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Polynomial Interpolation to Boundary Data on Triangles

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Abstract. Boolean sum interpolation theory is used to derive a polynomial interpolant which interpolates a function $F \in C^N(\bar{T})$, and its derivatives of order N and less, on the boundary ∂T of a triangle T. A triangle with one curved side is also considered.

1. Introduction. Boolean sum interpolation theory** was first used on triangles by Barnhill, Birkhoff, and Gordon [1] to derive rational functions interpolating the boundary data. The general theory of Boolean sum interpolation is briefly discussed in this paper and a *polynomial* Boolean sum interpolant is presented, which, for any positive integer N, interpolates a function $F \in C^N(\bar{T})$, and its derivatives of order N and less, on the boundary ∂T of a triangle T. The case $N = 0$ corresponds to an interpolant constructed by other means by Nielson [6]. The interpolant requires that certain derivatives of F be compatible at the vertices of T, but these conditions can be removed by adding suitable rational terms. The theory is generalized for a triangle with one curved side.

The interpolant can be used to define a piecewise function which is $C^N(\Omega)$ over a triangular subdivision of a polygonal region Ω. This has applications to computer aided geometric design and finite element analysis. Finite dimensional, piecewise defined, $C^N(\Omega)$ interpolants can be derived by taking the boundary data to be functions interpolating discrete data along the sides. Alternatively, the blending function can be incorporated with finite elements so as to match exactly a given boundary function on Ω; see, for example, Marshall and Mitchell [5], who interpolate over a polygonal region Ω. The general theory of interpolation to boundary data for a triangle with one curved side, presented in Section 5, permits essential boundary conditions to be satisfied exactly.

2. Boolean Sum Interpolation Theory. This section considers conditions which are sufficient for the application of Boolean sum interpolation theory. These conditions motivate the formulation of the projectors considered in Section 3. The interpolation

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**The expression "Boolean sum interpolation" is used in this paper, instead of "blending function interpolation", to emphasize the interpolation properties of the projectors *per se* in the Boolean sums.

of the function F is first discussed, and this is then generalized to the interpolation of the function F and its derivatives.

THEOREM 2.1. *Let Γ_1 and Γ_2 be two subsets of R^n, and let F be a function defined on $\Gamma_1 \cup \Gamma_2$. Let P_1 and P_2 be two interpolation projectors such that $P_iF = F$ on Γ_i, and P_iF is defined on $\Gamma_1 \cup \Gamma_2$, $i = 1, 2$. Then the Boolean sum function,*

$$(P_1 \oplus P_2)F = (P_1 + P_2 - P_1P_2)F, \tag{2.1}$$

(i) *interpolates F on Γ_1,*

(ii) *interpolates F on $\Gamma_2 - \Gamma_1$ if P_1F on $\Gamma_2 - \Gamma_1$ is a linear combination of function evaluations on Γ_2.*

Proof. (i) Since $I - P_1$ is null on Γ_1, where I is the identity operator, it follows that

$$F - (P_1 \oplus P_2)F \equiv (I - P_1)(I - P_2)F$$

is zero on Γ_1.

(ii) Also, since $(I - P_2)F = 0$ on Γ_2,

$$F - (P_1 \oplus P_2)F \equiv (I - P_2)F - P_1(I - P_2)F$$

is zero on $\Gamma_2 - \Gamma_1$ if P_1 on $\Gamma_2 - \Gamma_1$ is a linear combination of function evaluations on Γ_2. Q.E.D.

In practice, P_iF usually involves F only on Γ_i. The hypothesis of Theorem 2.1 then becomes that P_1F on $\Gamma_2 - \Gamma_1$ is a linear combination of function evaluations on $\Gamma_1 \cap \Gamma_2$, where it is a necessary condition that $\Gamma_1 \cap \Gamma_2$ is not null.

Remark. If the dual hypothesis holds for $(P_2 \oplus P_1)F$, that is, P_2F on $\Gamma_1 - \Gamma_2$ is a linear combination of function evaluations on Γ_1, then

$$(P_1 \oplus P_2)F = (P_2 \oplus P_1)F \quad \text{on } \Gamma_1 \cup \Gamma_2$$

and hence

$$P_1P_2F = P_2P_1F \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

We thus have sufficient conditions that the projectors satisfy the definition of weak commutativity of Gordon and Wixom [4].

The generalization of Theorem 2.1 to the interpolation of function and derivatives on $\Gamma_1 \cup \Gamma_2$ is the following:

THEOREM 2.2. *Let P_1 and P_2 be two interpolation projectors such that $D^\alpha P_iF = D^\alpha F$ on Γ_i and $D^\alpha P_iF$ is defined on $\Gamma_1 \cup \Gamma_2$, $i = 1, 2$, for all $|\alpha| \leq N$, where*

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Then

(i) *$D^\alpha(P_1 \oplus P_2)F = D^\alpha F$ on Γ_1 for all $|\alpha| \leq N$,*

(ii) *$D^\alpha(P_1 \oplus P_2)F = D^\alpha F$ on $\Gamma_2 - \Gamma_1$ for all $|\alpha| \leq N$ if $D^\alpha P_1F$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function and derivative evaluations on Γ_2 which are interpolated by P_2F.*

Proof. The proof is an extension of the proof of Theorem 2.1. The only complication is on $\Gamma_2 - \Gamma_1$ where

$$D^\alpha F - D^\alpha(P_1 \oplus P_2)F \equiv D^\alpha(I - P_2)F - D^\alpha P_1(I - P_2)F$$

is zero for all $|\alpha| \leqslant N$ if and only if $D^\alpha P_1(I - P_2)F = 0$. A sufficient condition for this to hold is that $D^\alpha P_1$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function and derivative evaluations on Γ_2. For $|\alpha| \geqslant 1$ some of these derivatives may be of order greater than N and thus we require that these be interpolated by P_2. Q.E.D.

Note. Since P_2F interpolates $D^\alpha F$ on Γ_2 for all $|\alpha| \leqslant N$, then, assuming its existence, $\partial^\beta/\partial s^\beta(D^\alpha F)$ is also interpolated on Γ_2, where $\partial/\partial s$ is any derivative along the set Γ_2. Such derivatives, assuming any necessary compatibility to allow change of order of differentiation, frequently include those required by Theorem 2.2.

Example of Rational Interpolation on Triangles. Consider the standard triangle T with vertices at $V_1 = (0, 1)$, $V_2 = (1, 0)$, and $V_3 = (0, 0)$, where the side opposite the vertex V_k is denoted by E_k. Rational Hermite projectors on T are defined by

$$(2.2)\quad T_1F = \sum_{i \leqslant N} \varphi_i\left(\frac{x}{1-y}\right)(1-y)^i F_{i,0}(0, y) + \sum_{i \leqslant N} \psi_i\left(\frac{x}{1-y}\right)(1-y)^i F_{i,0}(1-y, y),$$

$$(2.3)\quad T_2F = \sum_{i \leqslant N} \varphi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x, 0) + \sum_{i \leqslant N} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x, 1-x),$$

$$(2.4)\quad \begin{aligned} T_3F = &\sum_{i \leqslant N} \varphi_i\left(\frac{x}{x+y}\right)(x+y)^i\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(0, x+y) \\ &+ \sum_{i \leqslant N} \psi_i\left(\frac{x}{x+y}\right)(x+y)^i\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(x+y, 0), \end{aligned}$$

where the $\varphi_i(t)$ and $\psi_i(t) = (-1)^i\varphi_i(1-t)$ are the cardinal basis functions for Hermite two point Taylor interpolation on the interval $[0, 1]$. Boolean sum interpolation using these projectors was first considered by Barnhill, Birkhoff, and Gordon [1]. Application of Theorem 2.2 gives the following theorem.

THEOREM 2.3. *The Boolean sum functions,* $(T_i \oplus T_j)F = (T_i + T_j - T_iT_j)F$, $i \neq j$; $i, j = 1, 2, 3$, *interpolate* $F \in C^N(\overline{T})$ *and its derivatives of order N and less on* ∂T, *provided that F satisfies the compatibility conditions*

$$(2.5)\qquad \left(\frac{\partial^{m+n}F}{\partial s_i^m \partial s_j^n}\right)(V_k) = \left(\frac{\partial^{n+m}F}{\partial s_j^n \partial s_i^m}\right)(V_k), \qquad m, n \leqslant N;\ m + n > N,$$

where V_k *is the vertex with adjacent sides* E_i *and* E_j, *and* $\partial/\partial s_l$ *denotes differentation along the side* E_l.

Proof. By affine transformation and symmetry, it is sufficient to consider the case $(T_1 \oplus T_2)F$. T_1F and T_2F interpolate on $\Gamma_1 = E_2 \cup E_3$ and $\Gamma_2 = E_1 \cup E_3$, respectively. With reference to the hypotheses of Theorem 2.2, $D^\alpha T_1F$, $|\alpha| \leqslant N$, on $\Gamma_2 - \Gamma_1 = E_2$ involves linear combinations of

(2.6) $$\left(\frac{\partial^{n+m}F}{\partial y^n \partial x^m}\right)(0, 0) \quad \text{and} \quad \left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1, 0), \qquad 0 \leqslant m, n \leqslant N.$$

The latter derivative is interpolated by T_2F, since tangential derivatives along the side are automatically interpolated. Also, T_2F interpolates $F_{0,n}(x, 0)$ and hence interpolates

$$\left(\frac{\partial^{m+n}F}{\partial x^m \partial y^n}\right)(0, 0), \qquad 0 \leqslant m, n \leqslant N.$$

Thus the hypotheses of Theorem 2.2 are satisfied if $F \in C^N(\bar{T})$ and satisfies the compatibility conditions (2.5) at the vertex $V_3 = (0, 0)$. Q.E.D.

Precision. The precision set is the set of polynomials for which the interpolant is exact and is important in that it indicates the order of accuracy of the interpolant. The precision set of the Boolean sum operator $P_1 \oplus P_2$ is at least that of P_2 since

$$I - (P_1 \oplus P_2) \equiv I - P_2 - P_1(I - P_2)$$

and $I - P_2$ is null on the precision set of P_2. Thus the Boolean sum operator $P_1 \oplus P_2$ has at least the interpolation properties of the projector P_1 and the precision set of the projector P_2.

3. Polynomial Interpolation on Triangles. By affine invariance it is sufficient to consider the standard triangle T defined above. Projectors P_1 and P_2 are considered, which satisfy the conditions of Theorem 2.2 and which respectively interpolate $F \in C^N(\bar{T})$ and its derivatives of order N and less, on the hypotenuse $\Gamma_1 = E_3$ and on the x and y axes $\Gamma_2 = E_1 \cup E_2$. These projectors involve suitable combinations of the Taylor projectors which interpolate F and its derivatives on the sides of the triangle T along parallels to the x and y axes. Explicitly the Taylor projectors are defined by

(3.1)
$$\begin{aligned} T_x^2F &= \sum_{i \leqslant N} x^{(i)} F_{i,0}(0, y), \\ T_x^3F &= \sum_{i \leqslant N} (x + y - 1)^{(i)} F_{i,0}(1 - y, y), \\ T_y^1F &= \sum_{j \leqslant N} y^{(j)} F_{0,j}(x, 0), \\ T_y^3F &= \sum_{j \leqslant N} (x + y - 1)^{(j)} F_{0,j}(x, 1 - x), \end{aligned}$$

where $x^{(i)} = x^i/i!$ and T_x^2 denotes the Taylor projector across the side E_2 along the line through (x, y) parallel to the x axis etc.

Let

(3.2)
$$\begin{aligned} P_2F = (T_x^2 \oplus T_y^1)F = \sum_{i \leqslant N} x^{(i)} F_{i,0}(0, y) + \sum_{j \leqslant N} y^{(j)} F_{0,j}(x, 0) \\ - \sum_{i,j \leqslant N} x^{(i)} y^{(j)} \left(\frac{\partial^{i+j}F}{\partial x^i \partial y^j}\right)(0, 0). \end{aligned}$$

Then it is easily shown that for $F \in C^N(\bar{T})$ the conditions of Theorem 2.2 are satisfied for the Boolean sum of the projectors T_x^2 on E_2 and T_y^1 on E_1 if

$$\left(\frac{\partial^{m+n}F}{\partial x^m \partial y^n}\right)(0, 0) = \left(\frac{\partial^{n+m}F}{\partial y^n \partial x^m}\right)(0, 0), \qquad m, n \leqslant N;\ m + n > N \tag{3.3}$$

(in which case the Taylor projectors are commutative). Thus for F satisfying the compatibility condition (3.3), P_2F interpolates F and its derivatives of order N and less on $\Gamma_2 = E_1 \cup E_2$. The precision set of P_2 is the union of those of the two Taylor projectors T_x^2 and T_y^1, namely

$$x^i y^j, \begin{cases} 0 \leqslant i \leqslant N & \text{for all } j, \\ 0 \leqslant j \leqslant N & \text{for all } i. \end{cases} \tag{3.4}$$

A projector P_1 is required which interpolates F and its derivatives on $\Gamma_1 = E_3$ and which satisfies the conditions of Theorem 2.2, namely that $D^\alpha P_1F$ on Γ_2 is a linear combination of function and derivative evaluations on Γ_2 which are interpolated by P_2F. This is accomplished by taking a suitable combination of the two hypotenuse Taylor projectors.

Linear Case. (Nielson's interpolant.) Let

$$P_1F = yF(x, 1 - x) + xF(1 - y, y), \tag{3.5}$$

then P_1F interpolates F on $\Gamma_1 = \{x + y = 1\}$. Also, on $x = 0$, $P_1F = yF(0, 1)$ and, on $y = 0$, $P_1F = xF(1, 0)$. Thus P_1F on Γ_2 is a linear combination of function evaluations on Γ_2, and these are interpolated by

$$P_2F = F(0, y) + F(x, 0) - F(0, 0). \tag{3.6}$$

The conditions of Theorem 2.1 are thus satisfied and

$$\begin{aligned}(P_1 \oplus P_2)F = {} & yF(x, 1 - x) + xF(1 - y, y) + F(x, 0) + F(0, y) - F(0, 0) \\ & - y\{F(0, 1 - x) + F(x, 0) - F(0, 0)\} \\ & - x\{F(0, y) + F(1 - y, 0) - F(0, 0)\}\end{aligned} \tag{3.7}$$

interpolates F on the boundary ∂T of the triangle T. This is a Boolean sum derivation of Nielson's polynomial interpolant.

If we let

$$\widetilde{F}(x, 0) = (1 - x)F(0, 0) + xF(1, 0), \qquad \widetilde{F}(0, y) = (1 - y)F(0, 0) + yF(0, 1),$$

and $\widetilde{F}(x, 1 - x) = F(x, 1 - x)$, then

$$(P_1 \oplus P_2)\widetilde{F} = yF(x, 1 - x) + xF(1 - y, y) + (1 - x - y)F(0, 0) \tag{3.8}$$

is an interpolation function which is linear on two sides of the triangle, whilst matching the function F on the other side. This interpolant could be incorporated with piecewise linear finite elements on a triangulated polygon so as to satisfy given boundary conditions exactly.

General Case. Let

$$\begin{aligned}
&\sum_{i,j\leqslant N} \alpha_{i,j}(x, y)\left(\frac{\partial^i}{\partial y^i}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^j F\right)(0, 1) \\
&\quad + \sum_{i,j\leqslant N} \beta_{i,j}(x, y)\left(\frac{\partial^i}{\partial x^i}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^j F\right)(1, 0) \qquad (3.9)\\
&\quad + \sum_{i,j\leqslant N} \gamma_{i,j}(x, y)\left(\frac{\partial^{i+j}F}{\partial x^i \partial y^j}\right)(0, 0),
\end{aligned}$$

where $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$ are the appropriate cardinal functions, be the polynomial interpolant over the $3(N + 1)^2$ dimensional set of polynomials which are of degree $2N + 1$ along parallels to the three sides of T. The case $N = 1$ is the tricubic polynomial interpolant of Birkhoff [3] and, for general N, the existence of this interpolant is implied by Lemma 4.1 of Barnhill and Mansfield [2]. Then $\alpha_{0,0}(x, y) + \beta_{0,0}(x, y) + \gamma_{0,0}(x, y) = 1$ and

$$(D^\alpha \alpha_{0,0})(E_1) = (D^\alpha \beta_{0,0})(E_2) = (D^\alpha \gamma_{0,0})(E_3) = 0 \quad \text{for all } |\alpha| \leqslant N,$$

where $(D^\alpha \alpha_{0,0})(E_1)$ represents $D^\alpha \alpha_{0,0}(x, y)$ evaluated on the side E_1 etc. Hence

$$(\alpha_{0,0} + \beta_{0,0})(E_3) = 1 \qquad (3.10)$$

and

$$(D^\alpha[\alpha_{0,0} + \beta_{0,0}])(E_3) = 0, \qquad 1 \leqslant |\alpha| \leqslant N. \qquad (3.11)$$

Thus

$$\begin{aligned}
P_1F &= \alpha_{0,0}(x, y)T_y^3F + \beta_{0,0}(x, y)T_x^3F \\
&= \alpha_{0,0}(x, y) \sum_{j\leqslant N} (x + y - 1)^{(j)}F_{0,j}(x, 1 - x) \qquad (3.12)\\
&\quad + \beta_{0,0}(x, y) \sum_{i\leqslant N} (x + y - 1)^{(i)}F_{i,0}(1 - y, y)
\end{aligned}$$

is a projector which interpolates $F \in C^N(\overline{T})$ and its derivatives of order N and less on $\Gamma_1 = E_3$. Also, for all $|\alpha| \leqslant N$, $D^\alpha P_1 F$ on $y = 0$ involves the derivatives,

$$\left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1, 0), \qquad 0 \leqslant m, n \leqslant N.$$

P_2F defined by Eq. (3.2) interpolates these values provided that $F \in C^N(\overline{T})$ satisfies the compatibility condition,

$$\left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1, 0) = \left(\frac{\partial^m}{\partial x^m}\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n F\right)(1, 0), \qquad (3.13)$$

$$m, n \leqslant N; \; m + n > N.$$

Similarly on $x = 0$ we require that

$$(3.14)\quad \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^m \frac{\partial^n F}{\partial y^n}\right)(0, 1) = \left(\frac{\partial^n}{\partial y^n}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^m F\right)(0, 1),$$

$$m, n \leqslant N;\ m + n > N.$$

The conditions of Theorem 2.2 are then satisfied and with (3.3) we have:

THEOREM 3.1. *Let $F \in C^N(\overline{T})$ and satisfy the compatibility conditions,*

$$(3.15)\qquad \left(\frac{\partial^{m+n}F}{\partial s_i^m \partial s_j^n}\right)(V_k) = \left(\frac{\partial^{n+m}F}{\partial s_j^n \partial s_i^m}\right)(V_k), \qquad m, n \leqslant N;\ m + n > N,$$

at each vertex V_k with adjacent sides E_i and E_j, where $\partial/\partial s_l$ denotes differentiation along the side E_l. Then the polynomial Boolean sum function, $(P_1 \oplus P_2)F = (P_1 + P_2 - P_1P_2)F$, where P_1 is defined by (3.12) *and P_2 is defined by* (3.2), *interpolates F and its derivatives of order N and less on the boundary ∂T of the triangle T.*

The precision set of the interpolant is that of the projector P_2; see (3.4).

Examples. (i) For $N = 0$,

$$(3.16)\qquad \alpha_{0,0}(x, y) = y \quad \text{and} \quad \beta_{0,0}(x, y) = x,$$

giving the linear case (3.5).

(ii) For $N = 1$,

$$(3.17)\qquad \begin{aligned} \alpha_{0,0}(x, y) &= y^2[3 - 2y + 6x(1 - x - y)] \quad \text{and} \\ \beta_{0,0}(x, y) &= x^2[3 - 2x + 6y(1 - x - y)]. \end{aligned}$$

This case is discussed further in Section 4.

4. Removal of Compatibility Conditions. The compatibility conditions (3.15) of Theorem 3.1 can be removed by adding suitable rational terms to the Boolean sum interpolant $(P_1 \oplus P_2)F$. We consider the rational Hermite projectors on the standard triangle T defined by Eqs. (2.2)–(2.4).

Firstly, since T_3 interpolates F on $E_1 \cup E_2$, the projector P_2, defined by (3.2), can be modified to

$$(4.1)\qquad \widetilde{P}_2F = P_2F + T_3(F - P_2F)$$

where $T_3(F - P_2F)$ is a rational compatibility correction term. We consider now the modified Boolean sum interpolant,

$$(P_1 \oplus \widetilde{P}_2)F = (P_1 \oplus P_2)F + (I - P_1)T_3(F - P_2F),$$

where P_1 is defined by (3.12). This interpolant requires the compatibility conditions (3.15) at the vertices $V_1 = (0, 1)$ and $V_2 = (1, 0)$. Then $F - (P_1 \oplus \widetilde{P}_2)F$ has compatible derivatives at the vertex $V_3 = (0, 0)$ and can thus be interpolated by either of the rational Boolean sum operators $T_1 \oplus T_2$ or $T_2 \oplus T_1$. Thus

$$(4.2)\qquad (P_1 \oplus \widetilde{P}_2)F + (T_1 \oplus T_2)[F - (P_1 \oplus \widetilde{P}_2)F]$$

interpolates $F \in C^N(\overline{T})$ and its derivatives of order N and less on ∂T, where $(T_1 \oplus T_2) \cdot [F - (P_1 \oplus \widetilde{P}_2)F]$ is another rational compatibility correction term. The rational terms are zero if the compatibility conditions (3.15) hold.

Example. For $N = 1$, the average of (3.2) with the dual expression for $(T_y^1 \oplus T_x^2)F$ gives the symmetric projector,

$$(4.3)\quad \begin{aligned} P_2F = {} & F(0,y) + xF_{1,0}(0,y) + F(x,0) + yF_{0,1}(x,0) \\ & - F(0,0) - yF_{0,1}(0,0) - xF_{1,0}(0,0) \\ & - \frac{xy}{2}\left\{\left(\frac{\partial^2 F}{\partial x \partial y}\right)(0,0) + \left(\frac{\partial^2 F}{\partial y \partial x}\right)(0,0)\right\}. \end{aligned}$$

Then

$$(4.4)\quad T_3(F - P_2F) = \frac{xy(x-y)}{2(x+y)}\left\{\left(\frac{\partial^2 F}{\partial x \partial y}\right)(0,0) - \left(\frac{\partial^2 F}{\partial y \partial x}\right)(0,0)\right\}$$

and the projector

$$(4.5)\quad \widetilde{P}_2F = P_2F + T_3(F - P_2F)$$

interpolates $F \in C^N(\overline{T})$ on $\Gamma_2 = E_1 \cup E_2$. Now

$$(4.6)\quad \begin{aligned} P_1F = {} & y^2[3 - 2y + 6x(1-x-y)]\,[F(x, 1-x) + (x+y-1)F_{0,1}(x, 1-x)] \\ & + x^2[3 - 2x + 6y(1-x-y)]\,[F(1-y, y) + (x+y-1)F_{1,0}(1-y, y)] \end{aligned}$$

and the Boolean sum $(P_1 \oplus \widetilde{P}_2)F = (P_1 + \widetilde{P}_2 - P_1\widetilde{P}_2)F$ can be determined from Eqs. (4.3)–(4.6) where

$$(4.7)\quad \begin{aligned} P_1\widetilde{P}_2F = {} & y^2[3 - 2y + 6x(1-x-y)] \\ & \cdot\left[(\widetilde{P}_2F)(x, 1-x) + (x+y-1)\left(\frac{\partial \widetilde{P}_2F}{\partial y}\right)(x, 1-x)\right] \\ & + x^2[3 - 2x + 6y(1-x-y)] \\ & \cdot\left[(\widetilde{P}_2F)(1-y, y) + (x+y-1)\left(\frac{\partial \widetilde{P}_2F}{\partial x}\right)(1-y, y)\right]. \end{aligned}$$

It can then be shown that

$$(4.8)\quad \begin{aligned} & (T_1 \oplus T_2)[F - (P_1 \oplus \widetilde{P}_2)F] \\ & = \frac{(x+y-1)^2x^2y(3-2x)}{x-1}\left\{\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]\frac{\partial F}{\partial x}\right)(1,0) - \left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]F\right)(1,0)\right\} \\ & + \frac{(x+y-1)^2xy^2(3-2y)}{y-1}\left\{\left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\frac{\partial F}{\partial y}\right)(0,1) - \left(\frac{\partial}{\partial y}\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]F\right)(0,1)\right\} \end{aligned}$$

giving the compatibly corrected interpolant, $(P_1 \oplus \widetilde{P}_2)F + (T_1 \oplus T_2)[F - (P_1 \oplus \widetilde{P}_2)F]$.

5. Triangle With One Curved Side. By affine transformation, it is sufficient to consider the triangle with vertices at $V_1 = (0, 1)$, $V_2 = (1, 0)$ and $V_3 = (0, 0)$ and two straight edges along the coordinate axes. We assume that the third side E_3 opposite the vertex V_3 is defined by the one-to-one functions,

$$y = f(x) \quad \text{and} \quad x = g(y),$$

where g is the inverse function of f. The Taylor projectors on E_3 are now

$$T_x^3 F = \sum_{i \leq N} [x - g(y)]^{(i)} F_{i,0}(g(y), y), \tag{5.1}$$

$$T_y^3 F = \sum_{j \leq N} [y - f(x)]^{(j)} F_{0,j}(x, f(x)). \tag{5.2}$$

The cardinal functions $\alpha_{0,0}(x, y)$ and $\beta_{0,0}(x, y)$ of Section 3 have the properties,

$$[\alpha_{0,0}(1 - f(x), y) + \beta_{0,0}(1 - f(x), y)](E_3) = 1$$

and

$$[D^\alpha \alpha_{0,0}(1 - f(x), y) + D^\alpha \beta_{0,0}(1 - f(x), y)](E_3) = 0,$$

for $1 \leq |\alpha| \leq N$. Thus

$$P_1 F = \alpha_{0,0}(1 - f(x), y) T_y^3 F + \beta_{0,0}(1 - f(x), y) T_x^3 F \tag{5.3}$$

is a suitable projector on E_3. The dual projector is

$$P_1 F = \alpha_{0,0}(x, 1 - g(y)) T_y^3 F + \beta_{0,0}(x, 1 - g(y)) T_x^3 F \tag{5.4}$$

or alternatively an average of these two can be considered.

The Boolean sum function $(P_1 \oplus P_2)F$, where P_2 is defined by (3.2), gives a blending function interpolant on the curved triangle.

Examples. For the case $N = 0$, (5.1), (5.2), (5.3) and (3.16) give the projector,

$$P_1 F = yF(x, f(x)) + [1 - f(x)] F(g(y), y). \tag{5.5}$$

From (3.2)

$$P_2 F = F(0, y) + F(x, 0) - F(0, 0), \tag{5.6}$$

so that

$$\begin{aligned} (P_1 \oplus P_2)F = {} & yF(x, f(x)) + [1 - f(x)] F(g(y), y) + F(0, y) + F(x, 0) \\ & - F(0, 0) - y[F(0, f(x)) + F(x, 0) - F(0, 0)] \\ & - [1 - f(x)] [F(0, y) + F(g(y), 0) - F(0, 0)]. \end{aligned} \tag{5.7}$$

For the case $N = 1$, (5.1)–(5.3) and (3.17) give the projector,

$$\begin{aligned} P_1 F = {} & \alpha_{0,0}(1 - f(x), y)\{F(x, f(x)) + [y - f(x)] F_{0,1}(x, f(x))\} \\ & + \beta_{0,0}(1 - f(x), y)\{F(g(y), y) + [x - g(y)] F_{1,0}(g(y), y)\}, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned}\alpha_{0,0}(1 - f(x), y) &= y^2[3 - 8y + 6f(x)\{1 + y - f(x)\}] \quad \text{and} \\ \beta_{0,0}(1 - f(x), y) &= [1 - f(x)]^2[1 + 2f(x) + 6y\{f(x) - y\}].\end{aligned} \tag{5.9}$$

From (3.2),

$$\begin{aligned}P_2F = {} & F(0, y) + xF_{1,0}(0, y) + F(x, 0) + yF_{0,1}(x, 0) \\ & - \{F(0, 0) + yF_{0,1}(0, 0) + xF_{1,0}(0, 0) + xyF_{1,1}(0, 0)\}.\end{aligned} \tag{5.10}$$

Hence

$$\begin{aligned}P_1P_2F = {} & \alpha_{0,0}(1 - f(x), y)\{F(0, f(x)) + xF_{1,0}(0, f(x)) + F(x, 0) + f(x)F_{0,1}(x, 0) \\ & - [F(0, 0) + f(x)F_{0,1}(0, 0) + xF_{1,0}(0, 0) \\ & + xf(x)F_{1,1}(0, 0)] \\ & + [y - f(x)]\,[F_{0,1}(0, f(x)) + xF_{1,1}(0, f(x)) \\ & + F_{0,1}(x, 0) - F_{0,1}(0, 0) - xF_{1,1}(0, 0)]\} \\ & + \beta_{0,0}(1 - f(x), y)\{F(0, y) + g(y)F_{1,0}(0, y) + F(g(y), 0) + yF_{0,1}(g(y), 0) \\ & - [F(0, 0) + yF_{0,1}(0, 0) + g(y)F_{1,0}(0, 0) \\ & + yg(y)F_{1,1}(0, 0)] \\ & + [x - g(y)]\,[F_{1,0}(0, y) + F_{1,0}(g(y), 0) \\ & + yF_{1,1}(g(y), 0) - F_{1,0}(0, 0) - yF_{1,1}(0, 0)]\}.\end{aligned} \tag{5.11}$$

Equations (5.8)–(5.11) completely define the Boolean sum interpolant $(P_1 \oplus P_2)F = (P_1 + P_2 - P_1P_2)F$.

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