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POLYNOMIAL AND REGULAR IMAGES OF $\mathbb{R}^n$

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ABSTRACT

We obtain new necessary conditions for an n-dimensional semialgebraic subset of $\mathbb{R}^n$ to be a polynomial image of $\mathbb{R}^n$. Moreover, we prove that a large family of planar bidimensional semialgebraic sets with piecewise linear boundary are images of polynomial or regular maps, and we estimate in both cases the dimension of their generic fibers.

1. Introduction

The present work continues the study of polynomial and regular images of euclidean spaces began in our [FG]. A map $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ is a **polynomial map** if each component f_i is a polynomial of $\mathbb{R}[x_1, \ldots, x_n]$. A subset S of $\mathbb{R}^m$ is a **polynomial image** of $\mathbb{R}^n$ if there exists a polynomial map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $S = f(\mathbb{R}^n)$.

Let S be a subset of $\mathbb{R}^m$. We define

$$\mathrm{p}(S) = \begin{cases} \text{least } p \geq 1 & \text{such that } S \text{ is a polynomial image of } \mathbb{R}^p, \\ \infty & \text{otherwise.} \end{cases}$$

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Ultimately, one of the aims of our efforts is to characterize in an efficient way those subsets S with finite $\mathrm{p}(S)$. At present, this seems, in most cases, a difficult matter. For instance, we do not even know if the invariant p is finite for very simple sets like $S = \{y > 0, y - x^2 - 1 < 0\} \subset \mathbb{R}^2$.

As we have announced above we also deal with regular images of euclidean spaces. A map $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ is a **regular map** if each component f_i is a regular function of $\mathbb{R}(x_1, \ldots, x_n)$, that is, each component $f_i = g_i/h_i$ is a quotient of polynomials such that the zero set of h_i is empty. A subset S of $\mathbb{R}^m$ is a **regular image** of $\mathbb{R}^n$ if it is the image $S = f(\mathbb{R}^n)$ of $\mathbb{R}^n$ by a regular map f. Analogously, for a subset $S \subset \mathbb{R}^m$ we define the invariant

$$\mathrm{r}(S) = \begin{cases} \text{least } r \geq 1 & \text{such that } S \text{ is a regular image of } \mathbb{R}^r, \\ \infty & \text{otherwise.} \end{cases}$$

Again a natural question is to characterize in an efficient way those subsets S with finite $\mathrm{r}(S)$.

In Real Algebraic Geometry we know about several problems for semialgebraic sets S which are either polynomial or regular images, that in a certain sense can be reduced to the case $S = \mathbb{R}^n$. Examples of such problems are:

- optimization of polynomial and/or regular functions on S,
- characterization of the polynomial or regular functions which are positive semidefinite on S,
- the study of the 17th Hilbert problem for S,
- computation of trajectories inside S which are parametrizable by polynomial maps.

For the benefit of the reader, we recall a few definitions and results that appear in our previous work [FG]. Firstly, a subset $T \subset \mathbb{R}^m$ is **irreducible** if its Zariski closure $\overline{T}^{\mathrm{zar}}$ is an irreducible algebraic set. We showed in [FG] that if $\mathrm{p}(S)$ is finite, then S must be a pure dimensional, connected, semialgebraic and irreducible set. Moreover, the image of such an S under a polynomial function $g\colon \mathbb{R}^m \to \mathbb{R}$ must be either unbounded or a singleton.

Obviously,

$$\mathrm{p}(S) \geq \mathrm{r}(S) \geq \dim S \text{ for every semialgebraic subset } S \subset \mathbb{R}^m,$$

and it seems interesting to determine under what conditions on the subset S these inequalities are strict. As we shall see later in this work the topology and the shape of S play a crucial role.

A particularly interesting case is the one when S is an open subset of $\mathbb{R}^n$ which is the image of a polynomial map $f\colon \mathbb{R}^n \to \mathbb{R}^n$. In [FG, 1.3.1] we showed that the **exterior boundary** $\delta S = \overline{S} \setminus S$ of S is included into the set S_f of points of the target space $\mathbb{R}^n$ at which f is not a proper map. This set S_f was introduced by Jelonek in [J1] where he proved that it is a subset of $\overline{f(\mathbb{R}^n)}$. He also gave the definition of a **parametric semiline** as a polynomial 1-dimensional image of $\mathbb{R}$. Clearly, a parametric semiline is an irreducible and unbounded set. Based on his ideas (see [J1] and [J2]), one easily concludes that if an open set $S \subset \mathbb{R}^n$ has $\mathrm{p}(S) = n$, then $\overline{\delta S}^{\mathrm{zar}}$ is a finite union of parametric semilines.

At this point, a natural question arises: *Are the given necessary conditions, also sufficient for an open set $S \subset \mathbb{R}^n$ fulfilling them to ensure that it is a polynomial image of $\mathbb{R}^n$?*

It is natural to begin by checking if some simple subsets of $\mathbb{R}^2$ are polynomial images of $\mathbb{R}^2$. This was done with the open half-plane and the open quadrant in our [FG].

In this work we shall see that the above-mentioned properties are not sufficient. Even more we find new necessary conditions, for which at present we do not know whether they are sufficient.

The paper is organized as follows. In Section 2 we obtain a factorization theorem for polynomial maps which we think has some interest on its own, and we apply it to show that if $S \subset \mathbb{R}^m$ is a semialgebraic curve and $\mathrm{p}(S) < \infty$, then $\mathrm{p}(S) \leq 2$, and $\mathrm{p}(S) = 1$ if and only if S is a closed subset.

The main results of this work are in Section 3. There, we find new necessary conditions to have the equality $\mathrm{p}(S) = \dim(S) = n$ for a subset $S \subset \mathbb{R}^n$.

In Sections 5, 6 and 7 we study the invariants $\mathrm{p}(S)$ and $\mathrm{r}(S)$ for particular types of semialgebraic sets with piecewise linear boundary. For this purpose, we find useful the fact that the set $\{x^2 + y^2 \geq 1\}$ is a polynomial image of $\mathbb{R}^2$, a fact proved in Section 4. Assume in particular that $S \subset \mathbb{R}^2$ is either a closed or open convex unbounded polygon with e linear sides. We prove that $\mathrm{p}(S)$ is finite if and only if S has non-parallel sides. Moreover, if S is open, then $\mathrm{p}(S) = 2$ if and only if $e \leq 2$. For $e \geq 3$, we have that

$$\mathrm{p}(S) \leq \begin{cases} 2(e-2) & \text{if } S \text{ is closed,} \\ 2(e-2)+1 & \text{if } S \text{ is open.} \end{cases}$$

In contrast, if S is again open, then

$$\mathrm{r}(S) \leq \begin{cases} \max\{2, e-1\} & \text{if } S \text{ has two parallel sides,} \\ \max\{2, e\} & \text{otherwise.} \end{cases}$$

It must be pointed out that this class of sets allows us to show that both invariants p and r can be different, even when both are finite. We also prove that several not necessarily convex sets with piecewise linear boundary are polynomial images, and we estimate their invariants p and r. The paper ends with the formulation of some selected open questions.

Most of the results of this work are also true if we change the field $\mathbb{R}$ of the real numbers by any other real closed field. This generalization is quite straightforward, and we will not enter here into its details. We refer the reader to [BCR].

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2. One-dimensional polynomial images

It is a difficult question to decide under what conditions a polynomial map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ factors through $\mathbb{R}^d$ for $d = \dim(f(\mathbb{R}^n))$. We prove that this always happens for $d = 1$, and that fact helps us to get a better understanding of one-dimensional polynomial images. We also see that for $d \geq 2$ there exist examples of maps that do not factorize.

PROPOSITION 2.1: *Let $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map whose image has dimension 1. Then f factors polynomially through $\mathbb{R}$, that is, there exist polynomial maps $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}^m$ such that $f = h \circ g$.*

Proof: Let $\mathbb{F} = \mathbb{R}(f_1, \ldots, f_m)$ be the smallest subfield of the field of rational functions $\mathbb{R}(x_1, \ldots, x_n)$ in n variables that contains $\mathbb{R}$ and $f_1, \ldots, f_m$. Note that

$$\operatorname{tr.deg}(\mathbb{F}|\mathbb{R}) = \dim \operatorname{im} f = 1,$$

and that some of the f_i's are not constant. Then, by [N, §13], [Sch, §3. Thm. 4], there exists a polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\mathbb{F} = \mathbb{R}(g)$.

Next, we are seeking polynomials $h_1, \ldots, h_m \in \mathbb{R}[t]$ such that $h_i(g) = f_i$. For that, since $f_i \in \mathbb{F} = \mathbb{R}(g)$, we have $f_i = P_i(g)/Q_i(g)$ for some coprime polynomials $P_i, Q_i \in \mathbb{R}[t]$. By Bezout's lemma, we can write $1 = P_iA_i + Q_iB_i$ for some $A_i, B_i \in \mathbb{R}[t]$. Substituting the variable t by g we get the polynomial identity

$$\begin{aligned} 1 &= P_i(g)A_i(g) + Q_i(g)B_i(g) = Q_i(g)f_iA_i(g) + Q_i(g)B_i(g) \\ &= Q_i(g)(f_iA_i(g) + B_i(g)); \end{aligned}$$

hence, $Q_i(g)$ is a nonzero constant, and so the polynomials $h_i = P_i(t)/Q_i(g)$ fit our situation.

Finally, the polynomial maps $h = (h_1, \dots, h_m)\colon \mathbb{R} \to \mathbb{R}^m$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ satisfy $f = h \circ g$. ∎

COROLLARY 2.2: *Let $S \subset \mathbb{R}^m$ be a 1-dimensional polynomial image of $\mathbb{R}^n$. Then, if S is closed in $\mathbb{R}^m$ it is a parametric semiline. Otherwise, the closure $\overline{S}$ of S in $\mathbb{R}^m$, which differs from S in just one point, is a parametric semiline.*

Proof: By 2.1, there exist polynomial maps $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}^m$ such that $S = h(g(\mathbb{R}^n))$. By [FG, 1.3(3)], $g(\mathbb{R}^n)$ is an unbounded interval of $\mathbb{R}$. Without loss of generality, we may assume that $g(\mathbb{R}^n)$ is one of the following sets: $\mathbb{R}$, $[0, +\infty)$ or $(0, +\infty)$.

Now, if S is closed, $h^{-1}(S)$ is a closed subset of $\mathbb{R}$ that contains $g(\mathbb{R}^n)$. Thus, $S = h(\overline{g(\mathbb{R}^n)})$, that is, S is either $h(\mathbb{R})$ or $h([0, +\infty))$. But since $[0, +\infty)$ is the image of the polynomial map $t \mapsto t^2$, we conclude that S is a parametric semiline.

Next, if S is not closed, then, h being a proper map, $g(\mathbb{R}^n) = (0, +\infty)$ and $\overline{S} = S \cup \{h(0)\}$ is a parametric semiline. ∎

Examples 2.3: (a) Note that there exist a parametric semiline Γ and a point $p \in \Gamma$, which does not disconnect Γ, such that $\Gamma \setminus \{p\}$ is not a polynomial image of $\mathbb{R}^n$ for any n. Take, for instance,

$$\Gamma = \{x^2 - y^2 + x^3 = 0\}.$$

This set is the image of the polynomial map $t \mapsto (t^2 - 1, t(t^2 - 1))$. However, $\Gamma \setminus \{(-1, 0)\}$ is not a polynomial image of $\mathbb{R}^n$. Otherwise, $\Gamma \setminus \{(-1, 0)\}$, which has two branches, would be a polynomial image of $(0, +\infty)$ which has just one, a contradiction.

(b) The closed and connected subset $S = \Gamma \cap \{x \geq 0\}$ of the parametric semiline Γ above is not a polynomial image of $\mathbb{R}^n$ for any n. If S were a polynomial image of $\mathbb{R}^n$, it would be a polynomial image of $\mathbb{R}$ (note that it is closed in $\mathbb{R}^2$, see 2.2). But since S is analytically reducible at the origin, it is not even an analytic image of $\mathbb{R}$, a contradiction.

(c) With the terminology of the introduction it follows from the proof of Corollary 2.2 that if S is a 1-dimensional set then either $\mathrm{p}(S) = \infty$ or $\mathrm{p}(S) \leq 2$, because the open interval $(0, +\infty)$ is the image of the polynomial map

$$\mathbb{R}^2 \to \mathbb{R}\colon (x, y) \to (xy - 1)^2 + y^2.$$

Remarks 2.4: The factorization property stated in Proposition 2.1 is no longer true for polynomial maps of rank ≥ 2. Take, for instance, the polynomial map

$$f\colon \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x_1, x_2, x_3) & \mapsto & x_3(x_1^2, x_2^2, x_1 x_2), \end{array}$$

whose image is the cone $S = \{xy = z^2\}$. This map does not admit a polynomial factorization through $\mathbb{R}^2$, that is, there are no polynomial maps

$$g = (g_1, g_2)\colon \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{and} \quad h = (h_1, h_2, h_3)\colon \mathbb{R}^2 \to \mathbb{R}^3$$

such that $f = h \circ g$. Otherwise, since the image of g should be 2-dimensional we deduce that $h_1 h_2 = h_3^2$. This implies that h_1 is a reducible polynomial because, on the contrary, h_1 would divide h_3 and so $x_1^2 x_3 = h_1(g)$ would divide the product $h_3(g) = x_1 x_2 x_3$, which is imposible. The same works for h_2.

Hence, h_1, h_2 are reducible and there exist nontrivial factorizations $h_i = F_i G_i$ where $F_i, G_i \in \mathbb{R}[u, v]$ and $i = 1, 2$. Substituting $u = g_1, v = g_2$ one gets

$$x_1^2 x_3 = h_1(g) = F_1(g) G_1(g),$$
$$x_2^2 x_3 = h_2(g) = F_2(g) G_2(g).$$

Thus, the essentially different possibilities are:

$$(F_1(g), G_1(g)) = \begin{cases} (x_1^2, x_3) & \text{or} \\ (x_1 x_3, x_1) & \end{cases}$$
$$(F_2(g), G_2(g)) = \begin{cases} (x_2^2, x_3) & \text{or} \\ (x_2 x_3, x_2). & \end{cases}$$

In any case, we conclude that

$$\mathbb{K} = \mathbb{R}(x_1^2, x_2^2, x_3) \subset \mathbb{E} = \mathbb{R}(g_1, g_2)$$

which is false, since $\operatorname{tr}\deg_{\mathbb{R}} \mathbb{K} = 3$ and $\operatorname{tr}\deg_{\mathbb{R}} \mathbb{E} = 2$.

In addition, note that the cone $S = \{xy = z^2\}$ is a polynomial image of $\mathbb{R}^2$ via the polynomial map $P\colon \mathbb{R}^2 \to \mathbb{R}^3, (s, t) \mapsto (t^3 - s) \cdot (s^2, t^2, st)$. In fact, a direct substitution shows that the image of P is contained in S. To check the converse we take a point $p = (a, b, c) \in S$ and we prove that it is in the image of P. For, if $b = 0$ we have that $P(\sqrt[3]{-a}, 0) = p$. Next, if $b \neq 0$ we consider the system of polynomial equations:

$$(t^3 - s)s^2 = a,$$
$$(t^3 - s)t^2 = b,$$
$$(t^3 - s)st = c.$$

Dividing the third equation by the second one, we deduce that $s = \frac{c}{b}t$. Substituting this value in the second equation, we obtain that

$$t^5 - \frac{c}{b}t^3 - b = 0.$$

This equation has a real root t_0, and $P(\frac{c}{b}t_0, t_0) = p$, as wanted.

3. A new obstruction to be a polynomial image

In this section we find general necessary conditions that must satisfy the exterior boundary of the image of a polynomial map $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Before that we have to introduce some terminology. Following [J2, 6.2], we recall that a subset $S \subset \mathbb{R}^n$ is **$\mathbb{R}$-uniruled** if for every point $a \in S$ there exists a parametric semiline L such that $a \in L \subset S$. Moreover, the set S is **generically $\mathbb{R}$-uniruled** if there is an open dense subset $U \subset S$, such that for every point $a \in U$ there is a parametric semiline L such that $a \in L \subset S$.

THEOREM 3.1: *Let $S \subset \mathbb{R}^n$ be a n-dimensional semialgebraic subset and let δS be its exterior boundary. Suppose that S is a polynomial image of $\mathbb{R}^n$. Then there exist two semialgebraic sets $S_0, S_1 \subset \mathbb{R}^n$ such that:*

(a) $\dim S_1 \leq n - 2$,

(b) *S_0 is either empty or it is a closed, $(n-1)$-dimensional and generically $\mathbb{R}$-uniruled set,*

(c) $\delta S \subset S_0 \cup S_1 \subset \overline{S} \cap \overline{\delta S}^{\text{zar}}$.

Proof: Since S is a polynomial image of $\mathbb{R}^n$ there exists a polynomial map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = S$. As we have recalled in the introduction, the set $\mathcal{S}_f$ of points at which f is not proper contains δS. By [J2, 6.4], the set $\mathcal{S}_f$ is closed, semialgebraic, $\mathbb{R}$-uniruled and $\dim \mathcal{S}_f \leq n-1$. Let $X_1, \ldots, X_\ell$ be the irreducible components of $\overline{\mathcal{S}_f}^{\text{zar}}$. We can suppose that $\dim X_i \cap \delta S \leq n-2$ if and only if $1 \leq i \leq r$ for some $r \leq \ell$. Let $S_1 = \bigcup_{i=1}^r X_i \cap \delta S$, which is a semialgebraic set of dimension $\leq n-2$.

If $\delta S \subset S_1$, then we take $S_0 = \emptyset$ and we are done. Otherwise, let

$$T = \left(\mathcal{S}_f \setminus \bigcup_{i=1}^r X_i\right) \cap \overline{\delta S}^{\text{zar}} = (\mathcal{S}_f \cap \overline{\delta S}^{\text{zar}}) \setminus \bigcup_{i=1}^r X_i.$$

Note that since $T \subset \mathcal{S}_f \cap \overline{\delta S}^{\text{zar}} \subset \overline{S} \cap \overline{\delta S}^{\text{zar}}$, also $\overline{T} \subset \overline{S} \cap \overline{\delta S}^{\text{zar}}$. Moreover, T and T are semialgebraic sets and T is an open subset of $\overline{T}$.

Now, we will see that for every $x \in T$ there exists a parametric semiline L such that $x \in L \subset \overline{T}$. Indeed, since $\mathcal{S}_f$ is an $\mathbb{R}$-uniruled set, for every $x \in T \subset \mathcal{S}_f$

there exists a parametric semiline L such that $x \in L \subset \mathcal{S}_f$. Since $x \in T$, L is not contained in any of the algebraic sets $X_1, \dots, X_r$, hence $L \cap \bigcup_{i=1}^r X_i$ is a finite set and $\overline{L \setminus \bigcup_{i=1}^r X_i} = L$. On the other hand, since $L \subset \mathcal{S}_f$ is irreducible then there exists $r+1 \leq j \leq \ell$ such that $L \subset X_j$. Moreover, since X_j is irreducible and $\dim X_j \cap \delta S = n-1 = \dim X_j$, we conclude that $L \subset X_j \subset \overline{\delta S}^{\text{zar}}$. Hence,

$$L \setminus \bigcup_{i=1}^r X_i \subset \left(\mathcal{S}_f \setminus \bigcup_{i=1}^r X_i\right) \cap \overline{\delta S}^{\text{zar}} = T$$

and therefore, $L = \overline{L \setminus \bigcup_{i=1}^r X_i} \subset \overline{T}$. Thus, the $(n-1)$-dimensional closed semialgebraic set $S_0 = \overline{T}$ is also generically $\mathbb{R}$-uniruled.

Finally, we see that $\delta S \subset S_0 \cup S_1$. Since $\delta S \subset \mathcal{S}_f \cap \overline{\delta S}^{\text{zar}}$, if there exists some $x \in \delta S \setminus S_1$ then

$$x \in \delta S \setminus \bigcup_{i=1}^r X_i \subset \left(\mathcal{S}_f \setminus \bigcup_{i=1}^r X_i\right) \cap \overline{\delta S}^{\text{zar}} = T \subset S_0,$$

and this concludes the proof. ∎

Remark 3.2: (a) The set S_0 in the previous Theorem is generically $\mathbb{R}$-uniruled, but we do not know if it is in fact $\mathbb{R}$-uniruled. As far as we know the problem of determining which generically $\mathbb{R}$-uniruled semialgebraic sets are $\mathbb{R}$-uniruled is still open.

(b) Clearly, the union of $\mathbb{R}$-uniruled sets is also $\mathbb{R}$-uniruled. However, the irreducible components of an $\mathbb{R}$-uniruled algebraic set $X \subset \mathbb{R}^n$ have not to be even generically $\mathbb{R}$-uniruled. Consider, for instance, the union $X \subset \mathbb{R}^3$ of the irreducible algebraic sets

$$X_1\colon z^2 - x(x^2+y^2-1)^2 = 0,$$
$$X_2\colon x^2+y^2-1 = 0.$$

Obviously, the cylinder X_2 is $\mathbb{R}$-uniruled. On the other hand, X_1 is the union of the circumference $C = \{z = 0, x^2+y^2 = 1\}$ and the image of the map $(s,t) \mapsto (t^2, s, t(t^4+s^2-1))$. Note that for each point $a \in X_1 \setminus \{x < 0\}$ there exists a parametric semiline $L \subset X_1$ through a. Hence, X is $\mathbb{R}$-uniruled. However, for each point $b \in X_1 \cap \{x < 0\} = C \cap \{x < 0\}$ there is no parametric semiline through b contained in X_1; hence, X_1 is not even generically $\mathbb{R}$-uniruled. ∎

Next, we show that any real algebraic subset of a hyperplane of $\mathbb{R}^n$ is the union of some of the irreducible components of the Zariski closure of the exterior boundary of some polynomial image $S \subset \mathbb{R}^n$ of $\mathbb{R}^n$. Hence, it seems that there should not be many restrictions for the irreducible components of codimension ≥ 2 of the Zariski closure of the exterior boundary δS.

PROPOSITION 3.3: *Let $F_1, \ldots, F_r \in \mathbb{R}[u_1, \ldots, u_n]$ be polynomials in n variables which do not vanish simultaneously at the origin. Then there is a polynomial map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ whose image is the set $S = \mathbb{R}^{n+1} \setminus X$, where*

$$X = \{F_1 = 0, \ldots, F_r = 0, u_{n+1} = 0\} \cup \{u_1 = 0, \ldots, u_n = 0, u_{n+1} < 0\}.$$

Proof: Let $f = (f_1, \ldots, f_n, f_{n+1})\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the polynomial map defined by

$$f_i(x) = x_{n+1}x_i \quad \text{for } i = 1, \ldots, n;$$
$$f_{n+1}(x) = x_{n+1}^2 \sum_{k=1}^{r} F_k(x_{n+1}x_1, \ldots, x_{n+1}x_n)^2 + x_{n+1} \sum_{i=1}^{n} x_i^2.$$

An straightforward computation shows that the polynomials $f_1, \ldots, f_{n+1}$ satisfy the equality

$$(*) \qquad x_{n+1}^3 \sum_{k=1}^{r} F_k(f_1, \ldots, f_n)^2 - x_{n+1}f_{n+1} + \sum_{i=1}^{n} f_i^2 = 0,$$

and that $f(x_1, \ldots, x_n, 0) = 0$, and so $0 \in \operatorname{im} f$.

Now, let us see that $\operatorname{im} f = S$. For each point $u = (u_1, \ldots, u_n, u_{n+1}) \in \mathbb{R}^{n+1}$, consider the polynomial

$$P_u(T) = T^3 \sum_{k=1}^{r} F_k(u_1, \ldots, u_n)^2 - u_{n+1}T + \sum_{i=1}^{n} u_i^2.$$

We claim that this polynomial has a nonzero root in $\mathbb{R}$ if and only if $u \in S \setminus \{0\}$. Indeed, if $u \in S$ and $\sum_{i=1}^n u_i^2 \neq 0$ then P_u is a polynomial of odd degree and $P_u(0) \neq 0$. Hence, it has a nonzero real root. If $\sum_{i=1}^n u_i^2 = 0$, and $u \in S \setminus \{0\}$, then $u_{n+1} > 0$ and so

$$\theta = \sqrt{\frac{u_{n+1}}{\sum_{k=1}^r F_k(0)^2}}$$

is a nonzero root of P_u.

Conversely, if $u \notin S$ then either P_u is a nonzero constant polynomial (which has no root in $\mathbb{R}$) or its only root is zero (this last happens if $u_1 = \cdots = u_n = 0$ and $u_{n+1} < 0$). If $u = 0$ then $P_u = \sum_{k=1}^r F_k(0)^2 T^3$ whose unique root is zero.

Using the identity $(*)$ one deduces that if $u \in \operatorname{im} f \setminus \{0\}$ then P_u has a nonzero root and therefore $u \in S \setminus \{0\}$. Thus, we only have to check the converse.

Let $u \in S \setminus \{0\}$ and let x_{n+1} be a nonzero root of P_u and $x_i = u_i/x_{n+1}$ for $i = 1, \ldots, n$. Computing a little and using that x_{n+1} is a nonzero root of P_u one deduces that $f(x_1, \ldots, x_{n+1}) = u$, as wanted. ∎

As an easy consequence of Theorem 3.1, we get the following obstruction for a n-dimensional semialgebraic subset of $\mathbb{R}^n$ to be a polynomial image of $\mathbb{R}^n$.

COROLLARY 3.4: *Let $S \subset \mathbb{R}^n$ be a n-dimensional semialgebraic set and let $X \subset \mathbb{R}^n$ be a $(n-1)$-dimensional algebraic set. Suppose that $X \cap \overline{S}$ is bounded and that $\dim(X \cap \delta S) = n-1$. Then S is not a polynomial image of $\mathbb{R}^n$.*

Proof: We proceed by the way of contradiction. Suppose that $S \subset \mathbb{R}^n$ is a polynomial image of $\mathbb{R}^n$. By 3.1 there exist two semialgebraic sets $S_0, S_1 \subset \mathbb{R}^n$ such that:

(a) $\dim S_1 \leq n-2$,

(b) S_0 is either empty or it is a closed, $(n-1)$-dimensional and generically $\mathbb{R}$-uniruled set,

(c) $\delta S \subset S_0 \cup S_1 \subset \overline{S} \cap \overline{\delta S}^{\text{zar}}$.

Since $\dim(X \cap \delta S) = n-1$ and $\delta S \subset S_0 \cup S_1$, also $\dim(X \cap \delta S \cap S_0) = n-1$. Hence, X and $\overline{S_0}^{\text{zar}}$ share an irreducible component Y of dimension $n-1$ such that $\dim(Y \cap \delta S) = n-1$. Let $x \in Y \cap \delta S \cap S_0$ be a point such that x does not belong to any of the other irreducible components of $\overline{S_0}^{\text{zar}}$ (such a point exists because $\dim(Y \cap \delta S) = n-1$). Since S_0 is a generically $\mathbb{R}$-uniruled set, we can choose x such that there exists a parametric semiline $L \subset S_0$ through it.

Next, we want to see that $L \subset Y \subset X$. Since $L \subset S_0$, we have that $\overline{L}^{\text{zar}} \subset \overline{S_0}^{\text{zar}}$. L being irreducible, we deduce that $x \in L \subset Z$ for some irreducible component Z of $\overline{S_0}^{\text{zar}}$. In fact, we necessarily have $Z = Y$. Hence, we get that $L \subset S_0 \cap X \subset \overline{S} \cap X$ which is a bounded set, a contradiction because the parametric semilines are unbounded. ∎

Remark 3.5: The previous result does not extend to an algebraic set X of codimension ≥ 2. Take, for instance, the octant

$$S = \{x_1 > 0, x_2 > 0, x_3 > 0\} \subset \mathbb{R}^3,$$

which by [FG, 1.6] is a polynomial image of $\mathbb{R}^3$. Now, we consider the line $X = \{x_3 = 0, x_1 + x_2 = 1\}$. Note that $\dim X \cap \delta S = 1$ and that

$$X \cap \overline{S} = \{x_3 = 0, x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\},$$

which is a bounded set.

Example 3.6: The semialgebraic sets

$$S = \{x_1 *_1 0, \dots, x_n *_n 0, x_1 + \cdots + x_n > 1\},$$

where $*_i$ denotes either $>$ or $\geq$, are not polynomial images of $\mathbb{R}^n$. Indeed, consider $X = \{x_1 + \cdots + x_n = 1\}$. It is clear that

$$X \cap \overline{S} = \{x_1 + \cdots + x_n = 1, x_1 \geq 0, \dots, x_n \geq 0\} = X \cap \delta S$$

is bounded and has dimension $n-1$. Then by 3.4 we conclude that such sets S are not polynomial images of $\mathbb{R}^n$.

Note that for $n = 2$ the previous example solves a question proposed in [FG, 4.1.1]. In fact, we have the following more general result:

COROLLARY 3.7: *Let $S \subset \mathbb{R}^2$ be an open convex polygon with linear sides. Then S is a polynomial image of $\mathbb{R}^2$ if and only if S has only two sides, that is, S is affinely equivalent to the open quadrant.*

Proof: First, suppose that S is a polygon with more than two sides. Then S has a bounded side ℓ. Since S is convex then $\overline{\ell}^{\text{zar}} \cap \overline{S} = \overline{\ell}$, which is a bounded segment. Thus, by 3.4, we conclude that S is not a polynomial image of $\mathbb{R}^2$.

Conversely, if S has only two sides, then S is affinely equivalent to the open quadrant. Now, it follows from [FG, 1.7] that S is a polynomial image of $\mathbb{R}^2$. ∎

Next, we show that Theorem 3.1 can be slightly improved for $n = 2$. This is a consequence of the fact that we know stronger properties for the set S_f of a polynomial map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ than for the general case; compare the results in [J2, 4.2] and [J2, 6.4]. In fact, Jelonek announced, in the 2003 *Network workshop on real algebra (Dortmund)*, a related result from which it follows that $\mathrm{p}(S) \geq 3$ for the set

$$S = \{x > 0, y > 0, x - y + 4 > 0\}$$

(a question proposed in [FG, 4.1.1]).

THEOREM 3.8: *Let $S \subset \mathbb{R}^2$ be a 2-dimensional semialgebraic subset and let δS be its exterior boundary. Suppose that S is a polynomial image of $\mathbb{R}^2$. Then δS is either empty or there exist a finite set F and a finite family of parametric semilines $L_1, \dots, L_r$ such that*

$$\delta S \subset F \cup \bigcup_{i=1}^{r} L_i \subset \overline{S} \cap \overline{\delta S}^{\text{zar}}.$$

Proof: Since S is a polynomial image of $\mathbb{R}^2$, there is a polynomial map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathbb{R}^2) = S$. By [J2, 4.2], the set $\mathcal{S}_f$ is either empty or a finite union of parametric semilines $L_1, \ldots, L_s$. We can suppose that there exists an integer r such that the set $L_i \cap \delta S$ is finite (or empty) if and only if $r < i \leq s$. Thus, the set $F = (L_{r+1} \cup \cdots \cup L_s) \cap \delta S$ is a finite subset of δS.

Now, we see that if $1 \leq i \leq r$ then $L_i \subset \overline{S} \cap \overline{\delta S}^{\mathrm{zar}}$. First, we recall that $\mathcal{S}_f \subset \overline{S}$. On the other hand, since $\overline{L_i}^{\mathrm{zar}}$ is an irreducible algebraic set of dimension 1 and $L_i \cap \delta S$ is infinite then $\overline{L_i}^{\mathrm{zar}} \subset \overline{\delta S}^{\mathrm{zar}}$. Hence, $L_i \subset \overline{S} \cap \overline{\delta S}^{\mathrm{zar}}$.

Finally, since $\delta S \subset \mathcal{S}_f$, we have

$$\begin{aligned}\delta S = \delta S \cap \mathcal{S}_f &= \bigcup_{i=1}^{s} (\delta S \cap L_i)\\ &= \bigcup_{i=1}^{r} (\delta S \cap L_i) \cup \bigcup_{i=r+1}^{s} (\delta S \cap L_i) \subset \bigcup_{i=1}^{r} L_i \cup F \subset \overline{S} \cap \overline{\delta S}^{\mathrm{zar}},\end{aligned}$$

as wanted. ∎

To finish this section, we show that for every integer $r \geq 1$ there exists a polynomial image $S \subset \mathbb{R}^2$ of $\mathbb{R}^2$ whose exterior boundary has r connected components which are simultaneously bounded and 1-dimensional.

PROPOSITION 3.9: *Let $J \subset \mathbb{R}$ be a finite union of (bounded or unbounded) disjoint closed proper intervals of $\mathbb{R}$. Then the set $S = \mathbb{R}^2 \setminus (J \times \{0\})$ is a polynomial image of $\mathbb{R}^2$.*

Proof: First, since J is a proper subset of $\mathbb{R}$, the (topological) boundary $\overline{J} \setminus \mathring{J}$ of J in $\mathbb{R}$ has at least one point. Let $a \in \mathbb{R}$ be such a point and let $F \in \mathbb{R}[T]$ be a polynomial such that $J = \{t \in \mathbb{R}\colon F(t)(a-t) \leq 0\}$. To construct such F, let $\mathrm{Fr}(J) = \{a, a_2, \ldots, a_r\}$, and

$$G(T) = \varepsilon(a-T)(T-a_2)\cdots(T-a_r)$$

where $\varepsilon = \pm 1$ is chosen so that G is nonpositive on J. Take the polynomial $F(T) = \varepsilon(T-a_2)\cdots(T-a_r)$. In particular, note that $F(a) \neq 0$.

Next, we consider the polynomial map $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned}f_1 &= a - x^2 F(a)(1-xy),\\ f_2 &= \frac{F(a - x^2F(a)(1-xy)) - F(a)(1-xy)}{x}.\end{aligned}$$

Note that f_2 is in fact a polynomial in the variables x, y. A straightforward computation shows that the polynomials f_1, f_2 satisfy the equality

$$(*) \qquad x^3 f_2 - x^2 F(f_1) + (a - f_1) = 0.$$

Now, let us see that $\operatorname{im} f = S \cup \{(a,0)\}$. Consider, for $(u,v) \in \mathbb{R}^2$, the polynomial

$$P_{(u,v)}(T) = T^3 v - T^2 F(u) + (a - u).$$

We claim that this polynomial has a nonzero root in $\mathbb{R}$ if and only if $(u,v) \in S$. Indeed, if $v \neq 0$ and $u \neq a$ then $P_{(u,v)}$ is a polynomial of odd degree such that $P_{(u,v)}(0) \neq 0$. Hence, it has a nonzero real root. If $(u,v) \in S$ and $u = a$ then $\theta = F(a)/v$ is a nonzero root of $P_{(u,v)}$. If $(u,v) \in S$ and $v = 0$ then $\theta = \sqrt{F(a)(a-u)}/F(a)$ is a nonzero root of $P_{(u,v)}$.

Conversely, if $(u,v) \notin S$ then $v = 0$ and $F(a)(a-u) \leq 0$, that is, $P_{(u,v)}$ has no real root or its unique root is 0 (the latter only happens if $u = a$).

Using the identity $(*)$ and that $F(a) \neq 0$ one deduces that $P_{(u,v)}$ has a nonzero root for each $(u,v) \in \operatorname{im} f \setminus \{(a,0)\}$ and therefore $(u,v) \in S$. Conversely, let $(u,v) \in S$ and let x be a nonzero root of $P_{(u,v)}$ and $y = (u - a + x^2 F(a))/x^3 F(a)$. Computing a little and using that x is a nonzero root of $P_{(u,v)}$ one deduces that $f(x,y) = (u,v)$.

Note that $\operatorname{im} f = S \cup \{(a,0)\}$ and that the fiber of the point $(a,0)$ is $\{(0,0)\}$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map whose image is $\mathbb{R}^2 \setminus \{(0,0)\}$ (which exists by [FG, 1.5]). The composition $g = f \circ \varphi$ has S as its image. ∎

Remark 3.10: (i) The previous result does not go against 3.4 because all the bounded connected components of δS are contained in the line $\{v = 0\}$, which intersects $\bar{S}$ at an unbounded set.

(ii) Note that the map f constructed in the last proposition has finite fibers on the points of $S \cap \{v = 0\}$. This allows us, again using [FG, 1.5], to degenerate some of the intervals of J into points.

4. Exterior of the disc

Let $T = \{x^2 + y^2 > 1\}$ be the complementary set of the closed disc of radius one centered at the origin of $\mathbb{R}^2$. We prove here that $\mathrm{p}(T) = 3$ and $\mathrm{p}(\overline{T}) = 2$. We will use this in the next section to estimate the invariant p for unbounded convex polygons.

PROPOSITION 4.1: *Let $D \subset \mathbb{R}^2$ be the open disc of radius one centered at the origin. Its complementary set $S = \mathbb{R}^2 \setminus D$ is a polynomial image of $\mathbb{R}^2$. In other words,* $\mathrm{p}(S) = 2$.

Proof: First, let us explain how we construct a suitable candidate for a polynomial map $f = (P, Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathbb{R}^2) = S$. Since we need that $P^2 + Q^2 \geq 1$, a natural choice is to look for $P, Q, R \in \mathbb{R}[x, y]$ such that $P^2 + Q^2 = 1 + R^2$. Hence,

$$(P-1)(P+1) = (R-Q)(R+Q).$$

Equivalently, there should exist auxiliary polynomials $\varphi, \psi, \ell, h \in \mathbb{R}[x, y]$ such that

$$P - 1 = \psi h, \quad P + 1 = \varphi \ell, \quad R - Q = \varphi \psi \quad \text{and} \quad R + Q = h\ell.$$

In particular, this implies that $2 = \varphi\ell - \psi h$. Thus, we need a pair polynomials $\varphi, \psi \in \mathbb{R}[x, y]$ whose common zero set in $\mathbb{C}^2$ is empty. After several trials we have taken $\varphi = xy - 1$ and $\psi = x\varphi - y$, because they simplify the rest of the proof.

Next, we choose $\ell, h \in \mathbb{R}[x, y]$ such that

$$2 = \varphi\ell - \psi h = \varphi\ell - (x\varphi - y)h = \varphi(\ell - xh) + yh.$$

In fact, for the sake of simplicity we look for $\ell, h \in \mathbb{R}[x]$ satisfying the previous equation. After expanding that expresion we get

$$2 = (x(\ell - xh) + h)y + xh - \ell.$$

Thus, we must take $h = 2x$ and $\ell = 2(x^2 - 1)$. Putting all together we get that

$$P = \psi h + 1, \quad Q = \frac{h\ell - \varphi\psi}{2} \quad \text{and} \quad R = \frac{h\ell + \psi\varphi}{2},$$

where

$$\varphi(x, y) = xy - 1, \quad \psi(x, y) = x(xy - 1) - y,$$
$$\ell(x, y) = 2(x^2 - 1) \quad \text{and} \quad h(x, y) = 2x.$$

Since $P^2 + Q^2 = 1 + R^2 \geq 1$, it is clear that $f(\mathbb{R}^2) \subset S$. Next, we check that in fact $f(\mathbb{R}^2) = S$. Since $f(0, -2b) = (1, b)$ and $f(1, 2b + 1) = (-1, b)$ for all $b \in \mathbb{R}$, we are only left to check that all $(a, b) \in S$ with $a^2 \neq 1$ belong to $f(\mathbb{R}^2)$. In what follows we always suppose that we are in this situation.

Consider the resultant $\Delta(X)$ of $P(X,Y)-a$, $Q(X,Y)-b$ with respect to Y, given by

$$\Delta(X)=\frac{1}{2}X(X^2-1)H(X),$$

where

$$H(X)=16X^2(X^2-1)^2-8bX(X^2-1)+1-a^2.$$

We claim that for all a,b as above, there exists a real root x of H different from $-1,0,1$. Suppose this is true for a moment. Then the univariate polynomial $P(x,Y)-a$ has degree one (note that $x\neq -1,0,1$) and has a common (real) root

$$y=\frac{1}{x}+\frac{a+1}{2x(x^2-1)}$$

with the polynomial $Q(x,Y)-b$ (recall that $\Delta(x)=0$). Thus, $f(x,y)=(a,b)$.

Next, we check that H has a real root, necessarily different from $-1,0,1$ because $a^2\neq 1$. Consider the polynomial

$$G(Z)=16Z^2-8bZ+1-a^2$$

which satisfies the identity $G(X(X^2-1))=H(X)$. Note that the real number

$$z=\frac{b+\sqrt{a^2+b^2-1}}{4}$$

is a nonzero root of G, because $a^2\neq 1$. Finally, it is enough to observe that there exists $x\in\mathbb{R}$ such that $x(x^2-1)=z$, and we are done. ∎

Remark 4.2: In [FG, 1.4(i)] it was pointed out that the exterior of the closed unitary disc $T=\{x^2+y^2>1\}$ is not a polynomial image of $\mathbb{R}^2$. However, it is a polynomial image of $\mathbb{R}^3$. Indeed, by 4.1, there exists a polynomial map $g\colon \mathbb{R}^2\to\mathbb{R}^2$ such that $g(\mathbb{R}^2)=\overline{T}$. By [FG, 1.4(iv)], there exists a polynomial map $\mathbb{R}^2\to\mathbb{R}^2$ whose image is the upper open half-plane $\mathbb{R}\times(0,+\infty)$. Since $\mathbb{R}^3=\mathbb{R}\times\mathbb{R}^2$ and $\mathbb{R}\times(\mathbb{R}\times(0,+\infty))=\mathbb{R}^2\times(0,+\infty)$, there exists a polynomial map $\mathbb{R}^3\to\mathbb{R}^3$ whose image is $\mathbb{R}^2\times(0,+\infty)$. Hence, everything reduces to checking that T is the image of $\mathbb{R}^2\times(0,+\infty)$ under the polynomial map

$$h\colon\ \begin{array}{ccc}\mathbb{R}^3=\mathbb{R}^2\times\mathbb{R} & \to & \mathbb{R}^2\\ (x,t) & \mapsto & (1+t)g(x).\end{array}$$

Thus, we conclude that $\mathrm{p}(T)=3$.

5. Convex unbounded polygons and sawsets

The main purpose of this section is to prove that the closed (resp. open) convex unbounded polygons of $\mathbb{R}^2$ with nonparallel linear sides are polynomial images of some $\mathbb{R}^n$. We also see that other, not necessarily convex, sets with piecewise linear boundary are polynomial images of some $\mathbb{R}^n$. We begin with the closed case.

THEOREM 5.1: *Let $S \subset \mathbb{R}^2$ be a closed convex unbounded polygon with $e \geq 3$ linear sides. Suppose that S has nonparallel sides. Then there is a polynomial map $f\colon \mathbb{R}^{2(e-2)} \to \mathbb{R}^2$ such that $f(\mathbb{R}^{2(e-2)}) = S$.*

Proof: The proof runs by induction on e. For $e = 3$, after an affine change of coordinates we can assume that S is a convex unbounded polygon of $\mathbb{R}^2$ whose two unbounded sides are contained in the lines $x = 0$, $y = 0$, and $S \subset \{x \geq 0, y \geq 0\}$. Thus, the bounded side of S lies on a line with equation $x/a + y/b = 1$ for positive real numbers a, b. Then S is the image of the map $g = (aP^2, bQ^2)$, where $(P, Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map whose image is $\{x^2 + y^2 \geq 1\}$ (such a map exists as we have seen in 4.1).

Suppose now $e \geq 4$. Again we can assume that the two unbounded sides of S are contained in the lines $x = \lambda, y = \mu$ for some negative real numbers λ, μ, and that S is contained in $\{x \geq \lambda, y \geq \mu\}$. We can also suppose that $a_0 < 0 = a_1 < \cdots < a_{e-2}$ are the first coordinates of the vertices $V_0, V_1, \ldots, V_{e-2}$ of the boundary of S and that the origin is the vertex V_1 corresponding to $a_1 = 0$. We write $V_j = (a_j, b_j)$ for $j = 0, \ldots, e-2$.

Consider the convex polygons $S_1 = S \cap \{y \geq 0\}$ and $S_2 = S \cap \{x \geq 0\}$. By the induction hypothesis, we can find polynomial maps $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $h\colon \mathbb{R}^{2(e-3)} \to \mathbb{R}^2$ such that $g(\mathbb{R}^2) = S_1$ and $h(\mathbb{R}^{2(e-3)}) = S_2$. Consider the polynomial map

$$f\colon \quad \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^{2(e-3)} & \to & \mathbb{R}^2 \\ (u, v) & \mapsto & g(u) + h(v). \end{array}$$

We end by proving that S is its image. Indeed, it is enough to check that S is equal to the set

$$S_1 + S_2 = \{s_1 + s_2\colon s_1 \in S_1, s_2 \in S_2\}.$$

Since $(0,0) \in S_1 \cap S_2$, we have that $S = S_1 \cup S_2 \subset S_1 + S_2$. To prove the converse let $s_1 \in S_1$ and $s_2 \in S_2$; we have to see that $s_1 + s_2 \in S$. If both $s_1, s_2 \in S_1 \cap S_2$ then it is clear that $s_1 + s_2 \in S_1 \cap S_2 \subset S$. Thus, we may assume that one of them is not in $S_1 \cap S_2$. We analyze the most involved case, that is, $s_2 \in S_2 \setminus S_1$.

Let $\ell_j = \{y = b_j\}$ be the horizontal line through the vertex V_j, and let k be such that s_2 belongs to the band $R = \{b_{k+1} \le y < b_k\}$ as in Figure 1. Let A, B be the respective intersection points of ℓ_{k+1} with the lines joining the points s_1 and s_2 with the origin O. Let C be the intersection point of ℓ_{k+1} and the parallel ℓ through s_1 to the line OB.

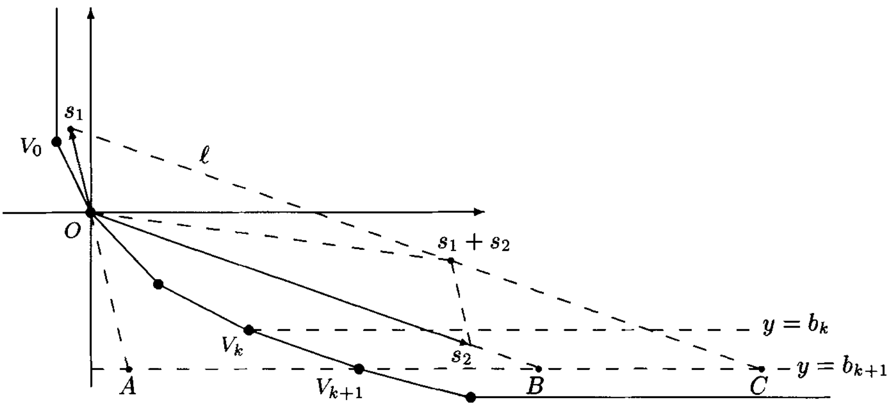

Figure 1.

Observe that $s_1 + s_2 \in \ell$. Since $s_1, C \in S$ and S is convex, it is enough to check that $s_1 + s_2$ is in the segment $\overline{s_1C}$, or equivalently, that $\overline{Os_2}$ is shorter than $\overline{s_1C}$.

For that, note that the sides of the triangles $\triangle OAB$ and $\triangle s_1AC$ are parallel. Thus, since $\overline{AB}$ is shorter than $\overline{AC}$, also $\overline{OB}$ is shorter than $\overline{s_1C}$. But $\overline{Os_2}$ is shorter that $\overline{OB}$, and we are done. ∎

COROLLARY 5.2: *Let $T \subset \mathbb{R}^2$ be an open convex unbounded polygon with $e \ge 3$ linear sides. Suppose that T has nonparallel sides. Then there is a polynomial map $g\colon \mathbb{R}^{2(e-2)+1} \to \mathbb{R}^2$ such that $g(\mathbb{R}^{2(e-2)+1}) = T$.*

Proof: First, by [FG, 1.4(iv)], there exists a polynomial map $\mathbb{R}^2 \to \mathbb{R}^2$ whose image is the upper open half-plane $\mathbb{R} \times (0, +\infty)$. Let $m = 2(e-2)$; since

$$\mathbb{R}^{m+1} = \mathbb{R}^{m-1} \times \mathbb{R}^2 \quad \text{and} \quad \mathbb{R}^{m-1} \times (\mathbb{R} \times (0, +\infty)) = \mathbb{R}^m \times (0, +\infty),$$

it is enough to check that T is the image of the restriction to $\mathbb{R}^m \times (0, +\infty)$ of a polynomial map $h\colon \mathbb{R}^{m+1} \to \mathbb{R}^2$.

We can assume that the two unbounded sides of T are contained in the lines of equations $x = 0, y = 0$, and $T \subset \{x > 0, y > 0\}$. By 5.1, there exists a polynomial map $f\colon \mathbb{R}^m \to \mathbb{R}^2$ such that $f(\mathbb{R}^m) = \overline{T}$. Next, consider the map

$$\begin{array}{rcll} h\colon & \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R} & \to & \mathbb{R}^2 \\ & (x,t) & \mapsto & f(x) + t(1,1). \end{array}$$

A straightforward computation shows that $T = h(\mathbb{R}^m \times (0, +\infty))$, as wanted. ∎

Remarks 5.3: (a) Note that if an unbounded convex polygon S with linear sides had two parallel sides then it would not be a polynomial image of any $\mathbb{R}^n$. This is so because, S being convex, the parallel sides had to be the unbounded ones. Hence, the projection of S in that direction (over an orthogonal line) would be a bounded set. This would imply, see [FG, 1.3.(3)], that S is not a polynomial image.

(b) For $e = 3$ in 5.2, we have that each open convex unbounded polygon S with three linear sides is a polynomial image of $\mathbb{R}^3$. This contrasts with 3.7, where we prove that these kinds of sets are not polynomial images of $\mathbb{R}^2$. In other words, $\mathrm{p}(S) = 3$.

(c) Using the invariant p we can summarize the preceding results as follows. Let S be a convex unbounded polygon with $e \geq 3$ linear and nonparallel sides. We have:

$$\mathrm{p}(S) \leq \begin{cases} 2(e-2) & \text{if } S \text{ is closed,} \\ 2(e-2)+1 & \text{if } S \text{ is open.} \end{cases}$$

In what follows in this section, we will prove that other, not necessarily convex, sets with piecewise linear boundary are also polynomial images of some $\mathbb{R}^n$. To that end, we need to introduce the following terminology.

Definition 5.4: For a given $n \geq 2$, a semialgebraic set $S \subset \mathbb{R}^2$ is said to be a n-**generating set** if it satisfies the following properties:

- It is a polynomial image of $\mathbb{R}^n$ and its boundary $\overline{S} \setminus \overset{\circ}{S}$ is piecewise linear.
- It is either an open or a closed subset of $\mathbb{R}^2$ with nonempty interior.
- There is a positive real number $N > 0$ such that for every vector $w = (x, y)$ of the **wedge** $W = \{y - Nx \geq 0, y + Nx \geq 0\}$ the set $S + w$ is contained in S.

Under certain conditions, we will show (see Theorem 5.7 below) that if S is a n-generating set the union of finitely many images of S under translations is a polynomial image of $\mathbb{R}^{n+1}$. These kinds of unions will be called **sawsets** because of the shape of their boundary. Before that, we provide some examples of n-generating sets to give an idea of the aim of Theorem 5.7.

Examples 5.5: (i) The open quadrant

$$Q = \{y - x > 0, y + x > 0\}$$

is a 2-generating set. The same holds for the closed quadrant

$$\overline{Q} = \{y - x \geq 0, y + x \geq 0\}.$$

(ii) Let $S \subset \mathbb{R}^2$ be an open convex unbounded polygon with $e \geq 3$ linear sides. Suppose that S contains a point $p = (p_1, p_2)$ and two half-lines r_1, r_2 with origin at p of the form

$$r_1 = \{(y - p_2) + N(x - p_1) = 0, \quad y \geq p_2\}$$
$$\text{and} \quad r_2 = \{(y - p_2) - N(x - p_1) = 0, \quad y \geq p_2\},$$

for some $N > 0$. Then S is a $(2e-3)$-generating set.

By 5.2, it is enough to prove that for every w in the wedge

$$W = \{y - Nx \geq 0, y + Nx \geq 0\}$$

the set $S + w$ is contained in S. Note that for each $w \in W$ we have $w = u + v$ where $p + u \in r_1$ and $p + v \in r_2$. Thus, if $S + u \subset S$ and $S + v \subset S$ we have that

$$S + w = (S + u) + v \subset S + v \subset S,$$

and we will be done. Hence, it is enough to check that $S + \xi \subset S$ for each vector $\xi \in (r_1 - p) \cup (r_2 - p)$.

Notice that S is a finite intersection of open half-planes, that is,

$$S = \bigcap_{k=1}^{n} \{l_k(x, y) > 0\}$$

where each l_k is a polynomial of degree one. Thus, it suffices to check that for each k and each $q \in S$ we have $l_k(q + \xi) > 0$. We write $\vec{l_k} = l_k - l_k(0,0)$. Note that since for all $\lambda \geq 0$ the point $p + \lambda\xi \in S$, we have

$$0 < l_k(p + \lambda\xi) = l_k(p) + \lambda\vec{l_k}(\xi).$$

Moreover, since this is so for arbitrarily large $\lambda > 0$, we conclude that $\vec{l_k}(\xi) \geq 0$. Hence, for all $q \in S$ we have $l_k(q + \xi) = l_k(q) + \vec{l_k}(\xi) > 0$, and we are done.

(iii) Let $S \subset \mathbb{R}^2$ be a closed convex unbounded polygon with $e \geq 3$ linear sides. Suppose that S contains a point $p = (p_1, p_2)$ and two half-lines r_1, r_2 with origin at p of the type

$$\begin{aligned} r_1 &= \{(y - p_2) + N(x - p_1) = 0, \quad y \geq p_2\} \\ \text{and} \quad r_2 &= \{(y - p_2) - N(x - p_1) = 0, \quad y \geq p_2\}, \end{aligned}$$

for some $N > 0$. Then S is a $(2e - 4)$-generating set.

The proof of this fact is analogous to the previous one, using now Theorem 5.1 instead of Corollary 5.2.

Before proving Theorem 5.7, we need the following technical result.

LEMMA 5.6: *Let $a_1 < a_2 < \cdots < a_\ell$ be real numbers and let $N > 0$ be a positive real number. For $1 \leq j \leq \ell$, consider the polynomials of degree 1*

$$\begin{aligned} r_j(x, y) &= y - N(x - a_j), \\ s_j(x, y) &= y + N(x - a_j). \end{aligned}$$

There exists a polynomial map $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ whose image has cusps at the points $p_j = (a_j, 0)$, and it is contained in the semialgebraic set

$$\mathcal{W} = \bigcup_{j=1}^{\ell} \{r_j \geq 0, s_j \geq 0\}.$$

Proof: First, note that there exists a monic polynomial P (of degree $\leq 4\ell$) such that

$$P(j) = a_j, \ P'(j) = P''(j) = 0, \ P'''(j) = 6 \quad \text{for each } j = 1, \ldots, \ell.$$

Now, we check that for $\varepsilon > 0$ small enough P is an injective function in the intervals $(j - \varepsilon, j + \varepsilon)$. For each j, we can write $P(t) = a_j + (t - j)^3 \lambda_j(t)$ where $\lambda_j = 1 + (t - j)u_j(t)$ for some $u_j \in \mathbb{R}[t]$. Then, since $\lambda_j(j) = 1 > 0$, the derivative

$$P'(t) = (t - j)^2 (3\lambda_j(t) + (t - j)\lambda_j'(t))$$

is positive in the open set $(j - \varepsilon, j + \varepsilon) \setminus \{j\}$ for all small enough $\varepsilon > 0$.

The polynomial $Q(t) = \prod_{j=1}^{\ell} (t - j)^2$ is positive semidefinite in $\mathbb{R}$ and has double roots at the values $j = 1, \ldots, \ell$. Hence, for $\varepsilon > 0$ small enough, Q is a decreasing function in the intervals $(j - \varepsilon, j)$ and is increasing in the intervals $(j, j + \varepsilon)$.

For each pair (m, d) of positive integers consider the polynomial map

$$\varphi_{m,d} = (P, m(Q(1 + Q^d))) \colon \mathbb{R} \to \mathbb{R}^2$$

and its image $\Gamma_{m,d} = \varphi_{m,d}(\mathbb{R})$. From the properties of P and Q we deduce that the map $\varphi_{m,d}$ satisfies the following:

- Its second coordinate $m(Q(1+Q^d))$ is a decreasing function in the intervals $(j - \varepsilon, j)$ and is increasing in the intervals $(j, j + \varepsilon)$ for small enough $\varepsilon > 0$.
- For each $j = 1, \ldots, \ell$ we have $\varphi_{m,d}^{-1}(p_j) = \{j\}$. This follows from the choice of the second coordinate of $\varphi_{m,d}$.
- The curve $\Gamma_{m,d}$ has at each p_j a singularity analytically equivalent to the cusp given by the parametrization $s \mapsto (s^3, s^2)$. Indeed, if we write $t - j = s$ we have

  $$(*) \qquad \begin{cases} P = a_j + v_1(s)^3 \\ mQ(1 + Q^d) = v_2(s)^2 \end{cases}$$

  for some analytic series v_1, v_2 over $\mathbb{R}$ in the variable s such that $v_i(0) = 0$, $v_i'(0) > 0$. Our claim follows by classification of planar curve singularities.

Next, we observe that for $\varepsilon > 0$ small enough the image by $\varphi_{m,d}$ of the open interval $(j - \varepsilon, j + \varepsilon)$ is contained in the wedge contained in the upper half-plane $y > 0$ and delimited by the half-lines joining the point p_j with the points $\varphi_{m,d}(j - \varepsilon)$ and $\varphi_{m,d}(j + \varepsilon)$; see Figure 2. This is a straightforward consequence of the expression $(*)$ of $\varphi_{m,d}$. More precisely, the clue is that $P - a_j$ has order 3 (with respect to s) and $mQ(1 + Q^d)$ has order 2.

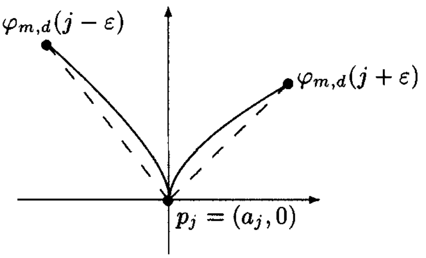

Figure 2.

In what follows we fix $d \geq 1$ such that $\deg(P) < \deg(Q(1+Q^d))$ (for instance, $d = 1 + \deg(P)$) and we look for m such that $\alpha = \varphi_{m,d}$ satisfies the desired conditions. We fix also $0 < \varepsilon < 1/2$ satisfying all the previous conditions for the map $\beta = \varphi_{1,d} = (\beta_1, \beta_2)$.

Let us define $\eta_1 = \min\{\beta_2(j-\varepsilon), \beta_2(j+\varepsilon)\colon j = 1,\dots,\ell\}$. Let

$$K = \bigcup_{j=1}^{\ell-1} [j+\varepsilon, j+1-\varepsilon]$$

and $\eta_2 = \min\{\beta_2(t)\colon t \in K\}$. Both η_1, η_2 are strictly positive real numbers since $\beta_2^{-1}(0) = \{1,2,\dots,\ell\}$. Let $0 < \eta < \min\{\eta_1,\eta_2\}$ and let

$$0 < \mu < \min\Big\{\Big|\frac{\beta_2(j+\varepsilon)}{\beta_1(j+\varepsilon)-\beta_1(j)}\Big|, \Big|\frac{\beta_2(j-\varepsilon)}{\beta_1(j-\varepsilon)-\beta_1(j)}\Big|\colon j = 1,\dots,\ell\Big\}.$$

From our choice of ε, η and μ, we have that (Figure 3)

$$\beta([1-\varepsilon, \ell+\varepsilon]) \subset \{y > \eta\} \cup \bigcup_{j=1}^{\ell} \{y - \mu(x-a_j) \geq 0, y + \mu(x-a_j) \geq 0\}.$$

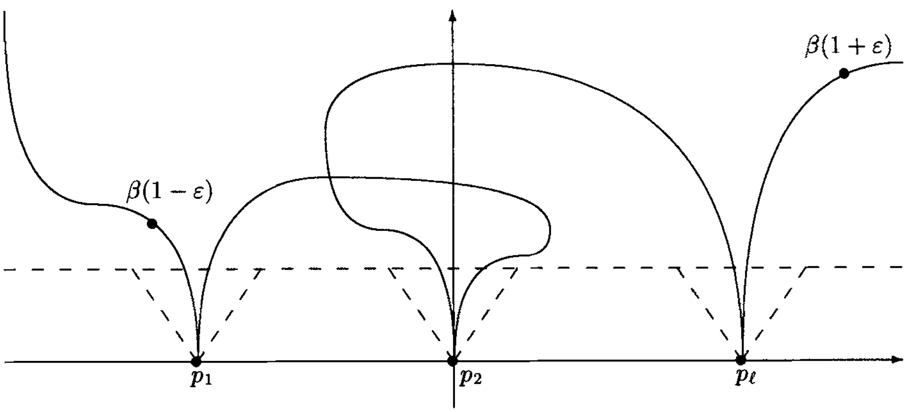

Figure 3.

Since $\deg(\beta_2) > \deg(\beta_1)$, there exists $M > 0$ such that if $|t| > M$ then

$$\beta_2(t) + \mu(\beta_1(t) - a_1) > 0 \quad \text{and} \quad \beta_2(t) - \mu(\beta_1(t) - a_\ell) > 0.$$

Consider the compact sets

$$K' = (K \cup [-M, 1-\varepsilon] \cup [\ell+\varepsilon, M]) \cap \beta_1^{-1}((-\infty, a_1 - \eta/\mu]),$$
$$K'' = (K \cup [-M, 1-\varepsilon] \cup [\ell+\varepsilon, M]) \cap \beta_1^{-1}([a_\ell + \eta/\mu, +\infty)),$$

and the positive numbers

$$\zeta_1 = \min_{t\in K'} \Big\{\frac{\beta_2(t)}{\mu(a_1 - \beta_1(t))}\Big\} \quad \text{and} \quad \zeta_2 = \min_{t\in K''} \Big\{\frac{\beta_2(t)}{\mu(\beta_1(t) - a_\ell)}\Big\}.$$

We denote by S the union $S = \{y > \eta, a_1 - \eta/\mu < x < a_\ell + \eta/\mu\} \cup T_\mu$, where

$$T_\mu = \bigcup_{j=1}^{\ell} \{y - \mu(x - a_j) \geq 0, y + \mu(x - a_j) \geq 0\}.$$

For any positive integer $m > \nu = \max\{1/\zeta_1, 1/\zeta_2\}$ the map $\varphi_{m,d}$ has the property $\varphi_{m,d}(\mathbb{R}) \subset S$. On the other hand, for $\rho > 0$ small enough the set S is contained in the set

$$T_\rho = \bigcup_{j=1}^{\ell} \{y - \rho(x - a_j) \geq 0, y + \rho(x - a_j) \geq 0\};$$

hence, for $m > \nu$ we have $\varphi_{m,d}(\mathbb{R}) \subset T_\rho$. It is straightforward to see that if $m > N\nu/\rho$ then (Figure 4)

$$\varphi_{m,d}(\mathbb{R}) \subset T_N = \bigcup_{j=1}^{\ell} \{y - N(x - a_j) \geq 0, y + N(x - a_j) \geq 0\}.$$

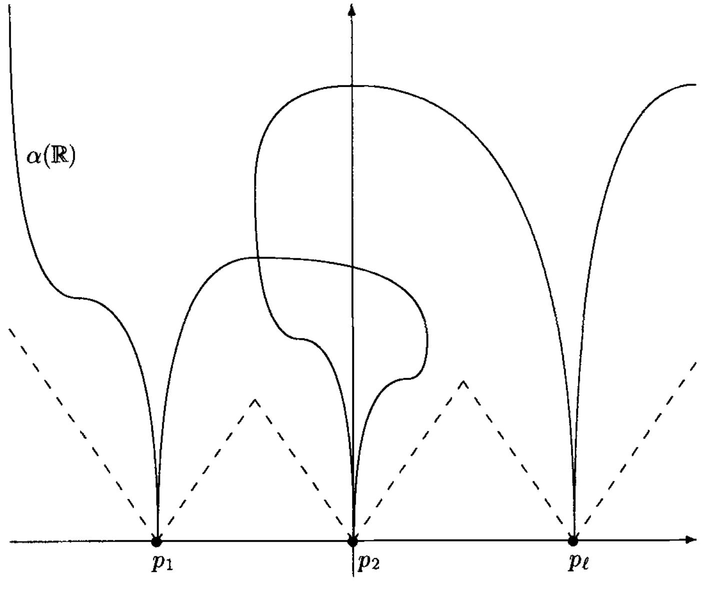

Figure 4.

Taking $\alpha = \varphi_{m,d}$ and $\mathcal{W} = T_N$ we are done. ∎

THEOREM 5.7: *Let $S \subset \mathbb{R}^2$ be a n-generating set and let p be a point of S. Let $a_1 < a_2 < \cdots < a_\ell$ be real numbers and consider the points $p_j = (a_j, 0)$ for $j = 1, \ldots, \ell$. Then the sawset $T = \bigcup_{j=1}^{\ell}(S + \overrightarrow{pp_j})$ is a polynomial image of $\mathbb{R}^{n+1}$.*

Proof: Let $g: \mathbb{R}^n \to \mathbb{R}^2$ be a polynomial map whose image is S. Since S is a n-generating set, there exists a positive real number $N > 0$ such that for every vector $w = (x, y)$ of the wedge $W = \{y - Nx \geq 0, y + Nx \geq 0\}$ the set $S + w$ is contained in S.

By the previous Lemma 5.6, there exists a polynomial map $\alpha: \mathbb{R} \to \mathbb{R}^2$ whose image has cusps at the points p_j, $1 \leq j \leq \ell$, and it is contained in the semialgebraic set

$$\mathcal{W} = \bigcup_{j=1}^{\ell} \{y - N(x - a_j) \geq 0, y + N(x - a_j) \geq 0\}.$$

Then T is the image of the polynomial map

$$f: \mathbb{R}^{n+1} \to \mathbb{R}^2: (x, t) \to g(x) + \alpha(t) - p.$$

Indeed, we claim that for each $t \in \mathbb{R}$, we have $S + \alpha(t) - p \subset S + \overrightarrow{pp_j}$; hence,

$$f(\mathbb{R}^{n+1}) = \bigcup_{t \in \mathbb{R}} (S + \alpha(t) - p) \subset \bigcup_{j=1}^{\ell} (S + \overrightarrow{pp_j}) = T.$$

Since $\alpha(\mathbb{R}) \subset \mathcal{W}$, our claim follows from the identity

$$S + \alpha(t) - p = S + (\alpha(t) - p_{j(t)}) + (p_{j(t)} - p) = S + \overrightarrow{pp_{j(t)}}$$

where $1 \leq j(t) \leq \ell$ satisfies

$$\alpha(t) \in \{y - N(x - a_{j(t)}) \geq 0, y + N(x - a_{j(t)}) \geq 0\} = W + p_{j(t)}.$$

The other inclusion $T \subset f(\mathbb{R}^{n+1})$ is clear because the curve $\alpha(\mathbb{R})$ goes through the points $p_1, \ldots, p_\ell$, and the proof is finished. ∎

6. Regular images

In contrast, to be a regular image of some $\mathbb{R}^n$ is a less restrictive condition than to be a polynomial one. For instance, a regular image can have bounded projections.

In particular, there exist semialgebraic sets $S \subset \mathbb{R}^2$ such that $\mathrm{r}(S)$ is finite while $\mathrm{p}(S) = \infty$. Moreover, we will exhibit semialgebraic sets $S \subset \mathbb{R}^2$ such that $\mathrm{r}(S) < \mathrm{p}(S) < \infty$.

The main purpose of this section is to obtain a better estimation for the invariant $\mathrm{r}(S)$ of an open polygon S with linear sides than the one provided in 5.3.

PROPOSITION 6.1: *Let $S \subset \mathbb{R}^2$ be an open convex unbounded polygon with $e \geq 2$ linear sides. Suppose that S has nonparallel sides. Then there exists a regular map $f\colon \mathbb{R}^e \to \mathbb{R}^2$ such that $f(\mathbb{R}^e) = S$.*

Proof: First, after an affine change of coordinates, we can assume that the two unbounded sides of S are contained in the lines $x = 0, y = 0$, and S is contained in the open quadrant $\{x > 0, y > 0\}$. We proceed by induction on the number e of sides of S. The case $e = 2$ is proved in [FG, 1.7]. Suppose that $e > 2$ and let $\ell_1, \dots, \ell_e$ be the sides of S and let $V_1, \dots, V_{e-1}$ be the vertices of S denoted such that ℓ_1 is the side of S contained in $x = 0$, and that V_i is the common vertex of ℓ_i and ℓ_{i+1}.

Now, consider the sub-polygon S' of S with vertices $V_1, \dots, V_{e-2}$ and sides $\ell_1, \dots \ell_{e-2}$, ℓ'_{e-1} where ℓ'_{e-1} is the horizontal half-line contained in $S \cup \{V_{e-2}\}$ whose origin is the point V_{e-2} (Figure 5).

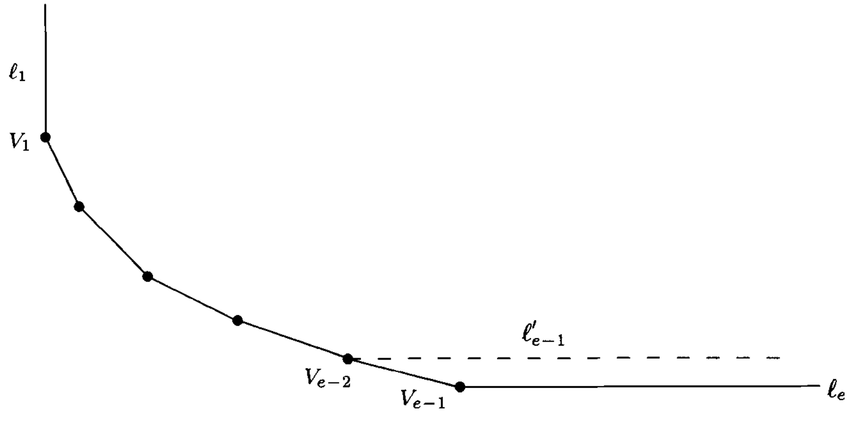

Figure 5.

Hence, by the induction hypothesis, $S' = g(\mathbb{R}^{e-1})$ for some regular map $g\colon \mathbb{R}^{e-1} \to \mathbb{R}^2$. Let v be the vector from V_{e-2} to V_{e-1} and consider the map

$$\begin{array}{rccc} f\colon & \mathbb{R}^e & \to & \mathbb{R}^2 \\ & (x_1, \dots, x_{e-1}, x_e) & \mapsto & g(x_1, \dots, x_{e-1}) + \frac{1}{1+x_e^2} v. \end{array}$$

We claim that $f(\mathbb{R}^e) = S$. First, notice that since the image of the regular function $x_e \mapsto 1/(1+x_e^2)$ is the interval $(0,1]$ then $f(\mathbb{R}^e) = \bigcup_{\lambda\in(0,1]}(S'+\lambda v)$. Since S is convex, we have the inequality $\measuredangle(\overrightarrow{(1,0)}, \ell_i) < \measuredangle(\overrightarrow{(1,0)}, \ell_{e-1})$ for all $i = 1,\dots,e-2$. From this it follows that $f(\mathbb{R}^e) \subset S$. Let us see now that $S \subset f(\mathbb{R}^e)$. Let $p \in S$ and let l be the vertical line going through p. We distinguish three different situations:

(a) If l cuts ℓ_i or passes through V_i for some $i = 1,\dots,e-2$ then we have $p \in S' \subset f(\mathbb{R}^e)$.

(b) If l cuts ℓ_e (and does not passes through V_{e-1}) then $p \in S' + v \subset f(\mathbb{R}^e)$.

(c) If l cuts ℓ_{e-1} or passes through V_{e-1} we proceed as follows. Let q be the intersection point of l and ℓ_{e-1} or $q = V_{e-1}$ depending on the situation in which we are. Let w be the vector from V_{e-2} to q. There exists $\lambda \in (0,1]$ such that $w = \lambda v$. Finally, using the fact that S' is convex, one concludes almost straightforwardly that $p \in S' + w = S' + \lambda v \subset f(\mathbb{R}^e)$. ∎

In the open case also the polygons with parallel sides, either bounded or not, are regular images. In fact, we get

THEOREM 6.2: *Let $S \subset \mathbb{R}^2$ be an open convex polygon with e linear sides. Then,*

$$\mathrm{r}(S) \leq \begin{cases} \max\{2, e-1\} & \text{if } S \text{ has two parallel sides,} \\ \max\{2, e\} & \text{otherwise.} \end{cases}$$

Proof: Let ℓ be a side of S and $l \subset \mathbb{R}^2$ be the affine line that contains this side. After an affine change of coordinates (which preserves lines and convexity) we can suppose l: $x = 0$ (using coordinates (x,y)). Now consider S embedded in the projective plane $\mathbb{P}^2(\mathbb{R})$ via the map

$$\mathbb{R}^2 \hookrightarrow \mathbb{P}(\mathbb{R}^2)\colon (x,y) \mapsto (1:x:y).$$

Consider the homography

$$\varphi\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R}), (x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2)$$

which transforms respectively the projective lines $x_1 = 0, x_0 = 0$ into the projective lines $x_0 = 0, x_1 = 0$. This map induces the birational map $\widetilde{\varphi}\colon \mathbb{R}^2 \dashrightarrow \mathbb{R}^2$, defined by $(x,y) \mapsto (1/x, y/x)$ which is regular outside the line $x = 0$ and preserves lines and convexity there. Since S is open and $S \cap \{x=0\} = \emptyset$ then $\widetilde{\varphi}(S)$ is a convex unbounded open polygon of $\mathbb{R}^2$ with r linear sides, where $r = e$ or $r = e-1$ depending on the case. Moreover, $\widetilde{\varphi}(S)$ has nonparallel sides.

Now, by 6.1 (if $r \geq 2$) and [FG, 1.4 (iv)] (if $r = 1$, that is $\widetilde{\varphi}(S)$ is a half-plane) we know that $\widetilde{\varphi}(S)$ is the image of a regular map $f\colon \mathbb{R}^s \to \mathbb{R}^2$, where

$s = \max\{2, e-1\}$ if the polygon S has two parallel sides and $s = \max\{2, e\}$ otherwise. Hence, $S = \widetilde{\varphi}(f(\mathbb{R}^s))$, as wanted. ∎

Remark 6.3: (a) The open disc $\mathbb{D} = \{u^2 + v^2 < 1\}$ (which can be seen as a limit of convex open regular bounded polygons) is a regular image of $\mathbb{R}^2$. Indeed, let $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map whose image is the upper half-plane $\mathbb{H} = \{v > 0\}$. With complex notation, the Möbius transform

$$\begin{array}{cccc} \phi\colon & \mathbb{H} & \to & \mathbb{R}^2 \\ & z = u + iv & \mapsto & (z-i)/(z+i) \end{array}$$

maps $\mathbb{H}$ onto $\mathbb{D}$. Thus $\phi \circ P$ is a regular map whose image is $\mathbb{D}$.

(b) Let $S = \{x > 0, y > 0, x - y + 4 > 0\} \subset \mathbb{R}^2$. This set is the image of the band $\mathbb{B} = \{u > 0, -1 < v < 1\}$ under the map

$$\begin{array}{cccc} \eta\colon & \mathbb{B} & \to & \mathbb{R}^2 \\ & (u,v) & \mapsto & (2u, (u+v+1)(v+1)). \end{array}$$

On the other hand, by 6.2, the band $\mathbb{B}$ is a regular image of $\mathbb{R}^2$. Hence, the same holds for S. However, by 3.7, S is not a polynomial image of $\mathbb{R}^2$ although, by 5.2, it is a polynomial image of $\mathbb{R}^3$. In other words, $2 = \mathrm{r}(S) < \mathrm{p}(S) = 3$.

7. More examples and open questions

In this section we will prove that the set

$$S = \{x > -2, x - y > 0\} \cup \{2x - y > 0, x - y \le 0\}$$

is a polynomial image of $\mathbb{R}^5$. Notice that it satisfies the known necessary conditions to be a polynomial image of $\mathbb{R}^2$, but we do not know if it is in fact a polynomial image of $\mathbb{R}^2$. Of course, the natural aim is to decide if such conditions are also sufficient.

For that, we start by fixing the following notations. Given two nonzero vectors $u = (u_1, u_2)$, $v = (v_1, v_2)$ of $\mathbb{R}^2$ we consider its oriented angle $-\pi \le \measuredangle(u,v) \le \pi$, that is, the angle is measured from the first vector to the second in the counter-clockwise direction. Similarly, given two half-lines r_1, r_2 with common origin, we define the oriented angle $\measuredangle(r_1, r_2)$ of r_1, r_2 as the oriented angle of their directional vectors. Also, we denote by $u^\perp = (-u_2, u_1)$ the vector of the same length as u and such that $\measuredangle(u, u^\perp) = \pi/2$. The announced result is a particular case of the following:

PROPOSITION 7.1: *Let $w = (w_1, w_2) \in \mathbb{R}^2$ be a nonzero vector satisfying the following inequalities $\pi/4 < \measuredangle((1,0), w^{\perp}) \leq \pi/2$. Then the open semialgebraic subset*

$$S = \{x > -2, x - y > 0\} \cup \{w_1 x + w_2 y > 0, x - y \leq 0\}$$

is a polynomial image of $\mathbb{R}^5$.

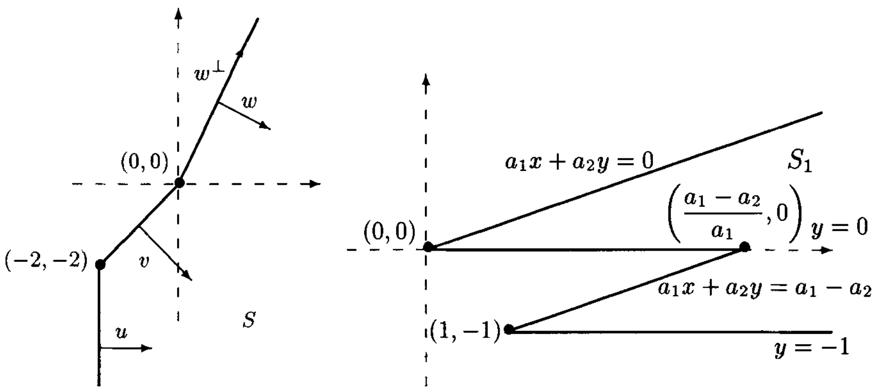

Figure 6.

Proof: We denote $u = (1,0)$ and $v = (1,-1)$. Let $a = (a_1, a_2)$ be a nonzero vector such that $\measuredangle(u, a^{\perp}) = \measuredangle(u, w^{\perp})/3$. Note that $a_1 > 0$ and

$$\measuredangle(u, a^{\perp}) = \frac{\measuredangle(u, w^{\perp})}{3} \leq \frac{\pi}{6}.$$

In particular, $a_1 - a_2 > 0$. Consider the set

$$S_1 = \{y > -1, a_1 x + a_2 y > a_1 - a_2\} \cup \{y > 0, a_1 x + a_2 y > 0\},$$

which is a polynomial image of $\mathbb{R}^3$ as a consequence of 5.7 and 5.5 (i). Let $f_1 \colon \mathbb{R}^3 \to \mathbb{R}^2$ be a polynomial map such that $f(\mathbb{R}^3) = S_1$ (Figure 6).

Let ℓ_1 be the unbounded side of δS_1 contained in the line $a_1 x + a_2 y = 0$, ℓ_2 the bounded side of δS_1 contained in the line $y = 0$, ℓ_3 the bounded side of δS_1 contained in the line $a_1 x + a_2 y = a_1 - a_2$ and ℓ_4 the unbounded side of δS_1 contained in $y = -1$.

Next, consider the map

$$\begin{array}{rccccccc} f_2\colon & \mathbb{R}^2 & \cong & \mathbb{C} & \to & \mathbb{C} & \cong & \mathbb{R}^2, \\ & (x,y) & \equiv & x + iy & \mapsto & (x+iy)^3 & \equiv & (x^3 - 3xy^2, 3x^2y - y^3). \end{array}$$

The image S_2 of S_1 under f_2 is the set contained in $x > -2$ and delimited by the curves $f_2(\ell_1), f_2(\ell_2), f_2(\ell_3)$ and $f_2(\ell_4)$, which are the following:

- $f_2(\ell_1)$ is the half-line $\{(-w_2t, w_1t)\colon t > 0\}$.
- $f_2(\ell_2)$ is the segment that connects the points $(0,0)$ and $(\frac{a_1-a_2}{a_1}, 0)$.
- $f_2(\ell_3)$ is the oriented curve parametrized by $\alpha(t) = f_2(-a_2t+1, a_1t-1)$ where $t \in [0, 1/a_1)$. The sign of the curvature $\kappa_\alpha(t)$ in the point $\alpha(t)$ of the oriented curve parametrized by α coincides with the one of the third coordinate of the vector product $(\alpha', 0) \wedge (\alpha'', 0)$, whose value is

$$18(a_1-a_2)(a_1^2+a_2^2)((a_1t-1)^2+(a_2t-1)^2)$$

  which is strictly positive in the interval $[0, 1/a_1)$, since $a_2 - a_1 > 0$ (as we have seen above). Thus, when we go along this curve from $t = 0$ to $t = 1/a_1$ the curve always turn left. Moreover, the tangent vector $\alpha'(0)$ to $f_2(\ell_3)$ in the point $f_2(1,-1) = \alpha(0)$ is $b = 6(a_1, a_2)$.

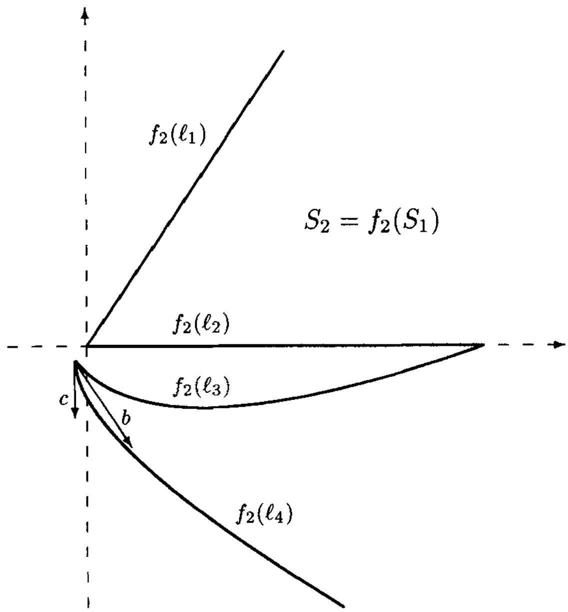

Figure 7.

- $f_2(\ell_4)$ is the oriented curve parametrized by $\beta(t) = f_2(t+1, -1)$ where $t \geq 0$. The sign of the curvature $\kappa_\beta(t)$ in the point $\beta(t)$ of the curve parametrized by β coincides with the one of the third coordinate of the vector product $(\beta', 0) \wedge (\beta'', 0)$, whose value is

$$18((t+1)^2+1)$$

which is strictly positive in the interval $[0, +\infty)$. Thus, when we go along this curve from $t = 0$ to $+\infty$ the curve always turn left. Moreover, the tangent vector $\beta'(0)$ to the curve $f_2(\ell_4)$ in the point $f_2(1, -1) = \beta(0)$ is $c = (0, -6)$; see Figure 7.

Next consider the polynomial maps:

- $\widehat{f_1}\colon \mathbb{R}^5 \to \mathbb{R}^4\colon (x, y, z, u, v) \mapsto (f_1(x, y, z), u, v)$ whose image is $S_1 \times \mathbb{R}^2$.
- $\widehat{f_2}\colon \mathbb{R}^4 \to \mathbb{R}^4\colon (x, y, u, v) \mapsto (f_2(x, y), u, v)$. Note that the image of $S_1 \times \mathbb{R}^2$ under $\widehat{f_2}$ is $S_2 \times \mathbb{R}^2$.
- $f_3\colon \mathbb{R}^4 \to \mathbb{R}^3\colon (x, y, u, v) \to (x, y - u^2, v)$. We have that the image of $S_2 \times \mathbb{R}^2$ under f_3 is $S_3 \times \mathbb{R}$ where $S_3 = \bigcup_{\lambda \le 0}(S_2 + (0, \lambda))$, that is, the set obtained when we slide S_2 along the half-line $\{(0, y)\colon y \le 0\}$. One can check that

$$S_3 = S_2 \cup \{x > -2, a_2x - a_1y > 2a_1 - 2a_2\} \cup \{x > 0, y < 0\}.$$

- $f_4\colon \mathbb{R}^3 \to \mathbb{R}^2\colon (x, y, v) \to (x + v^2, y + v^2)$. Note that the image of $S_3 \times \mathbb{R}$ under f_4 is $S_4 = \bigcup_{\lambda \ge 0}(S_3 + (\lambda, \lambda))$, that is, the set obtained when we slide S_3 along the half-line $\{(x, x)\colon x \ge 0\}$. One can check that

$$S_4 = S_3 \cup \{x > -2, x - y > 0\} = S.$$

Hence, S is the image of the polynomial map $f_0 = f_4 \circ f_3 \circ \widehat{f_2} \circ \widehat{f_1}\colon \mathbb{R}^5 \to \mathbb{R}^2$. ∎

Remarks 7.2: (i) Let p_1 be the common origin of two half-lines r_1, r_2 that do not lie in the same line, and let T_1 be the open convex region bounded by r_1 and r_2. Choose a point $p_2 \in r_2$ different from p_1 and a third half-line r_3 with origin at p_2 which does not intersect T_1 and such that $|\measuredangle(r_1, r_2)| + |\measuredangle(r_2, r_3)| \le \pi$. Let T_2 be the open convex region bounded by r_2 and r_3 (Figure 8). The set $T = T_1 \cup T_2$ is affinely equivalent to one of the sets S of the previous Proposition 7.1; hence, T is a polynomial image of $\mathbb{R}^5$.

Indeed, let l_1, l_2 and l_3 be the lines that respectively contain r_1, r_2 and r_3. We distinguish two cases:

- If the lines l_1 and l_3 are parallel, let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an affine equivalence that satisfies:

$$\begin{array}{ccc} p_1 & \mapsto & (-2, -2), \\ p_2 & \mapsto & (0, 0), \\ l_1 & \mapsto & \{x = -2\}. \end{array}$$

  One can check that, after composing with the symmetry with respect to the line $x - y = 0$ if necessary,

$$g(T) = \{x > -2, x - y > 0\} \cup \{x > 0, x - y \le 0\} = S.$$

- If the lines l_1 and l_3 are not parallel, let $p_3 = l_1 \cap l_3$. Consider the affine equivalence $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{array}{rcc} p_1 & \mapsto & (-2,-2), \\ p_2 & \mapsto & (0,0), \\ p_3 & \mapsto & (-2,-4). \end{array}$$

One can verify that

$$g(T) = \{x > -2, x - y > 0\} \cup \{2x - y > 0, x - y \leq 0\} = S.$$

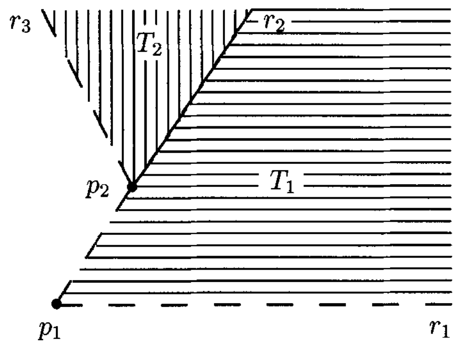

Figure 8.

(ii) According to our Definition 5.4, the previous set T is a 5-generating set for a suitable choice of the directions of the half-lines r_1, r_2, r_3. The proof is similar to the case of the polygons (see Example 5.5 (ii)).

(iii) The previous Proposition 7.1 and the remarks (i) and (ii) can be reformulated for the closures of the involved sets.

(7.3) Some open questions.

1. The set $S = \mathbb{R}^3 \setminus \{x \geq 0, y \geq 0, x + y \leq 1\}$ satisfies also the necessary known conditions to be a polynomial image of $\mathbb{R}^2$. However, we do not even know if S is a polynomial image of some $\mathbb{R}^n$.

2. As we have seen in 3.9, there exist polynomial images $S \subset \mathbb{R}^2$ of $\mathbb{R}^2$ whose exterior boundary has as many connected components as desired. Nevertheless, is there a bound for the number of connected components of the topological exterior $\mathbb{R}^2 \setminus \overline{S}$ of a polynomial image $S \subset \mathbb{R}^2$ of $\mathbb{R}^2$? In particular, is the open set $\{y > 0, y - x^2 - 1 < 0\}$ a polynomial image of $\mathbb{R}^2$?

3. In Section 4, we showed that the semialgebraic set $\{x_1^2 + x_2^2 \geq 1\}$ is a polynomial image. We ask if the same holds for $S = \{x_1^2 + \cdots + x_n^2 \geq 1\}$ for $n \geq 3$.

Notice that a priori the proof for $n = 2$ is difficult to be generalized because it uses strongly the fact that $x_1^2 + x_2^2$ is a sum of two squares of polynomials.

4. With respect to regular images, we recall that the open convex polygons with $e \geq 2$ linear sides are regular images of $\mathbb{R}^e$ (see 6.2). The natural question here is if they are regular images of $\mathbb{R}^2$.

5. For two fixed positive integers $d \leq m$, we define

$$\mathrm{p}(d,m) = \sup\{\mathrm{p}(S)\colon S \subset \mathbb{R}^m, \dim S = d, p(S) < \infty\}.$$

For instance, by 2.3(c), we have $\mathrm{p}(1,m) = 2$ for all $m \geq 1$. The question is to estimate this invariant for $d \geq 2$. An analogous problem can be formulated for regular images.

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