Publicacions Matemàtiques, Vol 33 (1989), 85–97.

POLYNOMIAL RINGS OVER JACOBSON–HILBERT RINGS

CARL FAITH

Abstract

All rings considered are commutative with unit. A ring R is $SISI$ (in Vámos' terminology) if every subdirectly irreducible factor ring R/I is self-injective. $SISI$ rings include Noetherian rings, Morita rings, and almost maximal valuation rings ([**V1**]). In [**F3**] we raised the question of whether a polynomial ring $R[x]$ over a $SISI$ ring R is again $SISI$. In this paper we show this is not the case.

1. Introduction

The counter–example to the above is provided by the theorem below proved in $\oint 4$.

1.0. Theorem. *For every field K and local injective module E of $K[x]$ (= the injective hull of a simple $k[x]$- module), the split-null extension $A = (K[x], E)$ is subdirectly irreducible, and a factor ring of $R[x]$, where R is the split- null extension (K, N) and N is any vector space over K of dimension not less than that of E.*

By Example 3.4B, R is $SISI$, but $R[x]$ is not. (See $\oint 4$ for the proof.)

This leaves open the question: Is $R[x] SISI$ for a Vámos, or even Morita, ring R?. We also settle another question raised in [**F3**] by showing that not every $SISI$ ring is "Monica". To define this term, we need two other concepts: (1) an ideal I of co–subdirectly irreducible (= $COSI$) if R/I is subdirectly irreducible ring; (2) an ideal I of $R[x]$ is *monic* if it contains a monic polynomial $\neq 0$ (equivalently, $R[x]/I$ is a finitely generated canonical R–module.)

In connection with (2), we first remark:

1.1. Theorem. *R is a Jacobson–Hilbert ring iff every maximal ideal of $R[x]$ is monic.*

The necessity is stated as an exercise in $[K]$.

Now define *Monica* ring to a ring R such that every $COSI$ ideal of $R[x]$ is monic. As a corollary to Theorem 1.1, we prove:

1.2. Theorem. *A Noetherian ring R is Monica iff R is Jacobson–Hilbert.*

Since any Noetherian ring R is Vamosian (see ∮3), hence $SISI$, this shows these rings need not be Monica. (Cf. Theorem 6.1 which asserts that Von Neumann regular rings are Monica.)

We call a ring R an H–ring (after Camillo) if every factor ring modulo a $COSI$ ideal is a local ring. Obviously any $SISI$ ring is an H–ring. Moreover:

1.3. Theorem. *A Jacobson–Hilbert ring R is Monica if $R[x]$ is an H–ring.*

In Theorem 1.0, R is trivially an H–ring (since R is local) but $R[x]$ is not (since A is not a local ring). See Theorem 2.5 for a proof of Theorem 1.3.

Every Morita ring is 1.c. (See ∮3.) A conjecture of Zelinsky–Mueller–Vámos ($ZMVC$) is that every 1.c. ring R is Morita. In [$F5$] we prove that $ZMVC$ is equivalent to the assertion that every 1.c. ring is $SISI$.

2. Proofs of Theorems 1.1 and 1.2

A domain R is a *G–domain* if its quotient field $K = Q_C(R)$ has the equivalent properties:

(G-1) $K = R[a_1, \ldots, a_n]$, for finitely many elements $a_1, \ldots, a_n$

(G-2) $K = R[a]$, for some $a \in K$.

An ideal I is a *G–ideal* iff I satisfies the equivalent properties:

(G-A) R/I is a G–domain

(G-A) Some maximal ideal M of $R[x]$ contracts to I.

Let rad A denote the Jacobson radical of A for any ring A, i.e. the intersection of all maximal ideals of A. An ideal I is a J–ideal if the e.c.'s hold:

(1) rad $(R/I) = 0$

(2) I is an intersection of maximal ideals.

Let nil rad R denote the maximal nil ideal of R, i.e.

$$\text{nil rad} R = \{a \in R \mid \exists_n a^n = 0\}$$

If I is an ideal, then $\sqrt{I}$ is the classical radical of I, namely the ideal such that

$$\sqrt{I}/I = \text{ nil rad } (R/I).$$

R is a *Jacobson–Hilbert ring* if R satisfies the e.c.'s:

(J-1) Every G-ideal is maximal

(J-2) $\forall$ ideals I, $\sqrt{I}$ is a J–ideal, i.e.

$$\sqrt{I}/I = \text{ nil rad } R/I = \text{ rad } R/I$$

(J-3) Every (semi) prime ideal is a J–ideal

(J-4) Every G–ideal is a J–ideal

(J-5) For all maximal ideals M of $R[x]$, $M \cap R$ is a maximal ideal of R.

See, e.g., [K], or [G] and [Kr]. (A Jacobson–Hilbert ring is called a Hilbert ring in [K], and *Jacobson* ring in [Kr], and in Bourbaki.)

2.1. Proposition. *A maximal ideal M of $R[x]$ is monic iff M contracts to a maximal ideal of R.*

Proof: Let $M_O = M \cap R$ be maximal in R. Now $M \supseteq M_O[x]$, and $M \neq M_O[x]$ that is, $M_O[X]$ is not maximal in $R[X]$, since

$$S = R[x]/M_O[x] \approx R/M_O[x]$$

a polynomial ring over a field; thus $M/M_O[x]$ is a monic ideal of S, hence M is monic in $R[x]$.

Conversely, let M be a monic ideal of $R[x]$. The domain $\bar{R} = R/M_O$ embeds canonically in the field $A = R[x]/M$. Let

$$p(x) = x^{n+1} - \alpha_O x^n - \cdots - \alpha_n \in M$$

be a monic polynomial with $\alpha_i \in R$. For $0 \neq \bar{d} \in \bar{R}$, $p(d^*) \in M$ where $dd^* \equiv 1(\operatorname{mod} M)$, i.e. $\bar{d}^* = \bar{d}^1$; hence

$$d^{-(n+1)} = \bar{\alpha}_0 d^{-n} + \cdots + \bar{\alpha}_n$$

so

$$1 = (\bar{\alpha}_0 + \cdots + \bar{\alpha}_n \bar{d}^n)\bar{d}(\ \operatorname{mod}\ M)$$

that is,

$$\bar{d}^{-1} = \bar{\alpha}_0 + \cdots + \bar{\alpha}_n \bar{d}^n \in \bar{R}$$

so $\bar{R}$ is a field, hence M_O is maximal, as required. ■

We say that a ring R is a *maxmonica* ring if all maximal ideals of $R[x]$ are monic.

2.2. Theorem. *A ring R is a maxmonica ring iff R is Jacobson–Hilbert.*

Proof: By J-5 and Proposition 2.1, any Jacobson–Hilbert ring is a maxmonica ring. Conversely, if R is maxmonica, every G–ideal of R is maximal by the proposition, i.e. R is Jacobson–Hilbert (J-1). ■

2.3. Theorem. *A Noetherian ring R is Monica iff R is Jacobson–Hilbert.*

Proof: One way by the last theorem. Conversely, if R is Jacobson–Hilbert, and I in $R[x]$ is $COSI$, then $A = R[x]/I$ is QF (since Noetherian rings are $SISI$ by [**V1**]; also see 3.3B below in ∮3), hence A is Artinian with nilpotent radical M/I. Then M is a maximal ideal $= \sqrt{I}$, so M monic implies I monic. ■

2.4. Proposition. *If R is a local Jacobson–Hilbert ring with radical J, then an ideal I of $R[x]$ is contained in just finitely many maximal ideals (equivalently I is co-semilocal) iff I is monic.*

Proof: If $k = R/J$, then $k[x]$ is a principal ideal domain, and hence every nonzero ideal is co–semilocal in fact co–Artinian. Since $k[x] = R[x]/J[x]$, the same is true for any ideal K of $R[x]$ containing $J[x]$ properly. Now every maximal ideal of $R[x]$ containing I also contains $I + J[x]$, which properly contains $J[x]$ if I is monic, so there are only finitely of them.

The converse does not use the local ring hypothesis. Since R Jacobson–Hilbert implies that $R[x]$ is Jacobson–Hilbert, if $R[x]/I$ is semilocal, then $\sqrt{I}/I$ is the intersection of just finitely many maximal ideals $\{M_i/I\}_{i=1}^n$, so $\sqrt{I} = \cap_{i=1}^n M_i$ contains the product $\Pi_{i=1}^n M_i$. But each M_i is monic by Theorem 1.1, hence $\Pi_{i=1}^n M_i$ whence $\sqrt{I}$, whence I is monic. ■

2.5. Theorem. *The f.a.e.c.'s on R:*

(1) *R is Jacobson–Hilbert*
(2) *Every ideal I of $R[x]$ contained in a unique maximal ideal is monic.*
(3) *Every ideal I of $R[x]$ contained in just finitely many maximal ideals is monic.*

Proof: (1)⇒(3) by the proof of one part (the converse) of the last theorem, and (3)⇒(1) by Theorem 1.1. Similarly, for (1)⇔(2). ■

A corollary of Theorem 2.5 is Theorem 1.3 (See Proposition 5.4.)

3. Morita, Vamosian, and SISI Rings

A ring R is *Vamosian*, or a *Vámos* ring, provided that the injective hull $E(R/M)$ is *linearly compact* (l.c.) in the discrete topology for all maximal ideals M. See **[V1]** and **[F3]** for background, and the basic theorems. We list a few of these:

3.1. *Locally Noetherian rings*, i.e. R_M is Noetherian for $M \in \max R$ (see **[V1]**). Any polynomial ring $R[x]$ is then locally Noetherian (**[F3]**). The basic facts harken back to Matlis' classic paper **[Ma]**.

3.2. *Morita rings*, i.e. both R and the minimal injective cogenerator E over R are l.c. R–modules (Mueller **[Mu]**). An equivalent formulation:

$$R = \mathrm{End}_R F$$

canonically, where F is an injective cogenerator of mod–R (Morita **[Mo]**). Then there is a Morita duality induced by $\mathrm{Hom}_R(\ , F)$ on the l.c. R–modules. If R_M is Noetherian, then $A = \mathrm{End}_R E(R/M)$ is l.c., and $\mathrm{Hom}_A(\ , E(R/M))$ induces a Morita duality on the l.c. modules (**[Ma]**,**[Mo]** and **[Mu]**).

3.3A. *A* ring R is *right* PF provided R is an injective cogenerator as a right R–module, equivalently, R is right self–injective, and has finite essential right socle. Then, by Morita's theorem, there is a Morita duality, induced by $\mathrm{Hom}_R(\, , R)$, when R is a 2–sided PF ring.

3.3B. The QF (= *quasi–Frobenius rings* are the Artinian (or Noetherian) right (or left) self–injective rings. Every QF ring is right and left Artinian and right and left PF; and conversely a left or right Artinian or Noetherian right or left PF ring is QF. (See, e.g. **[F2]**, Chap. 24 ff.)

3.4A. Theorem (Vámos). *Morita rings are Vamosian, and Vámos rings are $SISI$, but not conversely.*

3.4B. Example (Vámos [V1]) Let R be any local ring with square–zero radical N. Then, R is $SISI$, and f.a.e.:

(1) R is Vámos

(2) R is Morita

(3) $\dim_{R/N} N < \infty$.

Proof: This is essentially in **[V1]**. If I is $COSI$ in R, then either $I = N$ or $I \subset N$. In the latter case N/I must be simple, so A/I is Artinian of length 2. Now any semilocal Vamosian ring has finite uniform or Goldie dimension **[V1]**, hence R is Vámos (Morita) if (3) holds. ■

3.4C. By **[V1]** and **[F3]**, R is locally $SISI$ iff every local endomorphism ring (= $\mathrm{End}_R E(V)$, where V is a simple R–module, and $E(V)$ is its injective hull) is commutative.

4. Proof of Theorem 1.0

A subdirect irreducible (injective) module is an (injective) module E with simple essential socle V. In case E is injective, then $E = E(V)$ is a *local* injective module (**[F3]**).

Proof: Theorem 1.0 is one of those increasingly familiar theorems in which the statement contains the proof (practically).

Any injective module E over a ring T is divisible by all regular elements of T, hence for a domain $T = K[x]$, E is divisible. This implies that $(0, E)$ is a waist in A since if $(a, x) \in A$, and $a \neq 0$, then

$$(a, x)(0, E) = (0, aE) = (0, E).$$

Since E is subdirect irreducible, this implies that A is also.

Let c be a cardinal of a generating set for E over $K[x]$, let F be the free $K[x]$–module on c letters, so there is an exact sequence of $K[x]$–modules

$$0 \to I \to F \to E \to 0. \tag{3.5.1}$$

Note that if $N = L^{(c)}$, the direct sum of c copies of K, then

$$F = K[x]^{(c)} \approx K^{(c)}[x] = N[x]. \tag{3.5.2}$$

Now N is a K–module and the split–null extension $R = (K, N)$ has the required property, namely, there is a ring epimorphism

$$R[x] \to A.$$

We use without proof the fact that there is a ring isomorphism

$$h\begin{cases} R[x] \approx (K[x], N[x]) \\ \sum_{i=0}^{t}(\alpha_i, n_i)x^i \to \left[\sum_{i=0}^{t} \alpha_i x^i, \sum_{i=0}^{t} n_i x^i\right] \end{cases}$$

for $\alpha_i \in K, n_i \in N, i = 0, \ldots, t < \infty$.

Then, we use the ring homomorphism

$$\begin{cases} (K[x], N[x]) \longrightarrow A = (K[x], E) \\ (f_1(x), f_2(x)) \longrightarrow (f_1(x), \overline{f_2(x)}) \end{cases}$$

where $f_1(x) \in K[x], f_2(x) \in N[x]$, and $f_2(x)$ maps onto $\overline{f_2(x)}$ under the $K[x]$–module homomorphism $N[x] \to E$ defined by (3.5.1) and (3.5.2). (Hint: use the fact that $\overline{f(x)g(x)} = \overline{f(x)}g(x)$ for $f(x) \in K[x]$ and $g(x) \in N[x]$, i.e. $N[x] \to E$ is a $K[x]$–module homomorphism.)

Finally, since A is not a local ring (also not self–injective), then $R[x]$ is neither a $SISI$, nor H ring. ■

We proof a partial converse of Theorem 1.0.

4.1. Theorem. *If I is a non–monic COSI ideal of a polynomial ring $R[x]$ over a Jacobson–Hilbert local ring R, then $I \subset N[x]$, where*

$$N = rad\, R = nil\ rad\, R.$$

Next assume $N^2 = 0$. Then $A = R[x]/I$ is the trivial extension $(K[x], E)$ where

$$E = \overline{N[x]} = N[x]/I$$

is divisible, hence injective, whence local injective $K[x]$–module.

Proof: A Jacobson–Hilbert local ring R has nil Jacobson radical since the nil radical N must be the (intersection of the) unique maximal ideal. Then $\sqrt{I}$ contains $N[x]$, and since

$$R[x]/N[x] \approx R/N[x]$$

and R/N is a Monica ring, then $\sqrt{I} = N[x]$, consequently $I \subseteq N[x]$. But $I \neq N[x]$, since $N[x]$ is not $COSI$.

Now let $N^2 = 0$. We first assume $I \cap R = 0$. If $0 \neq \alpha \in N$, then $\alpha \in I$, hence

$$\overline{\alpha R[x]} \supseteq V$$

where $V = \operatorname{soc} A$, and $\overline{f(x)}$ is the image of any $f(x) \in R[x]$, under the canonical map $R[x] \to A$. Let $V = (v)$, and write

$$v = \bar{\alpha}\overline{g(x)}$$

for some $g(x) \in R[x]$. If $V \approx R[x]/M$, where M is maximal in $R[x]$, then M is generated modulo $N[x]$ by a monic polynomial $m(x)$, and hence $\bar{M} = (\bar{m}, \overline{N[x]})$. Since $\overline{N[x]}^2 = 0$, then $\overline{g(x)} \notin N[x]$. Write

$$g(x) = m^t g_1 + h$$

where

$$(m, g_1) = 1 \text{ (modulo } N[x]),$$

and $h \in N[x]$, $t \geq 0$. Since $N^2 = 0$, then $\alpha h = 0$, hence

$$v = \overline{\alpha g(x)} = \alpha \bar{m}^t \bar{g_1}.$$

Now $\bar{m}v = 0$, hence

$$\overline{\alpha m}^{t+1}\overline{g_1} = 0.$$

But the regular elements of A are those $\overline{f(x)}$ with $f(x) \notin M$, i.e. $\overline{g_1}$ is regular, so $\overline{\alpha m}^{t+1} = 0$, that is, $\bar{\alpha}$ annihilates a power of $\bar{m}$.

Expressed otherwise,

$$N \approx \overline{N} \subseteq \cup_{n=1}^{\infty} (\bar{m}^n)^{\perp} \tag{1}$$

where $\bar{f}^{\perp}$ is the annihilator in A of any $\bar{f} \in A$.

It is easy to see that if $Q \supseteq I$ is such that

$$\bar{Q} = \cap \bar{m}^n,$$

then

$$\bar{Q} = \overline{mQ} = \bar{m}^n \bar{Q} \, \forall n \geq 0.$$

This follows, since if $\bar{q} \in \bar{Q}$, then

$$\bar{q} = \overline{m a_1} = \bar{m}^2 \bar{a}_2 = \ldots$$

for suitable $\bar{a}_i \in A$, and then

$$a_1 - \bar{m} a_2 \in \bar{m}^{\perp}.$$

Since

$$V = \bar{M}^{\perp} = \bar{m}^{\perp} \cap \overline{N[x]}^{\perp} = \bar{m}^{\perp} \cap \overline{N[x]} = \bar{m}^{\perp},$$

is contained in every ideal $\neq 0$ of A, the $\bar{m}^{\perp} \subseteq (\bar{m})$, so $\bar{a}_1 \in (\bar{m})$. By induction, every $\bar{a}_i \in (\bar{m})$, consequently $\bar{a}_1 \in (\bar{m}^i)\forall_i$. This proves that $\bar{a}_1 \in \bar{Q}$, hence that $\bar{Q} \subseteq \overline{mQ}$, that is, $\bar{Q} = \overline{mQ}$, whence $\bar{Q} = \bar{m}^n\bar{Q}\forall_n$.

Since I is not monic, $m^n \notin I$ hence $m^n + I \supseteq I$, so $\bar{m}^n \supseteq V\forall_n$. Thus $\bar{Q} \supseteq V$. Let $\bar{H}_n = (\bar{m}^n)^{\perp}$. Then

$$\bar{H}_1 = \bar{m}^{\perp} = \bar{m}^{\perp} \cap \overline{N[x]} = \bar{M}^{\perp} = V.$$

Suppose $\bar{H}_n \subseteq Q$, and let $\bar{u} \in \bar{H}_{n+1}$. Then $\bar{m}\bar{u} \in \bar{H}_n \subseteq \bar{Q} = \overline{mQ}$, so $\overline{mu} = \overline{mq}$ for some $\bar{q} \in Q$. Then $\bar{h} = \bar{u} - \bar{q} \in \bar{m}^{\perp} = V \subseteq Q$ hence

$$\bar{u} = \bar{h} + \bar{q} \in Q.$$

This proves that

$$\bar{H} = \cup_{n=1}^{\infty} \bar{H}_n \subseteq \bar{Q}. \tag{2}$$

Now, by (1) and (2),

$$\bar{N} \subseteq \overline{N[x]} \subseteq \bar{H} \subseteq \bar{Q}. \tag{3}$$

And since $\overline{N[x]}$ is a prime ideal, it contains $\bar{f}^{\perp}\forall \bar{f} \neq 0 \in A$, so

$$\bar{H} = \overline{N[x]} \subseteq \bar{Q}. \tag{4}$$

Since $R[x]/N[x] = K[x]$ is a polynomial ring over a field, then

$$\cap_{n=1}^{\infty}(m^n) \subseteq N[x]$$

so

$$\bar{H} = \bar{Q} = \cap(\bar{m}^n) = \overline{N[x]}. \tag{5}$$

Now $F = R[x]/M$ is a field, and $\bar{H}_{n+1}/\bar{H}_n$ is a vector space over F, hence divisible by every $0 \neq t \in F$, and therefore, $\overline{Hf} = \bar{H}$ for every $\bar{f} \in A/\bar{M}$. Since

$$\overline{Hm}^n = \overline{Qm}^n = \bar{Q} = \bar{H},$$

then $E = \overline{N[x]}$ is divisible by every $0 \neq f(x) \in K[x]$ (using (4)).

By the known theory of injective modules over a PID (see, e.g. [**F1**]), then E is injective. Since E is subdirect irreducible, then E is a local injective $K[x]$-module, and evidently A is the trivial extension $(K[x], E)$.

This completes the proof once we remove the condition $I \cap R = 0$: if $I_O = I \cap R$, then $I \supseteq I_O[x]$, and $A = R[x]/I$ is an epic image of the polynomial ring

$$R[x]/I_O[x] \approx (R/I_O)[x]$$

that is,

$$A \approx (R/I_O)[x]/(I/I_O)[x].$$

Moreover,

$$(I/I_O) \cap (R/I_O) = 0.$$

Now R/I_O is a local ring if R is, and also has square zero radical if R does. The conclusion of the theorem is therefore valid for any $COSI$ ideal I. ■

4.2. Theorem. *Let R be a semilocal ring with radical J. Then $R[x]$ is $SISI$ only if J/J^2 is finitely generated.*

Proof: $\bar{R} = R/J^2$ is a semiprimary ring, hence a finite product of radical square–zero local rings, hence assume $\bar{R}$ is a local ring. Since $\bar{R}[x]$ is a factor ring of $R[x]$, then $\bar{R}[x]$ is $SISI$ by **[F3]**, hence theorem 1.0 implies that $\dim_{\bar{R}} \bar{J} < \infty$, i.e. J/J^2 is finitely generated. ■

4.3. Theorem. *If R is a perfect ring, then $R[x]$ is $SISI$ iff R is Artinian.*

Proof: By a theorem of Osofsky [0] a perfect ring R is Artinian iff J/J^2 is finitely generated. By a theorem of Bass, a perfect ring R is semiperfect and has radical $J \neq J^2$. The theorem now applies to complete the proof. ■

A valuation ring R is *discrete* $VR(= DVR)$ provided that R satisfies the e.c.'s:

$(DVR1)$ R is a Principal ideal ring (PIR)

$(DVR2)$ R is Noetherian

In this case, $\cap_{n\epsilon\omega} J^n = 0$ by the Krull Intersection theorem.

Remark. *For convenience below, we allow the possibility that $J^n = 0$ for some n.*

4.4 Theorem. *If R is a $SISI$ VR, equivalently, an $AMVR$, and if $J = radR \neq J^2$, then $R[x]$ is $SISI$ only if $\bar{R}$ is Noetherian, that is, only if $\bar{R}$ is a DVR, where $\bar{R} = R/P$, and $P = \cap_{n\epsilon\omega} J^n$.*

Proof: If follows from theorem 4.3 that $\bar{R}[x]$ is $SISI$ iff R/J^2 is Artinian. Then $\bar{R}$ is a Noetherian VR, whence DVR, so $\bar{J} = \overline{xR}$ for some $x \in R$.

But, then $x \notin J^2$, hence $xR \supset J^2$, so $J = xR$.

This implies that $\bar{R}$ is Noetherian, whence a DVR. ■

4.5. Example. Let $R = (B, E)$ be the split–null extension of a DVR B and the least injective cogenerator E over B. Then R is an $AMVR$, and $\bar{R} = R/P$ is a DVR. Actually, in this case R is PF, by Theorem 2 of **[F4]**. Similarly, in Theorem 4.4, we have the:

4.6. Corollary. *Under the assumptions of the theorem, if $P \neq P^2$, then either*

(1) $P/P^2 \approx Q_C(\bar{R})$, or

(2) $E = P/P^2$ is the least injective cogenerator of $\bar{R}$, and

$R/P^2 \approx (\bar{R}, E)$ is PF.

Proof: It follows easily from the theorem that P is divisible by p^n for every n, so P/P^2 is divisible over $\bar{R}$, hence injective. Since P/P^2 is uniform, then

P/P^2 is indecomposable, and accordingly either torsion-free, or else torsion. Then (1) holds in the former case. IF $E = P/P^2$ is torsion, then it is the least injective cogenerator over $\bar{R}$. ■

Since $\bar{R}$ is Morita, then $\bar{R} = \text{End}_{\bar{R}}E$, so $R \approx (\bar{B}, E)$ is PF by Lemma 1 and Theorem 2 of [**F4**].

5. Polynomial rings over Morita rings

In this section we investigate $R[x]$ for R a Morita ring. By a theorem of Vámos [**V1**],[**V2**], if A is a ring extension of R, and if A is a l.c. R-module, e.g., if A is finitely generated R-module, then A is also Morita. This implies that $A = R[x]/I$ is Morita for any monic ideal I of $R[x]$.

5.1. Theorem. *1. If R is Jacobson-Hilbert, and if a factor ring $R[x]/I$ is l.c. ring, then I is monic.*

2. If R is a l.c. Jacobson-Hilbert ring, then for any monic ideal I, the factor ring $A = R[x]/I$ is l.c. as a ring.

Proof: Any l.c. ring is semilocal (in fact, semiperfect-see [**S**]). Then, A l.c. implies that I is monic by Corollary 2.3. In this case $R[x]/I$ is a finitely generated R-module, whence l.c. as an R-module, whenever R is. (See, e.g., [**V1**].) The converse is trivial if A is l.c. as an R-module. ■

5.2. Corollary. *If R is Jacobson-Hilbert, then for any ideal I of $R[x]$, $R[x]/I$ is Morita only if I is monic. This holds in particular, when $R[x]/I$ is PF (or QF).*

Proof: A Morita ring is l.c. and a PF (also QF) ring is a Morita ring. ■

(Part of the next result is Theorem 1.3 of the Introduction.)

5.3. Proposition. *For a Jacobson-Hilbert ring R, consider the following 3 conditions:*

(1) *$R[x]$ is $SISI$*
(2) *$R[x]$ is an H-ring*
(3) *Every $COSI$ ideal I of $R[x]$ is contained in just finitely many maximal ideals*
(4) *R is Monica;*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) and conversely if R is a Morita ring.

Proof: Any $SISI$ ring is an H-ring, so (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is trivial. Next, (3) $\Rightarrow$ (4) by Theorem 2.5.

Now assume (4). By the introduction to this section, if R is Morita, then $A = R[x]/I$ is Morita, hence $SISI$ for any $COSI$ ideal I, and therefore self-injective. Thus $R[x]$ is $SISI$, so (4) $\Rightarrow$ (1), assuming R is Morita. ■

Any Monica ring is Jacobson-Hilbert, so we also have:

5.4. Corollary. *If R is a Monica Morita ring, then $R[x]$ is $SISI$.*

6. Von Neumann regular rings are Monica

By a theorem of Kaplansky, a von Neumann regular (VNR) ring R has the (characterizing) property that R_M is a field for each maximal ideal M, and hence, as Vámos pointed out in [**VI**], is Vamosian, whence $SISI$ (see ∮ 3.1). Moreover, R is Jacobson–Hilbert since every prime ideal is maximal.

6.1. Theorem. *Any VNR ring R is Monica.*

Proof: R is Jacobson–Hilbert, and by 3.1 the ring $R[X]$ is $SISI$, hence R is Monica by Prop. 5.3. ∎

A ring R is *Prüfer* (also called *Arithmetical*) iff R_M is a VR for all maximal ideals M. Any semihereditary ring is a Prüfer ring, since then R_M is a valuation domain (VD) for every maximal ideal.

6.3. Corollary. *A Prüfer ring R is $SISI$ iff R_M is an $AMVR$ for all maximal ideals M.*

Proof: By [**VI**], R is $SISI$ iff R_M is $SISI$ $\forall M$. By [**F3**], any $SISI$ VR is an $AMVR$, so the corollary follows. ∎

6.2. Corollary. *If R is a von Neumann regular (VNR) ring, then $R[x]$ is Prüfer and $SISI$, hence $R[x]_M$ is an $AMVD$ for all maximal ideals M.*

Proof: Over a VNR ring R, the polynomial ring $R[x]$ is semihereditary hence Prüfer. Since R is locally Noetherian (in fact locally a field) then so is $R[x]$, so $R[x]$ is $SISI$. (See ∮ 3.1). Since $R[x]_M$ is Noetherian, it is a DVD. I have Dr. P. Pillay to thank for noting this. ∎

7. Open Problems

In this paper, we have shown that a polynomial ring over a $SISI$ Jacobson–Hilbert local ring need not be $SISI$, in fact need not be an H-ring. Does the corresponding hold for Vámos or Morita rings? Also similar questions may be asked for a 1.c. R, i.e. when is $R[x]/I$ also a 1.c. ring, other than when I is monic?

Characterize R such that all $COSI$ (or maximal) ideals of $R[x]$ are faithful. These include Monica (maxmonica) rings.

Note

Hilbert rings are so-called because of their connections with the Hilbert Nullstellensatz (see [K] for a lucid exposition of Goldman's [G] and Krull's [Kr] results.) Jacobson rings are named by [**Kr**] because of their characterizations via the condition that the nilradical equals the Jacobson radical in a any factor ring.

Some of these same ideas have been extended to polynomial rings over von Neumann regular rings by Gentle [Ge]. (Cf. Theorem 6.1 which implies that VNR's are Monica rings.)

References

[C]. CAMILLO, V.P., Homological independence of injective hulls of simple modules over commutative rings, *Comm. Alg.* **6(14)** (1978), 1459-69.

[F1]. FAITH, C., "*Algebra I: Rings, Modules, and Categories*," Springer-Verlag, Berlin, Heidelberg, New York, 1973. corrected reprint, 1981

[F2]. FAITH, C., "*Algebra II: Ring Theory*," Springer-Verlag, 1976.

[F3]. FAITH, C., Linearly compact modules and a theorem of Vámos, *Publ. Sec. Mat. Univ. Autònoma Barcelona* **30** (1986), 127–148.

[F4]. FAITH, C., Self-injective rings, *Proc. Amer. Math. Soc.* **77** (1979), 157–164.

[F5]. FAITH, C., A note on Morita and linearly compact rings. to appear

[G]. GOLDMAN, O., Hilbert rings and the Hilbert Nullstellensatz, *Math. Z.* **54** (1951), 136–140.

[Ge]. GENTLE, R., Comment on the Nullstellensatz for regular rings, *Can. Bull. Math.* **30** (1987), 124–128.

[K]. KAPLANSKY, I., Commutative Rings, *Univ. of Chicago Press* (1974).

[Kr]. KRULL, W., Jacobonsche Rings, Hilbertscher Nullstellensatz, Dimensiontheorie, *Math. Z.* **54** (1951), 354–387.

[Ma] MATLIS, E., Injective modules over Noetherian rings, *Pac. J. Math* **83** (1958), 207–19.

[Mu] MUELLER, B., Linear compactness and Morita duality, *J. Algebra* **16** (1970), 60–66.

[Mo] MORITA, K., Duality for modules and its applications to the theory of rings with minimum condition, *Sci. Reports* **6** (1959), 83-142, Tokyo Kyoiku Daigaku.

[V1]. VÁMOS, P., Classical rings, *J. Algebra* **34** (1959), 114–129.

[V2]. VÁMOS, P., Rings with duality, *London Math. Soc.* **35** (1977), 275–289.

[V3]. VÁMOS, P., The decomposition of finitely generated modules and fractional self-injective rings, *J. London Math. Soc. (2)* **16** (1977), 209–220.

[S]. SANDOMIERSKI, F.L., Linearly compact modules and local Morita duality, in Ring Theory, *Academic Press* (1972), New York.

[Z]. ZELINSKY D., Linearly compact modules and rings, *Amer. J. Math.* (1953), 79–90.

Rutgers, The State University
New Brunswick
NJ 08903 U.S.A.

Institut d'Estudis Catalans
Centre de Recerca Matemàtica
Bellaterra, Apartat 50
08193 SPAIN

Rebut el 4 de maig de 1988