IEEE TRANSACTIONS ON PATTERN ANALYSIS AND MACHINE INTELLIGENCE,
VOL. 34, NO. 7, JULY 2012

# Polynomial Eigenvalue Solutions to Minimal Problems in Computer Vision

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# Minimal Problems

Minimal problems in computer vision arise when computing geometrical models from image data. They often lead to solving systems of algebraic equations.

P3P problem

5-pt relative pose problem

Hand-eye

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22 minimal problems are listed on the website http://cmp.felk.cvut.cz/minimal/

# Minimal Problems

Since the solutions of a minimal problem are roots of a system of polynomial equations, minimal problems can be solved by the following two frequently used methods:

1. Grobner Bases
2. Polynomial Eigenvalue Solution

# Contents

- Outline of the Polynomial Eigenvalue Solution of a system of polynomial equations
- Polynomial Eigenvalue Problems
- Transformation of Systems of Polynomial Equations to a PEP

  Macaulay's Resultant-Based Method

  Resultant-Based Method proposed in this paper

# Polynomial Eigenvalue Solution

Consider a system of equations

$$f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0,$$

which is given by a set of $m$ polynomials $\mathtt{F} = \{f_1, \ldots, f_m | \ f_i \in \mathbb{C}[x_1, \ldots, x_n]\}$ in $n$ variables $\mathbf{x} = (x_1, \ldots, x_n)$ over the field of complex numbers.

If the number of independent equations equals the number of unknowns, this system have a finite number of solutions. In this case, this system is called a zero-dimensional polynomial system.

# Polynomial Eigenvalue Solution

Let

$$f_i(\mathbf{x}) = \mathbf{c}_i(x_1)\mathbf{v}(x_2, \ldots, x_n)$$

Hide $x_1$

Then, the original polynomial equations can be written as:

$$\begin{cases} f_1(\mathbf{x}) = \mathbf{c}_1(x_1)\mathbf{v}(x_2, \ldots, x_n) \\ \vdots \\ f_m(\mathbf{x}) = \mathbf{c}_m(x_1)\mathbf{v}(x_2, \ldots, x_n) \end{cases}$$

$$\mathbf{c}(x_1)\mathbf{v} = \begin{pmatrix} \mathbf{c}_1(x_1) \\ \vdots \\ \mathbf{c}_m(x_1) \end{pmatrix} \mathbf{v} = \mathbf{0}$$

# Polynomial Eigenvalue Solution

Polynomial eigenvalue solution

$$\det(\mathbf{c}(x_1)) = 0$$

The roots are called eigenvalues.

$$\mathbf{c}(x_1)\mathbf{v} = \begin{pmatrix} \mathbf{c}_1(x_1) \\ \vdots \\ \mathbf{c}_m(x_1) \end{pmatrix} \mathbf{v} = \mathbf{0}$$

The null vectors are called eigenvectors. From the eigenvectors, solutions of other variables can be found

# Polynomial Eigenvalue Solution

A simple case

$$\begin{cases} xy - 1 = 0 \\ x + 1 - y = 0 \end{cases}$$

Hide x

$$\begin{pmatrix} x & 1 \\ 1 & x+1 \end{pmatrix} \begin{pmatrix} y \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} x & 1 \\ 1 & x+1 \end{pmatrix} = 0$$

$$x(x+1) - 1 = 0$$

Eigenvalues: $x_1 = \frac{-1+\sqrt{5}}{2}$

$$x_2 = \frac{-1-\sqrt{5}}{2}$$

Eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad y_1 = \frac{1+\sqrt{5}}{2}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} y_2 = \frac{1-\sqrt{5}}{2}$$

# Polynomial Eigenvalue Problems

Polynomial eigenvalue problems are problems of the form

$$\mathtt{C}(\lambda)\mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is a vector of monomials in all variables except for $\lambda$ and $\mathtt{C}(\lambda)$ is a matrix polynomial in variable $\lambda$ defined as

$$\mathtt{C}(\lambda) \equiv \lambda^l \mathtt{C}_l + \lambda^{l-1}\mathtt{C}_{l-1} + \cdots + \lambda \mathtt{C}_1 + \mathtt{C}_0, \tag{2}$$

with $n \times n$ coefficient matrices $\mathtt{C}_j$ [1].

[1] Z. Bai, J. Demmel, J. Dongorra, A. Ruhe, and H. van der Vorst, *Templates for the Solution of Algebraic Eigenvalue Problems*. SIAM, 2000.

# Polynomial Eigenvalue Problems

Solution:

From a PEP

$$\mathtt{C}(\lambda)\mathbf{v} = 0$$

to a Generalized Eigenvalue Problems(GEP)

$$\mathtt{A}\,\mathbf{y} = \lambda\,\mathtt{B}\,\mathbf{y}. \qquad (3)$$

# Polynomial Eigenvalue Problems

High order PEPs of degree $l$ ,

$$(\lambda^l \mathtt{C}_l + \lambda^{l-1}\mathtt{C}_{l-1} + \cdots + \lambda \mathtt{C}_1 + \mathtt{C}_0)\mathbf{v} = 0,$$

can be transformed to the generalized eigenvalue problem (3). Here,

$$\mathtt{A} = \begin{pmatrix} 0 & \mathtt{I} & 0 & \dots & 0 \\ 0 & 0 & \mathtt{I} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathtt{C}_0 & -\mathtt{C}_1 & -\mathtt{C}_2 & \dots & -\mathtt{C}_{l-1} \end{pmatrix},$$

$$\mathtt{B} = \begin{pmatrix} \mathtt{I} & & & \\ & \dots & & \\ & & \mathtt{I} & \\ & & & \mathtt{C}_l \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \mathbf{v} \\ \lambda \mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix}.$$

# Polynomial Eigenvalue Problems

$$\begin{pmatrix} 0 & \mathtt{I} & 0 & \dots & 0 \\ 0 & 0 & \mathtt{I} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathtt{C}_0 & -\mathtt{C}_1 & -\mathtt{C}_2 & \dots & -\mathtt{C}_{l-1} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathtt{I} & & & \\ & \dots & & \\ & & \mathtt{I} & \\ & & & \mathtt{C}_l \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix}$$

If $\mathbf{C}_l$ is nonsingular and well conditioned, multiply $\begin{pmatrix} \mathbf{I} & & & \\ & \dots & & \\ & & \mathbf{I} & \\ & & & \mathbf{C}_l^{-1} \end{pmatrix}$ on both sides.

Often, $\mathbf{C}_l$ is singular but $\mathbf{C}_0$ is regular and well-conditioned. In this case, let $\beta = 1/\lambda$ .

# Polynomial Eigenvalue Problems

$$\mathtt{A}\begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix}$$

$$\mathtt{A} = \begin{pmatrix} 0 & \mathtt{I} & 0 & \dots & 0 \\ 0 & 0 & \mathtt{I} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathtt{C}_0^{-1}\mathtt{C}_l & -\mathtt{C}_0^{-1}\mathtt{C}_{l-1} & -\mathtt{C}_0^{-1}\mathtt{C}_{l-2} & \dots & -\mathtt{C}_0^{-1}\mathtt{C}_1 \end{pmatrix}$$

# Polynomial Eigenvalue Solution

For the roots of the polynomial equations, the above equation holds, but not for all PES's are solutions of the original problem.

Find the solutions either by :

1. testing all monomial dependencies in v,

or

2. by substituting the solutions to the original equations and checking if they are satisfied.

*Screen out the pseudo solutions of this relaxation method, not the pseudo solutions of the original problem.

# Transformation of a System of Polynomial Equations to a PEP

Consider a system of equations

$$f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0,$$

In some cases, for some $x_j$ , let say $x_1$ , the above system can directly be rewritten to a polynomial eigenvalue problem:

$$\mathtt{C}(x_1)\mathbf{v} = 0,$$

where $\mathtt{C}(x_1)$ is a matrix polynomial with square $m \times m$ coefficient matrices and $\mathbf{v}$ is a vector of $s$ monomials in variables $x_2, \ldots, x_n$, i.e., monomials of the form $\mathbf{x}^\alpha = x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_n^{\alpha_n}$. In this case, the number of monomials $s$ is equal to the number of equations $m$, i.e., $s = m$.

# Transformation of a System of Polynomial Equations to a PEP

However, we are not always lucky. The number of monomials may be larger than that of the polynomials. New linearly independent polynomial equations are needed.

# Transformation of a System of Polynomial Equations to a PEP

Macaulay's Resultant-Based Method

$$f_1(x_1,\ldots,x_n)=\cdots=f_n(x_1,\ldots,x_n)=0.$$

$$f_1,\ldots,f_n\in(\mathbb{C}[x_1])[x_2,\ldots,x_n].$$

Let the degrees of these equations in variables $x_2,\ldots,x_n$

be $d_1,d_2,\ldots,d_n$, respectively.

Homogenization: $F_i=x_{n+1}^{d_i}\; f_i(\frac{x_2}{x_{n+1}},\ldots,\frac{x_n}{x_{n+1}}).$

# Transformation of a System of Polynomial Equations to a PEP

Macaulay's Resultant-Based Method

Let $$d = \sum_{i=1}^{n} (d_i - 1) + 1 = \sum_{i=1}^{n} d_i - n + 1.$$

take the set of all monomials $\mathbf{x}^\alpha = x_2^{\alpha_2} x_3^{\alpha_3} \dots x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}} \quad \sum_{i=2}^{n+1} \alpha_i = d$

$$S_1 = \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1} | \mathbf{x}^\alpha\},$$
$$S_2 = \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1} \not| \, \mathbf{x}^\alpha \text{ but } \ x_3^{d_2} \mid \mathbf{x}^\alpha\},$$
$$\dots$$
$$S_n = \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1}, \dots, x_n^{d_{n-1}} \not|; \mathbf{x}^\alpha \ \text{ but } \ x_{n+1}^{d_n} \mid \mathbf{x}^\alpha\},$$

# Transformation of a System of Polynomial Equations to a PEP

Macaulay's Resultant-Based Method

Generalize the original system

$$\mathbf{x}^\alpha / x_2^{d_1} \; F_1 = 0 \quad \text{for all} \;\; \mathbf{x}^\alpha \in S_1$$

$$\cdots$$

$$\mathbf{x}^\alpha / x_{n+1}^{d_n} \; F_n = 0 \quad \text{for all} \;\; \mathbf{x}^\alpha \in S_n.$$

Dehomogenization:

$$x_{n+1} = 1$$

# Transformation of Systems of Polynomial Equations to a PEP

Macaulay's Resultant-Based Method

Disadvantage: designed for dense and small problems.

# Transformation of Systems of Polynomial Equations to a PEP

Resultant-Based Method proposed by Kukelova et al

$$
\begin{aligned}
S_1 &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1} | \mathbf{x}^\alpha\}, \\
S_2 &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1} \not| \mathbf{x}^\alpha \text{ but } x_3^{d_2} \mid \mathbf{x}^\alpha\}, \\
&\dots \\
S_n &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1}, \dots, x_n^{d_{n-1}} \not|; \mathbf{x}^\alpha \text{ but } x_{n+1}^{d_n} \mid \mathbf{x}^\alpha\},
\end{aligned}
$$

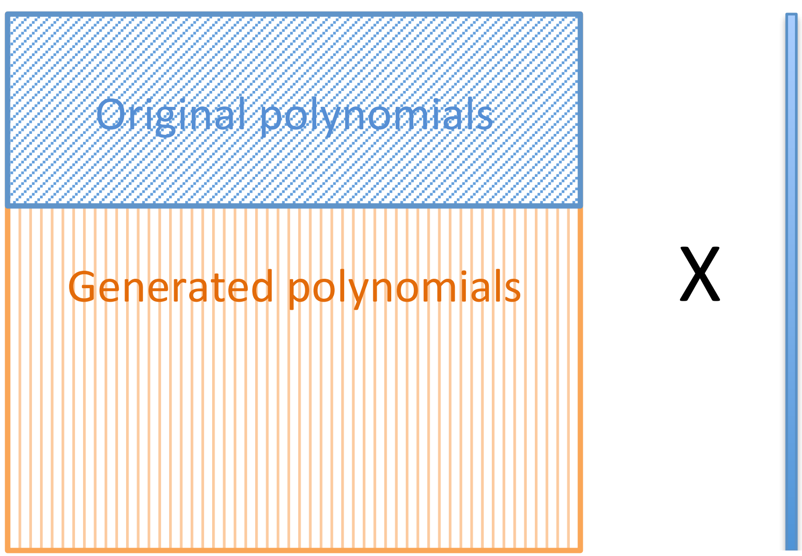
$$\downarrow$$

$$
\begin{aligned}
\overline{S_1} &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_2^{d_1} \mid \mathbf{x}^\alpha\}, \\
\overline{S_2} &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_3^{d_2} \mid \mathbf{x}^\alpha\}, \\
&\dots \\
\overline{S_n} &= \{\mathbf{x}^\alpha : |\alpha| = d,\ x_{n+1}^{d_n} \mid \mathbf{x}^\alpha\}.
\end{aligned}
$$

# Transformation of Systems of Polynomial Equations to a PEP

**Reducing the Size of the Polynomial Eigenvalue Problem**

Removing Unnecessary Polynomials




$$\mathbf{c}(x_1) \qquad \mathbf{v}$$

# Transformation of Systems of Polynomial Equations to a PEP

**Reducing the Size of the Polynomial Eigenvalue Problem**

Removing Zero Eigenvalues

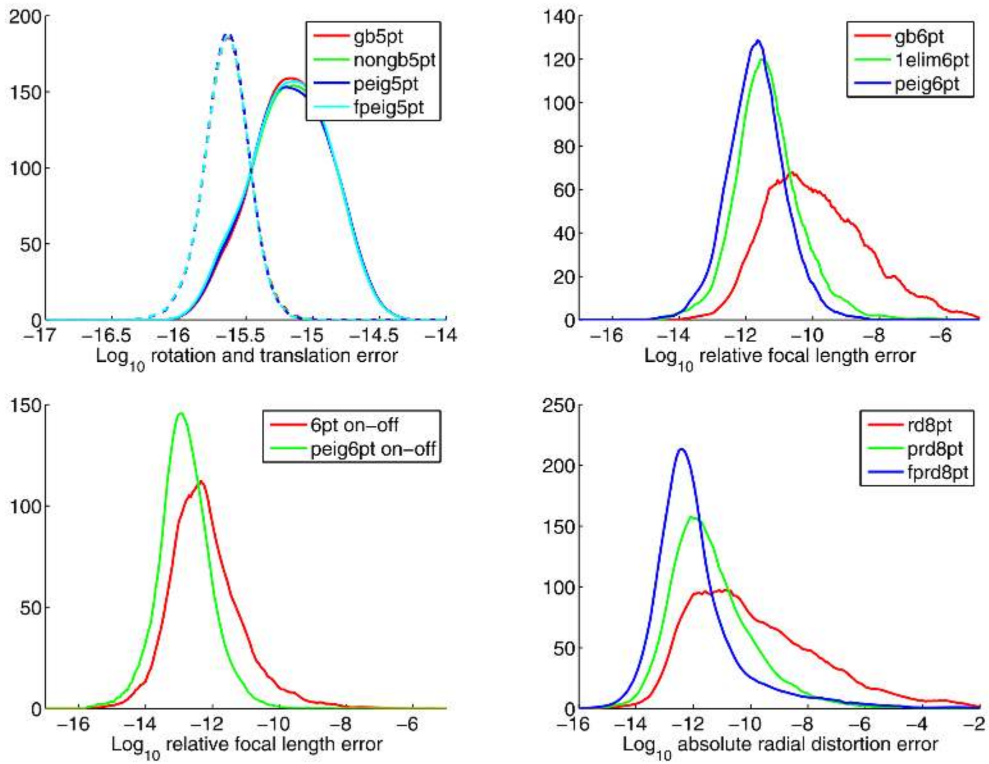
$$\mathtt{A} = \begin{pmatrix} 0 & \mathtt{I} & 0 & \dots & 0 \\ 0 & 0 & \mathtt{I} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathtt{C}_0^{-1}\mathtt{C}_l & -\mathtt{C}_0^{-1}\mathtt{C}_{l-1} & -\mathtt{C}_0^{-1}\mathtt{C}_{l-2} & \dots & -\mathtt{C}_0^{-1}\mathtt{C}_1 \end{pmatrix}$$

$$\mathtt{A} \begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{v} \\ \lambda\mathbf{v} \\ \dots \\ \lambda^{l-1}\mathbf{v} \end{pmatrix}$$

# Transformation of Systems of Polynomial Equations to a PEP

Evaluation

stability




Time: ~us. Faster than Grobner Bases

# The End, Thanks!