POLYNOMIAL REPRESENTATIONS FOR INITIAL-BOUNDARY-VALUE PROBLEMS INVOLVING THE INVISCID PROUDMAN–JOHNSON EQUATION

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[Received 2 June 2006. Revise 5 September 2006]

Summary

The central aim of this paper is to show how two-point Hermite interpolation can be used to construct polynomial representations of solutions to some initial-boundary-value problems for the inviscid Proudman–Johnson equation. This classic equation of fluid dynamics can be regarded as first-order hyperbolic, and an important by-product of our analysis is an understanding of how Hermite interpolation can be utilized for such equations. Different types of boundary conditions may result in finite time blow-up and/or large time approach to the steady state depending on the value of a parameter appearing in the problem.

1. Introduction

In this paper we consider some initial-boundary-value problems for the inviscid Proudman–Johnson equation

$$\frac{\partial^3 u}{\partial t \partial z^2} = u \frac{\partial^3 u}{\partial z^3} - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}. \tag{1.1}$$

This equation has a long and extensive history being related to the two-dimensional Euler equation through a separability assumption on the stream function. The so-called Proudman–Johnson equation (**1**) is the viscous counterpart of (1.1) and has an equally extensive literature associated with it dating back to Riabouchinsky (**2**).

We are going to analyse (1.1) on the interval z in [0, 1] for different types of boundary conditions using two-point interpolating polynomials—sometimes known as Hermite interpolating polynomials. The main aim of the paper is to provide a general semi-analytic technique for constructing polynomial solutions to the initial-value problem for (1.1) for a variety of boundary conditions.

It is important to observe that, with $v = \partial^2 u/\partial z^2$, (1.1) can be written in the form

$$\frac{\partial v}{\partial t} - u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z} = 0, \tag{1.2}$$

which can be regarded as first-order hyperbolic in $v(z, t)$ with a single set of characteristics

$$\frac{dz}{dt} = -u(z, t). \tag{1.3}$$

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Q. Jl Mech. Appl. Math, Vol. 59. No. 4
doi:10.1093/qjmam/hbl019

Hitherto in two previous papers (**3**, **4**), Hermite interpolation has been applied to initial-boundary-value problems for parabolic partial differential equations. The primary motivation for the current work is to extend these ideas to first-order hyperbolic equations with boundary and initial data of which (1.2) with (1.3) is a particularly important example. The main challenge here, and in hyperbolic systems in general, is that the boundary and initial data have contiguous domains of influence in the (z, t)-plane along which certain consistency conditions involving derivatives have to be satisfied. This is a feature that does not appear in applications to parabolic equations and leads to new and distinctly different outcomes. As we shall see our method is specifically suited to dealing with derivatives at spatial boundaries so the application to hyperbolic equations turns out to be particularly pertinent. The problems we consider in the paper are chosen to illustrate two contrasting evolutionary behaviours, namely global blow-up in finite time and large time evolution to steady states. Our method copes well with global blow-up which is a feature that has no counterpart in the parabolic problems we have studied in previous papers. So this is new and provides a further reason for the present study. Although we did observe local blow-up in some reaction–diffusion problems this is a difficult effect to analyse using two-point interpolation.

The outline of the paper is as follows. In section 2 we explain briefly what we mean by Hermite interpolation. In section 3 we consider two problems which exhibit blow-up in finite time wherein our results are compared with an exact solution which can be obtained in implicit closed form for a certain class of initial data. In section 4 we consider a problem in which the characteristics enter the domain of interest. Here, depending on the value of a parameter appearing in the problem, the solution might blow-up in finite time or approach a steady state as $t \to \infty$.

We must add a caveat at the outset concerning blow-up in this context since (1.1) arises from boundary-value problems for the Euler equation on infinite domains and so our analysis offers no direct contribution to the question of blow-up of the Euler equation on finite domains.

Throughout this paper we make extensive use of MAPLE as a manipulative and computational tool. In the tables the use of bold digits is intended to give the reader an indication of the convergence properties of the results. Bold digits represent 'unconverged' digits.

2. Two-point Hermite interpolation

We first explain what we mean by Hermite interpolation. Essentially this is a generalization of interpolation using Taylor polynomials and for that reason Hermite interpolation is sometimes referred to as two-point Taylor interpolation. The idea is to interpolate a function $f(z)$ by a polynomial $p(z)$ in which values of $f(z)$ and any number of its derivatives at given points are fitted by the corresponding function values and derivative of $p(z)$. In this paper we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval say $[0, 1]$ wherein a useful and succinct way of writing a Hermite interpolant $p_n(z)$ of degree $2n + 1$ was given for example by Phillips (**5**) as

$$p_n(z) = \sum_{j=0}^{n} \{ f^{(j)}(0) q_j(z) + (-1)^j f^{(j)}(1) q_j(1 - z) \}, \tag{2.1}$$

where

$$q_j(z) = \frac{z^j}{j!}(1 - z)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} z^s, \tag{2.2}$$

so that (2.1) with (2.2) satisfies

$$f^{(r)}(0) = p_n^{(r)}(0), \quad f^{(r)}(1) = p_n^{(r)}(1), \quad r = 0, 1, 2, \ldots, n. \tag{2.3}$$

The error on [0, 1] is given by

$$f(x) - p_n(x) = \frac{(-1)^{n+1} x^{n+1} (1-x)^{n+1} f^{(2n+2)}(\xi)}{(2n+2)!}, \tag{2.4}$$

where $0 < \xi < 1$ and $f^{(2n+2)}$ is assumed to be continuous.

As an example of such a Hermite interpolant we may take $n = 2$ so that (2.1) with (2.3) becomes the quintic

$$\begin{aligned} p_2(z) = {} & (1-z)^3(1+3z+6z^2) f(0) + z^3(10-15z+6z^2) f(1) \\ & + z(1-z)^2(1+3z) f'(0) - z^3(1-z)(4-3z) f'(1) \\ & + \tfrac{1}{2} z^2 (1-z)^3 f''(0) + \tfrac{1}{2} z^3 (1-z)^2 f''(1), \end{aligned} \tag{2.5}$$

satisfying

$$\begin{aligned} p_2(0) &= f(0), \quad p_2'(0) = f'(0), \quad p_2''(0) = f''(0), \\ p_2(1) &= f(1), \quad p_2'(1) = f'(1), \quad p_2''(1) = f''(1). \end{aligned} \tag{2.6}$$

The reader is referred to Davis (**6**) for further details, references and error analyses.

We observe that (2.1) fits an equal number of derivatives at each end point but, as we shall see later in the paper, it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval. There are algorithms for constructing such polynomials and a convenient representation for such a construction which fits up to $n_0 - 1$ derivatives at $x = 0$ and $n_1 - 1$ derivatives at $x = 1$ is given by Stoer and Bulirsch (**7**). This can be written as

$$p_{n_0-1,n_1-1}(x) = \sum_{j=0}^{n_0-1} f^{(j)}(0) L_{0j}(x) + \sum_{j=0}^{n_1-1} f^{(j)}(1) L_{1j}(x) \tag{2.7}$$

where, with

$$l_{0j} = \frac{x^j (1-x)^{n_1}}{j!}, \quad 0 \leqslant j < n_0 \quad \text{and} \quad l_{1j} = \frac{(x-1)^j x^{n_0}}{j!}, \quad 0 \leqslant j \leqslant n_1,$$

$$L_{0,n_0-1} = l_{0,n_0-1}, \quad L_{1,n_1-1} = l_{1,n_1-1},$$

and, for $k_0 = n_0 - 2, n_0 - 3, \ldots, 0$

$$L_{0k_0} = l_{0k_0} - \sum_{\nu=k_0+1}^{n_0-1} l_{0k_0}^{(\nu)}(0) L_{0\nu}(x),$$

while for $k_1 = n_1 - 2, n_1 - 3, \ldots, 0$

$$L_{1k_0} = l_{1k_0} - \sum_{\nu=k_1+1}^{n_1-1} l_{1k_1}^{(\nu)}(1) L_{1\nu}(x).$$

Both of the representations (2.1) and (2.7) are systematically programmable and we will show later in the paper how they can be used to advantage in the present context. This has not been employed in previous applications of the method so it is important to see how this new flexibility can be exploited.

3. A problem with homogeneous boundary conditions

The first problem we consider is (1.1) together with the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \tag{3.1}$$

$$u(z,0) = g(z). \tag{3.2}$$

We choose this for a number of reasons: many features of the solution are known and it has a straightforward exact solution when $g(z) = Az(1-z)$ with which we can compare our results; see, for example, Childress *et al.* (**8**) and Grundy and Kay (**9**). A further feature which simplifies matters is that the domain of dependence of the initial data is the whole interval [0, 1] for *all* $t > 0$. This is in contrast to the situation in the third example, where the domains of dependence of the initial and boundary data both vary with time.

The first step is to construct power series for $u(z,t)$ about $z = 0$ and $z = 1$ which implicitly we assume exist. We have

$$u(z,t) = A_1(t)z + A_2(t)z^2 + \cdots + A_i(t)z^i + \cdots \tag{3.3}$$

and

$$u(z,t) = B_1(t)(z-1) + B_2(t)(z-1)^2 + \cdots + B_i(t)(z-1)^i + \cdots, \tag{3.4}$$

where the $A_i(t)$ and $B_i(t)$ are unknown but satisfy the differential equations

$$\dot{A}_2 = -A_2A_1, \quad \dot{A}_3 = -\tfrac{2}{3}A_2^2, \quad \dot{A}_4 = A_1A_4 - A_2A_3, \quad \text{etc.}, \tag{3.5}$$

$$\dot{B}_2 = -B_1B_2, \quad \dot{B}_3 = -\tfrac{2}{3}B_2^2, \quad \dot{B}_4 = B_1B_4 - B_2B_3, \quad \text{etc.} \tag{3.6}$$

To determine the differential equations for the remaining two unknown functions $A_1(t)$ and $B_1(t)$ we recast (1.1) with (3.1) into the following integral forms. Integrating (1.1) on $[0, z]$ and imposing the relevant boundary conditions at $z = 0$ gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right) - \dot{A}_1 - u\frac{\partial^2 u}{\partial z^2} + \left(\frac{\partial u}{\partial z}\right)^2 - A_1^2 = 0, \tag{3.7}$$

whence putting $z = 1$ we have

$$\dot{B}_1 - \dot{A}_1 + B_1^2 - A_1^2 = 0. \tag{3.8}$$

Going further and repeating this process for (3.7) yields

$$\dot{A}_1 + A_1^2 - 2\int_0^1 \left(\frac{\partial u}{\partial z}\right)^2 dz = 0. \tag{3.9}$$

We now replace $u(z, t)$ in (3.9) by a Hermite interpolant $p_n(z, t)$ constructed from (3.3) and (3.4); for example with $n = 2$ (3.9) gives $\dot{A}_1$ directly as

$$\begin{aligned}\dot{A}_1 = {} & \tfrac{2}{15}(A_2A_1 - B_2B_1) + \tfrac{4}{315}(A_2^2 + A_2B_2 + B_2^2) - \tfrac{19}{35}A_1^2 - \tfrac{2}{35}A_1B_1 \\ & + \tfrac{16}{35}B_1^2 + \tfrac{4}{105}(A_1B_2 - B_1A_2),\end{aligned} \tag{3.10}$$

whence (3.8) gives $\dot{B}_1$.

The initial conditions for the $A_i(t)$ and $B_i(t)$ are given by the initial data (3.2) expanded about $z = 0$ and $z = 1$. We therefore have

$$A_i(0) = \frac{g^{(i)}(0)}{i!} \quad \text{and} \quad B_i(0) = \frac{g^{(i)}(1)}{i!}. \tag{3.11}$$

In what follows we compare our results with the exact solution given for example by Childress *et al.* (**8**); see also (**9**), namely

$$u(x, t) = A\frac{\sinh^2(r)}{r^2}\left\{\frac{z - e^{-2r(1-z)} + (1 - z)e^{-2r}}{r(1 - e^{-2r})}\right\}, \tag{3.12}$$

where $r(t)$ is given implicitly by

$$t(r) = \frac{1}{A}\int_0^r \frac{\rho^2}{\sinh^2 \rho}\, d\rho, \tag{3.13}$$

and the initial data by

$$u(z, 0) \equiv g(z) = Az(1 - z).$$

The initial data on the A_i and B_i corresponding to this exact solution is, from (3.11),

$$\begin{aligned} &A_1(0) = A, \quad A_2(0) = -2A, \quad A_i(0) = 0, i > 2, \\ &B_1(0) = -A, \quad B_2(0) = -2A, \quad B_i(0) = 0, i > 2. \end{aligned} \tag{3.14}$$

We are now in a position to solve (3.5), (3.6), (3.10) and (3.8) with (3.14) for any n. Using the facilities of MAPLE 9.5 for all the algebraic manipulations and numerical integrations we compare our results with those of the exact solution in Tables 1 and 2. Note that for the purposes of comparison with (3.12) and (3.13) it is easier to compute at prescribed values of r—clearly no such inconvenience applies to our method.

Table 1 Convergence of end point derivatives for $A = 1$ and comparison with exact values

$r(t)$	t	A_1	B_1	
0·1	0·099889	0·969915	−1·036760	p_2
		0·969915	−1·036760	p_3
		0·969915	−1·036760	p_4
		0·969915	−1·036760	Exact
0·5	0·486516	0·9080**68**	−1·2642**60**	p_2
		0·908081	−1·264241	p_3
		0·908081	−1·264241	p_4
		0·908081	−1·264241	Exact
1·0	0·900859	0·948**374**	−1·81**4235**	p_2
		0·9487**70**	−1·8134**21**	p_3
		0·948765	−1·81342**9**	p_4
		0·948765	−1·813430	Exact
2·0	1·403333	1·5**12503**	−5·**113524**	p_2
		1·52**2115**	−5·05**3521**	p_3
		1·5215**33**	−5·0555**50**	p_4
		1·521554	−5·055504	Exact
3·0	1·582834	3·**701994**	**−20·263230**	p_2
		3·66**8986**	−18·**548447**	p_3
		3·66**0760**	−18·64**4508**	p_4
		3·661538	−18·640199	Exact
4·0	1·631176	**16·677699**	**−139·600341**	p_2
		11·**544600**	**−78·7664233**	p_3
		11·59**9360**	−81·**6676972**	p_4
		11·605316	−81·487131	Exact
5·0	1·642165	–	–	p_2
		39·979193	**−329·750548**	p_3
		44·**278208**	**−402·245687**	p_4
		44·028933	−396·460384	Exact

Our results confirm that the solution blows-up for $A = 1$—a result which in fact holds for any A. The trivial zero steady state is unstable and it can be confirmed that the solution may blow-up positively or negatively depending on the initial data. The convergence with n and agreement with the exact solution is clearly apparent although this agreement deteriorates as the blow-up time is approached due to the effect of the large higher derivatives on the error. Nevertheless it appears that the blow-up times are reliably estimated using the overflow criterion in the ordinary differential equation solver.

Table 2 Convergence of blow-up time estimates for $A = 1$ which are computed using overflow criterion on the MAPLE ordinary differential equation solver

	T
p_2	1·6**391**
p_3	1·64**57**
p_4	1·6449
Exact	1·6449

Table 3 Convergence of blow-up times for the inhomogeneous problem with n for (3.15) and initial data (a)$g(z) = -z^2$ with positive blow-up and (b) $g(z) = \frac{1}{2}z^2 - \frac{3}{2}z$ with negative blow-up. This suggests convergence for p_4 to four decimal places

	(a)	(b)
	T	T
p_2	1·49**02**	1·37**57**
p_3	1·506**7**	1·3765
p_4	1·5062	1·3765

3.1 *A related inhomogeneous problem*

It is appropriate at this stage to mention a second problem, closely related to the above, which is the inviscid counterpart to a problem which has received considerable attention in recent years; see, for example, (**10**, **11**). This can be written as (1.1) subject to (3.2) and the boundary values

$$u(0, t) = 0, \quad u(1, t) = -1. \tag{3.15}$$

The point here is that characteristics from the $z = 1$ boundary do not enter the domain [0, 1] and so the two conditions (3.15) are sufficient to determine the solution. We can now follow a similar procedure to reveal an analogous structure. The trivial steady state solution, $u = -z$, is unstable and the solution may blow-up positively or negatively depending on the initial data. The blow-up times, which are dependent on the initial data, can be accurately computed as before and examples are given in Table 3 for the indicated initial data.

4. Inhomogeneous boundary conditions—vorticity input at a boundary

4.1 *The formulation*

Encouraged by the outcome of the analysis of section 3 it is tempting to become a little more ambitious and consider a second problem for (1.1) which is much more involved but provides us with a possible general procedure for solving hyperbolic problems. We now consider (1.1) together with the boundary conditions

$$u(0,t)=0,\ u(1,t)=1,\ \frac{\partial^2 u}{\partial z^2}(1,t)=-\alpha. \tag{4.1}$$

The viscous counterpart of this problem was first considered by Grundy and McLaughlin (**12**) whose conclusions were reaffirmed by Okamoto and Zhu (**13**). The problem is an interesting one which has its origins in incompressible magnetohydrodynamics, specifically in magnetic field annihilation in a current sheet formed by two regions of oppositely directed magnetic field lines. The condition on the second derivative in (4.1) is an attempt to model the effect of various vorticity levels on this process. We refer the reader to (**14**) for further details.

The mathematical difficulty here is that since all characteristics emanating from the boundary at $z=1$ enter $[0,1]$ for all $t>0$ we need the extra boundary condition in (4.1). This observation implies that the domain of dependence of the initial data is separated from the domain of dependence of the boundary conditions by the characteristic emanating from $z=1$ at $t=0$ along which the solution may lack regularity; we therefore have to deal with two regions instead of one. Since, however, our method naturally involves end point derivatives in intervals it is possible for us to impose any regularity conditions we wish along the bounding characteristic. In the situation here we impose continuity of u, $\partial u/\partial z$ and $\partial^2 u/\partial z^2$ along this bounding characteristic. We may of course impose less or more restrictive continuity conditions as is our wish but these are the most physically relevant and are the ones we impose here.

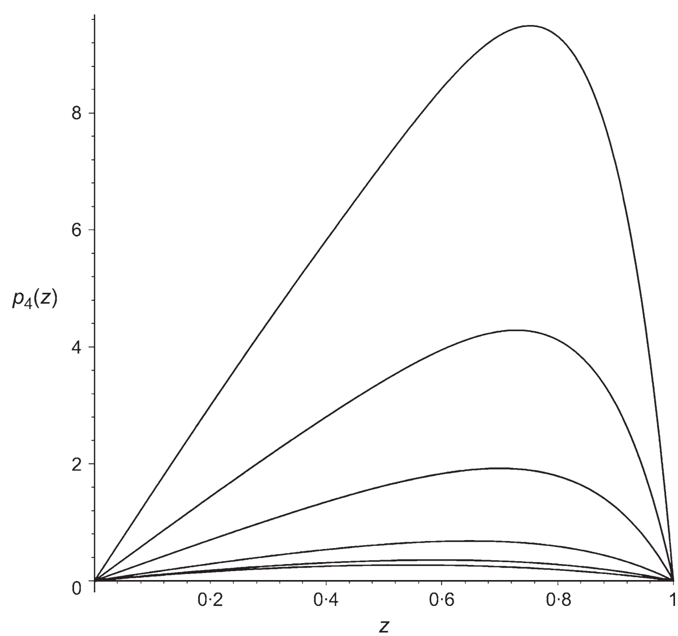

Fig. 1 Plot of $p_4(z,t)$ as an approximation to $u(z,t)$ to within 0·01 of the blow-up time with $A=1$. The times are $t=0{\cdot}5,\ t=1{\cdot}0,\ t=1{\cdot}4,\ t=1{\cdot}58,\ t=1{\cdot}62,\ t=1{\cdot}635$

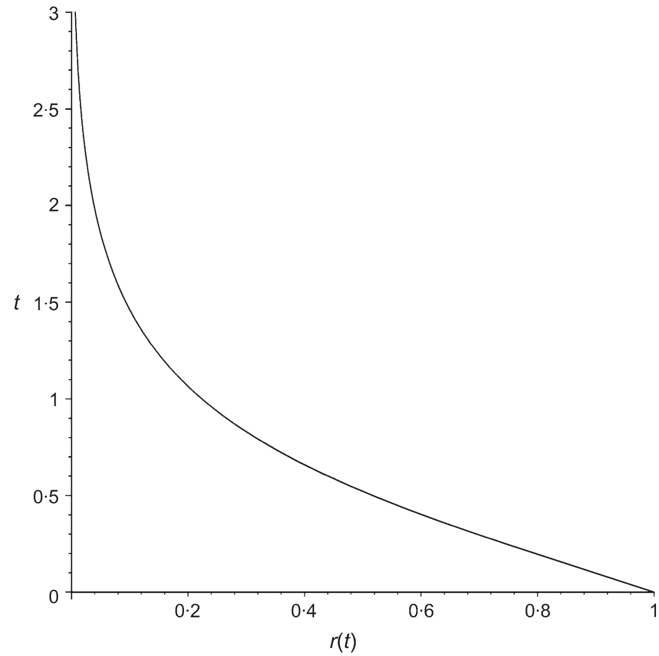

Fig. 2 The bounding characteristic $z = r(t)$ for $\alpha = 3$

Thus the hyperbolic nature of the problem means that we have to deal with two regions defined by A: $0 \leqslant z < r(t)$ and B: $r(t) < z \leqslant 1$, where $z = r(t)$ is the unknown equation of the characteristic through $z = 1$, $t = 0$. A plot of $r(t)$ is shown for a representative value of α in Fig. 2.

We now consider the problem as one involving the two regions separated by an unknown moving boundary $z = r(t)$, along which certain regularity conditions have to be imposed.

Region A: $0 \leqslant z < r(t)$
To recast the problem as one with a fixed boundary we define a new independent variable

$$Z = z/r(t), \tag{4.2}$$

so that $0 \leqslant Z \leqslant 1$ in A. With this change of variable (1.1) becomes

$$r\frac{\partial v}{\partial t} - \{\dot{r}Z + u\}\frac{\partial v}{\partial Z} + v\frac{\partial u}{\partial Z} = 0, \tag{4.3}$$

where

$$v = \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}\frac{\partial^2 u}{\partial Z^2}. \tag{4.4}$$

We now construct power series for $u(z, t)$ about $Z = 0$ and $Z = 1$ of the form

$$u = A_1(t)Z + A_2(t)Z^2 + \cdots + A_i(t)Z^i + \cdots \tag{4.5}$$

and

$$u = B_0(t) + B_1(t)(Z-1) + B_2(t)(Z-1)^2 + B_3(t)(Z-1)^3 + \cdots + B_i(t)(Z-1)^i + \cdots, \quad (4.6)$$

where

$$r\dot{A}_2 = A_2(2\dot{r} - A_1), \quad r\dot{A}_3 = \tfrac{1}{3}(9\dot{r}A_3 - 2A_2^2), \quad r\dot{A}_4 = A_4(4\dot{r} + A_1) - A_2A_3, \quad \text{etc.} \quad (4.7)$$

In (4.6) $B_0(t)$, $B_1(t)$ and $B_2(t)$ are unknown functions but since $Z = 1$ is a characteristic

$$r\dot{B}_3 = \tfrac{1}{3}(9\dot{r}B_3 - 2B_2^2), \quad r\dot{B}_4 = B_4(B_1 - 4B_0) - B_2B_3, \quad \text{etc.} \quad (4.8)$$

Writing (1.1) in characteristic form we have in addition the characteristic equations

$$\dot{r} = -B_0 \quad (4.9)$$

and

$$r\dot{B}_2 = -B_2(2B_0 + B_1). \quad (4.10)$$

Region B: $r(t) < z \leqslant 1$
By the same token we make the change of variable

$$Y = \frac{z-r}{1-r}, \quad (4.11)$$

so that $0 \leqslant Y \leqslant 1$ in B. Equation (1.1) now becomes

$$(1-r)\frac{\partial v}{\partial t} - \{\dot{r}(1-Y) + u\}\frac{\partial v}{\partial Y} + v\frac{\partial u}{\partial Y} = 0, \quad (4.12)$$

where

$$v = \frac{\partial^2 u}{\partial z^2} = \frac{1}{(1-r)^2}\frac{\partial^2 u}{\partial Y^2}.$$

We now construct the series about $Y = 0$ and $Y = 1$ in the form

$$u = B_0(t) + C_1(t)Y + C_2(t)Y^2 + C_3(t)Y^3 + C_4(t)Y^4 + \cdots, \quad (4.13)$$

$$u = 1 + D_1(t)(Y-1) - \tfrac{1}{2}\alpha(1-r)^2(Y-1)^2 + D_3(t)(Y-1)^3 + D_4(t)(Y-1)^4 + \cdots, \quad (4.14)$$

where continuity of the first two z-derivatives of u across the characteristic gives

$$C_1 = (1-r)B_1/r \quad \text{and} \quad C_2 = (1-r)^2B_2/r^2. \quad (4.15)$$

In (4.13) and (4.14) we have that

$$\dot{C}_3 = -\frac{\{9r^4\dot{r}C_3 + 2(1-r)^4B_2^2\}}{3r^4(1-r)}, \quad (4.16)$$

$$\dot{C}_4 = -\frac{\{(4r\dot{r} - B_1(1-r))rC_4 + (1-r)^2B_2C_3\}}{r^2(1-r)}, \quad (4.17)$$

$$D_3 = -\frac{(1-r)^2\alpha D_1}{6} \quad (4.18)$$

and

$$D_4 = \frac{a(1-r)^2}{24}\{\alpha(1-r)^2 - (1-r)\dot{D}_1 - \dot{r}D_1\}. \tag{4.19}$$

We now recast the problem into various integral forms. First in region A we integrate (4.3) with (4.4) over $[0, Z]$ to give

$$r\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial Z}\right) - r\dot{A}_1 - \dot{r}Z\frac{\partial^2 u}{\partial Z^2} - \dot{r}\frac{\partial u}{\partial Z} + \dot{r}A_1 - u\frac{\partial^2 u}{\partial Z^2} + \left(\frac{\partial u}{\partial Z}\right)^2 - A_1^2 = 0, \tag{4.20}$$

and putting $Z = 1$ yields

$$r(\dot{B}_1 - \dot{A}_1) + \dot{r}(A_1 - 2B_2 - B_1) + B_1^2 - A_1^2 - 2B_0B_2 = 0. \tag{4.21}$$

Repeating this procedure on (4.20) gives

$$r(\dot{B}_0 - \dot{A}_1) + \dot{r}(A_1 - B_1) + 2\int_0^1 \left(\frac{\partial u}{\partial Z}\right)^2 dZ - A_1^2 - B_0B_1 = 0. \tag{4.22}$$

A final repetition of this prescription gives

$$\begin{aligned} &r\left\{2\int_0^1 \frac{\partial u}{\partial t}dZ - \dot{A}_1\right\} + \dot{r}\left(A_1 - 2B_0 + 2\int_0^1 u\,dZ\right) \\ &\quad + 4\int_0^1 (1-Z)\left(\frac{\partial u}{\partial Z}\right)^2 dZ - A_1^2 - B_0^2 = 0. \end{aligned} \tag{4.23}$$

We now repeat this procedure on (4.12) in region B to give the three equations

$$(1-r)\dot{D}_1 + D_1^2 - B_0D_1 + \alpha(1-r)^2 + L_1(t) = 0, \tag{4.24}$$

$$2\int_0^1 \left(\frac{\partial u}{\partial Y}\right)^2 dY - D_1 + L_1(t) + L_2(t) = 0 \tag{4.25}$$

and

$$\begin{aligned} &2(1-r)\int_0^1 \frac{\partial u}{\partial t}dY + 2\dot{r}\left(B_0 - \int_0^1 u\,dY\right) - 1 + B_0^2 \\ &\quad + 4\int_0^1 (1-Y)\left(\frac{\partial u}{\partial Y}\right)^2 dY + L_1 + 2L_2 = 0, \end{aligned} \tag{4.26}$$

where $L_1(t) = -\dfrac{(1-r)^2}{r^2}(r\dot{B}_1 + B_1B_0 + B_1^2)$ and $\dot{L}_2 = -(1-r)\dot{B}_0$.

The idea now is to construct different two-point polynomials $p_n(z, t)$ and $P_n(z, t)$ in regions A and B respectively which fit the various series that we have constructed. We then replace $u(z, t)$ in the various integral formulations by the appropriate polynomial. In this way we can construct systems of ordinary differential equations for the unknown coefficients which appear in the series—the details of these computations are given in the next section. Such systems have to be solved subject to initial data for the coefficients. The construction of this is somewhat involved and to avoid excessive congestion in the main body of the paper we relegate the details to the Appendix.

4.2 *The solution using seventh-degree polynomials*

We construct a solution in region A using the seventh-degree polynomial p_3 in which $B_0(t)$, $B_1(t)$, $B_2(t)$, $B_3(t)$, $A_1(t)$, A_2(t) and A_3(t) are unknown functions. In region B we again use a seventh-degree polynomial P_3 which incorporates the additional functions $D_1(t)$ and $C_3(t)$. Together with the unknown function $r(t)$ we thus have ten unknown functions which we determine from the closed system of ordinary differential equations provided by $(4.7)_{1,2}$, $(4.8)_1$, (4.9), (4.10), (4.21), (4.22), with $u(z,t)$ replaced by $p_3(z,t)$ together with (4.24) and (4.25), (4.26) with $u(y,t)$ replaced by $P_3(y,t)$. We take non-negative initial data for $u(z,t)$, satisfying the boundary conditions, by taking the simple choice

$$g(z) = \left(1 + \frac{\alpha}{2}\right) z - \frac{\alpha}{2} z^2,$$

so from (A.5), (A.7) and (A.14) we have

$$\begin{aligned} &A_1(0) = 1 - \tfrac{1}{2}\alpha, \quad A_2(0) = -\alpha, \quad A_3(0) = 0, \\ &B_0(0) = 1, \quad B_1(0) = 1 - \tfrac{1}{2}\alpha, \quad B_2(0) = -\tfrac{1}{2}\alpha, \quad B_3(0) = 0 \end{aligned}$$

and (4.27)

$$\begin{aligned} &D_1(t) = (1 - \tfrac{1}{2}\alpha)t + O(t^2), \quad C_3(t) \sim -\tfrac{1}{6}\alpha(1 - \tfrac{1}{2}\alpha)t^3 + O(t^4), \\ &r(0) \sim 1 - t + O(t^2) \end{aligned}$$

as $t \to 0$. The choice of the above initial data does place an artificial restriction on α for $g(z)$ to remain non-negative. However, computations done with more general non-negative initial data which does not impose a restriction on α suggests that the appropriate steady state is attained irrespective of the initial data.

The results of the numerical integrations are shown in Tables 4 and 5 and Figs 3 to 5. The values of the series coefficients obtained as a result of the above computations are given in Table 4. We will return to these later.

The integrations have to be initiated at some non-zero value of $t = \varepsilon$ due to the singular behaviour of the system of ordinary differential equations at $t = 0$: we took $\varepsilon = 0{\cdot}001$ in what follows. With values of α up to $\alpha = 9$ it is clear that in Region B the solution approaches a steady state as $t \to \infty$; the situation being typified in Fig. 3. Also since $r \to 0$ as $t \to \infty$, Region B eventually occupies the whole region $0 < z \leqslant 1$.

This steady state solution $u = u_s(z)$ has been written down many times and can be easily obtained in closed form by integrating (1.1) with $u(z,t) = u_s(z)$ and applying the boundary conditions (4.1) to obtain

$$u_s(z) = \begin{cases} \sin\left(z\sqrt{\alpha}\right)/\sin\left(\sqrt{\alpha}\right), & \alpha > 0, \\ \sinh\left(z\sqrt{-\alpha}\right)/\sinh\left(\sqrt{-\alpha}\right), & \alpha < 0, \\ z, & \alpha = 0. \end{cases} \tag{4.28}$$

Table 4 Values of the series coefficients as the solution approaches the steady state in region B for $\alpha = 3$ and $\alpha = 9$ using p_3 and P_3. We quote three significant figures throughout

	$\alpha = 3$	$\alpha = 9$
$A_1(5{\cdot}0)$	0·000352	$5{\cdot}58 \times 10^{-34}$
$A_2(5{\cdot}0)$	$-7{\cdot}24 \times 10^{-12}$	$-2{\cdot}94 \times 10^{-104}$
$A_3(5{\cdot}0)$	$-2{\cdot}78 \times 10^{-12}$	$-2{\cdot}24 \times 10^{-104}$
$B_0(5{\cdot}0)$	0·000352	$5{\cdot}58 \times 10^{-34}$
$B_1(5{\cdot}0)$	0·000352	$5{\cdot}58 \times 10^{-34}$
$B_2(5{\cdot}0)$	$-2{\cdot}29 \times 10^{-11}$	$-2{\cdot}88 \times 10^{-103}$
$B_3(5{\cdot}0)$	$-8{\cdot}90 \times 10^{-12}$	$-8{\cdot}27 \times 10^{-50}$
$C_3(5{\cdot}0)$	−0·876	−32·2
$D_1(5{\cdot}0)$	−0·282	−21·4
$r(5{\cdot}0)$	0·000201	$2{\cdot}58 \times 10^{-35}$

Table 5 Steady state parameters for evolution in Region B together with the exponential decay parameter obtained from the analysis

α		P_3	P_{43}	P_4	Exact
1	$S_3 = (\partial u/\partial z)_{z=1}$	0·642093	0·642093	0·642093	0·642093
	$S_0 = \delta = (\partial u/\partial z)_{z=0}$	1·188395	1·188395	1·188395	1·188395
	$B_0(10)/r(10)$	1·188395	1·188395	–	–
3	$S_3 = (\partial u/\partial z)_{z=1}$	−·28175**0**	−·28174**8**	−·281747	−·281747
	$S_0 = \delta = (\partial u/\partial z)_{z=0}$	1·754817	1·754817	1·754817	1·754817
	$B_0(10)/r(10)$	1·754817	1·754817	–	–
5	$S_3 = (\partial u/\partial z)_{z=1}$	−1·754**545**	−1·754**403**	−1·7543**91**	−1·754389
	$S_0 = \delta = (\partial u/\partial z)_{z=0}$	2·842**258**	2·8421**70**	2·84216**3**	2·842161
	$B_0(10)/r(10)$	2·842285	2·842162	–	–
7	$S_3 = (\partial u/\partial z)_{z=1}$	−4·89**7131**	−4·891**630**	−4·891**351**	−4·891252
	$S_0 = \delta = (\partial u/\partial z)_{z=0}$	5·56**6138**	5·56**1299**	5·561**054**	5·560967
	$B_0(10)/r(10)$	5·566138	5·561008	–	–
9	$S_3 = (\partial u/\partial z)_{z=1}$	−21·**8066**	−21·0**692**	−21·0**584**	−21·0458
	$S_0 = \delta = (\partial u/\partial z)_{z=0}$	22·**0120**	21·2**817**	21·2**710**	21·2585
	$B_0(10)/r(10)$	22·003834	21·252241	–	–

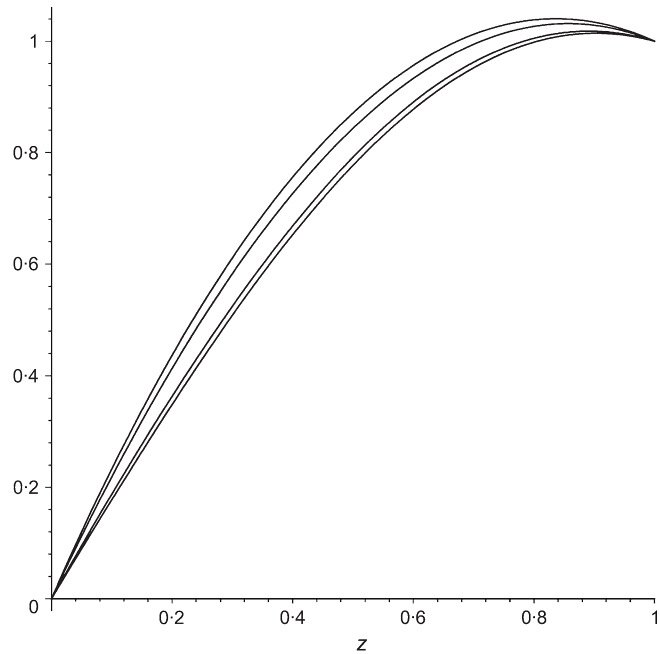

Fig. 3 The approach to the steady state for $\alpha = 3$ with p_3 and P_3. The five values of t are, going from the upper to the lower curves respectively, 0·01, 0·1, 0·5, 1·0. The ordinate represents $p_3(z)$, $0 \leqslant z < r(t)$ and $P_3(z)$, $r(t) < z \leqslant 1$ as an approximation to $u(z, t)$. The profiles for $t \geqslant 1{\cdot}0$ are graphically indistinguishable

For larger values of α something different happens—namely blow-up in finite time. This is revealed in Figs 4 and 5 which suggests that there exists a threshold value $\alpha = \alpha^*$ such that for $\alpha < \alpha^*$ the solution approaches a steady state as $t \to \infty$ while for $\alpha > \alpha^*$ the solution blows-up in finite time. We now show how we can compute such an α^*.

4.3 *The computation of* $\alpha*$

Motivated by the results presented in Table 4 we make the following scalings as the steady state is approached:

$$B_0(t) \sim r(t)S_0, \quad B_1(t) \sim r(t)S_0, \quad A_1(t) \sim r(t)S_0, \tag{4.29}$$

$$D_1(t) \sim S_3 \quad \text{and} \quad C_3(t) \sim S_4, \tag{4.30}$$

where the S_i are constants. The remaining variables are orders of magnitude smaller. We can now compute the derivatives of (4.29) to give

$$\dot{B}_0 \sim \dot{r} S_0 = -S_0^2 r(t) \tag{4.31}$$

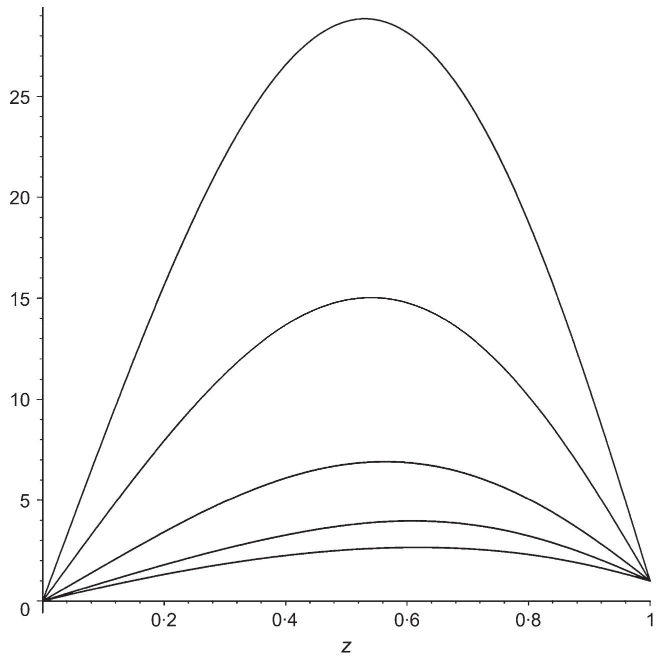

Fig. 4 Evolution towards blow-up for $\alpha = 15$ and $\varepsilon = 0{\cdot}001$ using p_3 and P_3. The values of t are $0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}45$ and the ordinate represents $p_3(z)$, $0 \leqslant z < r(t)$ and $P_3(z)$, $r(t) < z \leqslant 1$ as an approximation to $u(z, t)$

using (4.9). Similarly

$$\dot{B}_1 \sim -S_0^2 r(t), \;\; \dot{A}_1 \sim -S_0^2 r(t). \tag{4.32}$$

We also note that (4.9) implies that

$$r(t) \sim e^{-S_0 t}$$

as $t \to \infty$. If we now substitute (4.29), (4.30), (4.31) and (4.32) into our equations and take the limit $t \to \infty$ with $\dot{D}_1 = \dot{C}_3 = o(1)$, we obtain three equations for the three unknowns S_0, S_3 and S_4; the remaining equations are automatically satisfied in the limit.

Putting

$$S_3 = \tfrac{1}{2}(R_1 + R_2) \quad \text{and} \quad S_0 = \tfrac{1}{2}(R_2 - R_1),$$

we can eliminate R_1 via

$$R_1 = -\frac{a}{R_2} \tag{4.33}$$

to give two equations for R_2 and S_4. Finally elimination of S_4 gives an equation of degree eight for R_2. We can compute an α^* by finding the value of α for which the relevant root of the degree eight polynomial is zero. For p_3 this gives $\alpha^* = 9{\cdot}7781$. We shall come back to this calculation later.

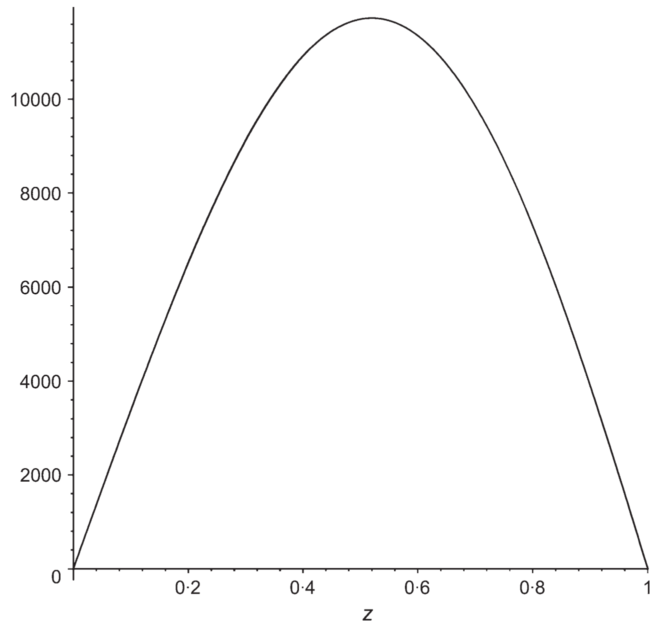

Fig. 5 Blow-up profile for $\alpha = 15$ using p_3 and P_3. The ordinate represents $p_3(z),\ 0 \leqslant z < r(t)$ and $P_3(z),\ r(t) < z \leqslant 1$ as an approximation to $u(z,t)$

4.4 *Computations with higher-degree polynomials*

We can follow a similar procedure to that followed in the previous section using polynomials p_4 and P_4 of degree nine. It turns out that we can extract a closed set of equations for the relevant unknown functions which we could integrate with the appropriate initial data to obtain a solution. Unfortunately the equation for D_1 now becomes second-order since D_4 involves $\dot{D}_1$ and the singular nature of the system near $t = 0$ is intensified, which hampers the initiation of the integration for $t > 0$. However, we can still use this system to estimate α^* since this only involves the steady state solution. To avoid the difficulty in initiating the integration at $t = 0$ for the time dependent system we may construct a polynomial P_{43} of degree eight in region B which fits four derivatives at $Y = 0$ but only three at $Y = 1$; we simply use the algorithm (2.6) with $n_0 = 5$ and $n_1 = 4$ to do this. In Region A we can still use the polynomial p_4 of degree nine without difficulty since this does not involve D_1. The initial data for all the variables can be obtained by extending the analysis of Appendix A.

We first compute the steady state parameters obtained from the analysis at the end of section 4.2. These are presented in Table 5 along with the estimate of the exponential decay parameter obtained from the actual computations using P_3 and P_{43}. We can see that the agreement is excellent and since from (4.9) with $r(t) \sim r_0 e^{-\delta t}$

$$\delta = -\lim_{t\to\infty}\left\{\frac{\dot{r}}{r}\right\} = \lim_{t\to\infty}\left\{\frac{B_0(t)}{r(t)}\right\}$$

Table 6 The dependence of the blow-up times on ε. As in previous examples the blow-up time T_ε is determined by overflow on the ordinary differential equation solver in MAPLE 9·5

α	$T_{0\cdot001}$	$T_{0\cdot0001}$	$T_{0\cdot00001}$	
	2·9250	2·9227	2·9224	$p_3 - P_3$
10	3·0204	3·0209	3·0209	$p_4 - P_{43}$
	1·4186	1·4159	1·4156	$p_3 - P_3$
11	1·4326	1·4333	1·4334	$p_4 - P_{43}$
	0·5149	0·5112	0·5109	$p_3 - P_3$
15	0·5184	0·5196	0·5197	$p_4 - P_{43}$

we confirm that the approach to the steady state in Region B is exponential with $\delta = S_0$. All these values are compared with their exact equivalents, computed from (4.28) and displayed in the last column in Table 5.

We can also refine our estimate of the threshold value α^*. We recall that the estimate for P_3 was $\alpha^* = 9{\cdot}7781$; we find that for P_{43}, $\alpha^* = 9{\cdot}8682$ and for P_4 , $\alpha^* = 9{\cdot}8689$. This suggests that α^* is converging to the value $\pi^2 = 9{\cdot}8696$ which coincides with the value of α for which the steady state solution (4.28) itself is singular.

As we have observed in section 4.2 the solution blows-up for $\alpha > \alpha^*$ in Region B using p_3 and P_3. We can reproduce this using higher-degree polynomials and results for blow-up times are shown in Table 6 for sample values of α. We also show the effect of different values of $\varepsilon \to 0$ on the blow-up time.

5. Discussion

In this paper we have shown how we can systematically construct polynomial solutions to initial-boundary-value problems for the inviscid Proudman–Johnson equation. Rather than consider the equation itself we recast the problems into various integral forms which include the boundary conditions, prior to replacement by a two point Hermite interpolant. Since the technique fits *derivatives* it is particularly adept at dealing with discontinuities of derivatives of any order, in this case along characteristics. This is illustrated particularly well by the example of section 4 which evinces the possibilities for hyperbolic equations in general. We are aware of no other comparable technique which deals with discontinuities in derivatives with such precision—in this way it could be regarded as a preferred method *of its kind* for hyperbolic problems. Although we have not produced a formal error analysis for the nonlinear problems encountered in the paper we have addressed this question of accuracy using the convergence properties of Hermite interpolating polynomials on finite intervals as the degree is increased. For the inviscid Proudman–Johnson equation considered in this paper the results appear to be satisfactory in this respect. The method appears to be able to provide accurate estimates of blow-up times, although it is not clear whether it can reproduce the detailed structure of the blow-up process. We have shown how in certain circumstances it may be convenient to fit different orders of derivative at each end of an interval. Systematically programmable algorithms are available in such situations and as we have seen can be used to good advantage. It is important to reiterate that we have used the symbolic computational package MAPLE to perform all the algebraic manipulation and also as an ODE solver. This is an indispensable tool and it would be difficult to make progress without such a facility—on a desktop machine or laptop the timings make interactive programming a straightforward matter.

Acknowledgment

The author would like to acknowledge numerous discussions and correspondence with Dr Tony Kay concerning the problems discussed in this paper; in particular ‘Euler blow-up driven by vorticity input at a boundary’, a private communication of 2002.

References

1. I. Proudman and K. Johnson, Boundary layer growth near a rear stagnation point, *J. Fluid. Mech* **12** (1962) 161–168.
2. D. Riabouchinsky, Quelques considerations sur les mouvements plans rotationnels d’un liquide. *C.R. Hebd. Acad. Sci.* **179** (1924) 1133–1136.
3. R. E. Grundy, The analysis of initial-boundary value problems using Hermite interpolation, *J. Comp. Appl Math*. **154** (2003) 63–95.
4. R. E. Grundy, The application of Hermite interpolation to the analysis of nonlinear diffusive initial-boundary value problems, *IMA J. Appl. Math.* **71** (2005) 814–838.
5. G. M. Phillips, Explicit forms for certain Hermite approximations, *BIT* **13** (1973) 177–180.
6. P. J. Davis, *Interpolation and Approximation* (Blaisdell, New York 1963).
7. J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis* (Springer, New York 1980).
8. S. Childress, G. R. Ierley, E. A. Spiegel and W. R. Young, Blow-up of unsteady two-dimensional Euler and Navier Stokes solutions having stagnation point form, *J. Fluid Mech.* **203** (1989) 1–22.
9. R. E. Grundy and A. Kay, The asymptotics of blow-up in inviscid Boussinesq flow and related systems. *IMA J. Appl. Math.* **68** (2003) 47–81.
10. S. M. Cox, Flow of a viscous fluid in a channel with porous walls, *J. Fluid Mech.* **227** (1991) 1–33.
11. J. R. King and S. M. Cox, Asymptotic analysis of the steady-state and time dependent Berman problem, *J. Engng Math.* **39** (2001) 87–130.
12. R. E. Grundy and R. McLaughlin, Global blow-up of separable solutions of the vorticity equation, *IMA J. Appl. Math*. **59** (1997) 287–307.
13. H. Okamoto and J. Zhu, Some similarity solutions to the Navier–Stokes equations and related topics, *Taiwanese Journal of Mathematics* **4** (2000) 65–103.
14. M. Jardine, H. R. Allen, R. E. Grundy and E. R. Priest, A family of two dimensional nonlinear solutions for magnetic field annihilation, *J. Geophys. Res.—Space Physics* **97** (1992) 4199–4207.

APPENDIX

The initial data for the inhomogeneous problem of section 4

We consider (1.1) with the boundary conditions (3.3), namely

$$u(0,t)=0, \quad u(1,t)=1, \quad \frac{\partial^2 u}{\partial z^2}(1,t)=-\alpha \tag{A.1}$$

and the initial data

$$u(z,0) = g(z), \tag{A.2}$$

where $g(z)$ satisfies the initial data compatibility condition

$$g(1) = 1, \quad g''(1) = -\alpha. \tag{A.3}$$

First it is clear that

$$r(0) = 1. \tag{A.4}$$

Now expanding $g(z)$ about $z = 0$ and comparing with (4.5) at $t = 0$ with Z replaced by $z/r(0)$ with $r(0) = 1$ we have

$$\sum_n A_n(0)z^n = \sum_n g^{(n)}(0)z^n/n!, \quad g(0) = 0$$

thus

$$A_n(0) = g^{(n)}(0)/n!, \quad n = 1, 2, 3 \ldots, \tag{A.5}$$

gives the initial conditions for the $A_n(t)$.

We now outline below what happens initially at $z = 1$. We first consider region A where we expand

$$u(z,t) = a_0 + t f_1(\eta) + t^2 f_2(\eta) + t^3 f_3(\eta) + \cdots \tag{A.6}$$

as $t \to 0$, η fixed, where $\eta = (z-1)/t$. Substituting this into equation (1.1) gives

$$f_1(\eta) = a_1\eta + b_1, \quad f_2(\eta) = a_2\eta^2 + b_2\eta + c_2,$$

$$f_3(\eta) = -a_1a_2 + a_3(a_0+\eta)^3 + b_3\eta + c_3, \text{ etc.}$$

Now (A.6) has to agree in the limit $t \to 0$ with the initial data expanded about $z = 1$, namely

$$u(z,0) \equiv g(z) = \sum_n g^{(n)}(1)(z-1)^n/n!$$

so that $a_0 = g(1)$, $a_1 = g'(1)$, $a_2 = g''(1)/2 = -\alpha/2$, $a_3 = g'''(1)/6$, etc. and comparison with (4.6) gives

$$B_0(0) = g(1), \quad B_1(0) = g'(1), \quad B_2(0) = -\alpha/2, \quad B_3(0) = g'''(1)/6, \quad \text{etc.} \tag{A.7}$$

which gives the initial conditions for the $B_n(t)$.

We now turn to Region B where as a consequence of the possible non-analyticity across $z = r(t)$ we need a different small time expansion corresponding to (A.6), namely

$$u(z,t) = k_0 + tF_1(\eta) + t^2F_2(\eta) + t^3F_3(\eta) + \cdots, \tag{A.8}$$

where

$$F_1(\eta) = k_1\eta + l_1, \quad F_2(\eta) = k_2\eta^2 + l_2\eta + m_2,$$

$$F_3(\eta) = -k_1k_2\eta^2 + k_3(k_0+\eta)^3 + l_3\eta + m_3, \text{ etc.}$$

This expansion has to agree with (4.13), including (4.15), and (4.14). Writing (4.14) in terms of z gives

$$u(z,t) = 1 + D_1(t)\frac{(z-1)}{(1-r)} - \frac{\alpha}{2}(z-1)^2 - \alpha D_1(t)\frac{(z-1)^3}{6(1-r)} + \cdots. \tag{A.9}$$

Furthermore if we write (A.8) in terms of z and t we have

$$u(z,t) = k_0 + l_1 t + m_2 t^2 + (m_3 + k_3 k_0^3) t^3 + \cdots + (z-1)\{k_1 + l_2 t + (l_3 + 3k_3 k_0^2) + \cdots\}$$
$$+ (z-1)^2\{k_2 + (3k_0 k_3 - k_1 k_2)t + \cdots\} + (z-1)^3\{k_3 + \cdots\} + \cdots . \tag{A.10}$$

Now (A.10) has to agree with (A.9) for all t so we must require

$$k_0 = 1, \quad l_1 = m_2 = 0, \quad m_3 + k_3 k_0^3 = 0 \tag{A.11}$$

and

$$k_2 = -\frac{\alpha}{2}, \quad 3k_0 k_3 - k_1 k_2 = 0. \tag{A.12}$$

Further we observe from (4.9) with $B_0(0) = 1$ that

$$r(t) = 1 - t + O(t^2)$$

as $t \to 0$; then the right-hand side of (A.9) can be written as

$$1 + \frac{D_1(t)}{t}(z-1) - \frac{\alpha}{2}(z-1)^2 - \frac{\alpha D_1(t)}{6t}(z-1)^3 + \cdots$$

so comparing with (A.10) in the same limit

$$D_1(t) \sim k_1 t \tag{A.13}$$

as $t \to 0$.

Finally we have to impose the required continuity conditions across the characteristic. To do this we put $z - 1 = -t + y$ in (A.10) and take the limit $t \to 0$ to obtain

$$u(y,t) \sim k_0 + k_1 y + k_2 y^2 + k_3 y^3 + \cdots .$$

Following the same procedure in (4.13) with (4.15) gives

$$u(y,t) = \lim_{t\to 0}\left\{B_0(t) + yB_1(t) + y^2 B_2(t) + \frac{C_3(t)y^3}{(1-r)^3} + \cdots\right\}$$

so equating these expressions gives

$$k_0 = 1, \quad k_1 = B_1(0) = g'(1), \quad k_2 = B_2(0) = -\frac{\alpha}{2}$$

and $C_3(t) \sim k_3 t^3$ as $t \to 0$. Thus combining with (A.11), (A.12) and (A.13) finally yields

$$D_1(t) \sim g'(1)t \quad \text{and} \quad C_3(t) \sim \frac{-\alpha g'(1)}{6} t^3 \tag{A.14}$$

as $t \to 0$. This completes the initial data for the coefficients up to $C_3(t)$ and $D_3(t)$. We can of course continue this process to obtain higher-order coefficients in each of the expansions as desired.