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POLYNOMIAL APPROXIMATION AND THE QUADRATURE PROBLEM OVER A SEMI-INFINITE INTERVAL

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INTRODUCTION

The polynomial approximation to a function in a semi-infinite interval is generally obtained by using Laguerre polynomials together with a suitable weight function of the form $\omega(x) = e^{-x}$. In the 1st part of this paper the authors have obtained a similar expansion of the function $f(x)$ over $(0, \infty)$ in terms of a variant of Chebyshev polynomials of the form $f(x) = \sum_{m=0}^{\infty}{}' a_m T_m^*(e^{-x})$ where $T_m^*(e^{-x}) = \cos m\theta$ with $2e^{-x} - 1 = \cos \theta$, the corresponding weight function being $\omega(x) = \sqrt{[(e^{-x})(1 - e^{-x})^{-1}]}$.

In the 2nd part of this paper methods for numerical evaluation of the integral $\int_0^\infty e^{-x} f(x)\,dx$ have been developed. The above integral which is usually solved by Laguerre Gauss quadrature method requires the use of Laguerre polynomials. However, in the present method the function $f(x)$ is first expressed in a series of a variant of Chebyshev polynomials as above and then the final evaluation is completed by integrating term by term. Also integrals of the form $\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx$ which may be reduced to the form $\int_0^\infty e^{-x^2} f(x)\,dx$ can be treated similarly. It may be mentioned in this connection that the method for solving the aforesaid integral over $(-\infty, \infty)$ which is evaluated with the help of Hermite polynomials is known as Hermite Gauss quadrature method. Numerical examples have been included to show the practical applications of the present method and to compare and contrast the results with the corresponding Laguerre Gauss and Hermite Gauss methods [1].

POLYNOMIAL APPROXIMATION

Let $f(x)$ be continuous over $(0, \infty)$ and let $T_m^*(e^{-x})$ be a variant of Chebyshev polynomials of degree m, where $T_m^*(e^{-x}) = T_m(2e^{-x} - 1) = \cos m\theta$ with $2e^{-x} - 1 = \cos \theta$.

Then the Chebyshev-Fourier expansion of $f(x)$ is

$$f(x) = \sum_{m=0}^{\infty}{}' a_m T_m^*(e^{-x}), \quad 0 < x < \infty, \tag{1}$$

where the prime indicates that the 1st term is to be halved. The polynomials $T_m^*(e^{-x})$ are orthogonal with respect to the weight function $\omega(x) = \sqrt{[(e^{-x})(1 - e^{-x})^{-1}]}$ and we get the following relations

$$\begin{aligned} \int_0^\infty \sqrt{[(e^{-x})(1 - e^{-x})^{-1}]}\, T_m^*(e^{-x})\, T_n^*(e^{-x})\, dx &= 0 \quad \text{for} \quad m \neq n, \\ &= \pi \quad \text{for} \quad m = n = 0, \\ &= \tfrac{1}{2}\pi \quad \text{for} \quad m = n \neq 0. \end{aligned} \tag{2}$$

The coefficients a_m of (1) are given by

$$a_m = \frac{2}{\pi} \int_0^\infty \sqrt{[(e^{-x})(1 - e^{-x})^{-1}]}\, T_m^*(e^{-x}) f(x)\, dx. \tag{3}$$

Assuming that the series (1) has faster rate of convergence an approximation to f may be taken as

$$f(x) \approx \sum_{k=0}^{N}{}' a_k T_k^*(e^{-x}). \tag{4}$$

The coefficients could be calculated from (3) but in practice even for quite simple functions it may be difficult to calculate exactly the integral involved. The approximate computation of the coefficients is done as follows.

The substitution $2e^{-x} = 1 + \cos\theta$ in (3) gives

$$a_k = \frac{2}{\pi} \int_0^\infty \cos k\theta f(\log \sec^2 \tfrac{1}{2}\theta)\, d\theta. \tag{5}$$

By using the mid-point quadrature formula in which the abscissae are taken mid-way between the equidistant points $\theta_i = \pi i/(N + 1)$ gives

$$a_k \approx \alpha_k = \frac{2}{N + 1} \sum_{i=0}^{N} \cos k\theta_i f(\log \sec^2 \tfrac{1}{2}\theta_i) \tag{6}$$

where

$$\theta_i = \frac{(2i + 1)\pi}{2(N + 1)}, \quad i = 0, 1, \ldots, N.$$

Thus

$$a_k \approx \alpha_k = \frac{2}{N + 1} \sum_{i=0}^{N} T_k^*(e^{-x_i}) f(x_i). \tag{7}$$

Again substituting this approximate expression for a_k in (4) we get the polynomial approximation to

(8) $$f(x) \approx \sum_{k=0}^{N}{}' \alpha_k \, T_k^*(e^{-x})$$

i.e.

$$f(x) \approx \sum_{i=0}^{N} \left[\frac{2}{N+1} \sum_{k=0}^{N}{}' T_k^*(e^{-x}) \, T_k^*(e^{-x_i}) \right] f(x_i) .$$

Also

(9) $$4e^{-x} \, T_r^*(e^{-x}) = T_{r-1}^*(e^{-x}) + 2T_r^*(e^{-x}) + T_{r-1}^*(e^{-x}) .$$

Putting

(10) $$\psi(x) = \sum_{k=0}^{N}{}' T_k^*(e^{-x_i}) \, T_k^*(e^{-x})$$

and employing (9) we obtain

(11) $$4e^{-x} \, \psi(x) = \sum_{k=0}^{N}{}' 4e^{-x} \, T_k^*(e^{-x}) \, T_k^*(e^{-x_i}) =$$
$$= 2e^{-x} + \sum_{k=1}^{N} [T_{k+1}^*(e^{-x}) + 2T_k^*(e^{-x}) + T_{k-1}^*(e^{-x})] \, T_x^*(e^{-x_i})$$

and

(12) $$4e^{-x_i} \, \psi(x) = 2e^{-x_i} + \sum_{k=1}^{N} [T_{k+1}^*(e^{-x_i}) + 2T_k^*(e^{-x_i}) + T_{k-1}^*(e^{-x_i})] \, T_k^*(e^{-x}) .$$

Now subtracting (12) from (11) we get

(13) $$\psi(x) = \frac{T_{N+1}^*(e^{-x}) T_N^*(e^{-x_i})}{4(e^{-x} - e^{-x_i})} .$$

Again

(14) $$e^{x_i} \, T_{N+1}^{*\prime}(e^{-x_i}) \, T_N^*(e^{-x_i}) = -2(N+1) .$$

Hence from (8), (10), (13) and (14) we obtain

(15) $$f(x) \approx \sum_{i=0}^{N} \left[\frac{T_{N+1}^*(e^{-x})}{\{1 - e^{-(x-x_i)}\} \, T_{N+1}^{*\prime}(e^{-x_i})} \right] f(x_i) .$$

QUADRATURE PROBLEM

The evaluation of the integral $\int_0^\infty e^{-x} f(x) \, \mathrm{d}x$ can be done in two ways. In the first case the function $f(x)$ is replaced by the expression contained in (4), whence we get

(16) $$\int_0^\infty e^{-x} f(x) \, \mathrm{d}x \approx \sum_{k=0}^{N}{}' a_k \int_0^\infty e^{-x} \, T_k^*(e^{-x}) \, \mathrm{d}x = \sum_{p=0}^{[N/2]}{}' \frac{a_{2p}}{1 - 4p^2} ,$$

where $[N/2]$ means the largest integer contained in $N/2$ for a given N, the coefficients a_k being calculated from (7).

In the other case we replace $f(x)$ by (15) so that

$$(17)\qquad \int_0^\infty e^{-x} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{N} \frac{f(x_i)}{e^{x_i} T^{*\prime}_{N+1}(e^{-x_i})} \int_0^\infty e^{-x}\, \frac{T^*_{N+1}(e^{-x})}{e^{-x_i} - e^{-x}}\,\mathrm{d}x\,.$$

Applying (10) and (13), (17) reduces to

$$(18)\qquad \int_0^\infty e^{-x} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{N} \left[\frac{2}{N+1} \sum_{k=0}^{N}{}' \, T^*_k(e^{-x_i}) \int_0^\infty e^{-x}\, T^*_k(e^{-x})\,\mathrm{d}x\right] f(x_i) = $$
$$= \sum_{i=0}^{N} C_i f(x_i)\,,$$

where

$$(19)\qquad C_i = \frac{2}{N+1} \sum_{p=0}^{[N/2]}{}' \, \frac{T^*_{2p}(e^{-x_i})}{1 - 4p^2}\,.$$

The same result is obtained if the function $f(x)$ in the previous integral is replaced by (8).

NUMERICAL EXAMPLES

We consider the following numerical examples:

$$(a)\qquad I = \int_0^\infty e^{-x} \frac{x\,\mathrm{d}x}{1 - e^{-2x}} = 1{\cdot}2337005\,,$$

$$(b)\qquad I = \int_0^\infty e^{-x} \sin x\,\mathrm{d}x = 0{\cdot}5\,,$$

$$(c)\qquad I = \int_{-\infty}^\infty e^{-x^2} \cos x\,\mathrm{d}x = 1{\cdot}3803884\,.$$

The numerical details of the above examples are contained in table 1, 2 and 3 respectively.

Remarks

(i) It may be seen from the above tables that to achieve the desired accuracy in some case larger number of points are required to evaluate the integral in the present method than in the corresponding Laguerre-Gauss quadrature and Hermite-Gauss quadrature methods. This is the only drawback of this method. But owing to the easy availability of a computer now-a-days such a defect should not be taken into account

Table 1

Present Method		Laguerre-Gauss Method	
N	I	N	I
3	1·2392836	3	1·2345388
6	1·2346744	6	1·2336694
8	1·2343299	8	1·2336918
10	1·2341360	10	1·2337020
11	1·2341142	11	1·2337010
12	1·2340182	12	1·2337016
13	1·2340000	13	1·2337000
14	1·2339420	14	1·2337014
15	1·2339276	15	1·2337008

Table 2

Present Method		Laguerre-Gauss Method	
N	I	N	I
3	0·4605961	3	0·49603015
4	0·4757321	4	0·50487947
5	0·4839439	5	0·49890318
7	0·4951350	7	0·50003902
8	0·4979664	8	0·49998787
9	0·4996647	9	0·50000151
10	0·5007259	10	0·50000014
11	0·5013793	11	0·49999969
13	0·5019106	13	0·49999988

Table 3

Present Method		Hermite-Gauss Method	
N	I	N	I
3	1·3705233	3	1·3820330
6	1·3820518	6	1·3803886
9	1·3803933	9	1·3803885
10	1·3803559	10	1·3803885
13	1·3803824	13	1·3803884
15	1·3803874	15	1·3803880
16	1·3803887	16	1·3803887

so seriously because it involves only a little more computing time in comparison to other methods. On the other hand the existing methods require the use of precomputed weight coefficients and the abscissae which should be known in advance, either in the form of a table. But no such previous data are required in the present method which is the advantage of it.

(ii) In the evaluation of the integral the formula (18) should be preferred to formula (16) because the weight coefficients C_i in (18) can be calculated beforehand from (19) for specified values of N and can be supplied in the form of a table. This saves a lot of computing time for a particular evaluation of an integral.

(iii) It appears from the above tables that although in some cases larger number of points are required in the present method as compared to Laguerre-Gauss of Laguerre-Hermite methods, as the case may be, the results obtained by the present method deviate less from the actual values than those of other methods. Thus by taking a few more points more accuracy in the solution is achieved.

(iv) No attempts have been made to obtain the error estimates both for the polynomial approximation and the integral evaluation. But simple estimates in these cases, if necessary, can be easily obtained by the methods given in [2].

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[2] *L. Fox* and *I. B. Parker:* Chebyshev Polynomials in Numerical Analysis. Oxford University Press, 1968, pp. 68, 90.

Souhrn

APROXIMACE POLYNOMY A PROBLÉM KVADRATURY NA POLONEKONEČNÉM INTERVALU

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V článku je vypracován způsob aproximace funkce na polonekonečném intervalu $(0, \infty)$ polynomy, při čemž je užita jistá modifikace Čebyševových polynomů. Metoda je aplikována na problém kvadratury na tomtéž intervalu.

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