the beginning and end of the series. The rate of "convergence" is like that of a theta series.

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Received March 25, 1969.

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