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Polynomial Wigner–Ville Distributions and Their Relationship to Time-Varying Higher Order Spectra

Boualem Boashash and Peter O'Shea

Abstract—**The Wigner–Ville distribution (WVD) has optimal energy concentration for linear frequency modulated (FM) signals. This paper presents a generalization of the WVD in order to effectively process nonlinear polynomial FM signals. A class of polynomial WVD's (PWVD's) that give optimal concentration in the time-frequency plane for FM signals with a modulation law of arbitrary polynomial form are defined. A class of polynomial time-frequency distributions (PTFD's) are also defined, based on the class of PWVD's. The optimal energy concentration of the PWVD enables it to be used for estimation of the instantaneous frequency (IF) of polynomial FM signals. Finally, a link between PWVD's and time-varying higher order spectra (TVHOS) is established. Just as the expected value of the WVD of a nonstationary random signal is the time-varying power spectrum, the expected values of the PWVD's have interpretations as reduced TVHOS.**

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I. Introduction

Classical Fourier transform-based analysis techniques have been shown to be often ineffective for analyzing nonstationary signals [1]. For such signals, much attention has been focused on the Wigner–Ville distribution (WVD) or on smoothed variants of it [9]. This attention has resulted largely because the WVD provides optimal energy concentration in the time-frequency ($t - f$) plane for linear frequency modulated (FM) (*quadratic* phase) signals [1].

This paper considers an extension of the WVD to higher orders, such that one can obtain good time-frequency energy concentration for FM signals with higher order polynomial phase laws. It is shown that the expected value of such distributions can be related to *time-varying higher order spectra* (TVHOS).

II. Polynomial Wigner–Ville Distributions

A. The Wigner–Ville Distribution

The WVD has optimal concentration in the $t - f$ plane for linear FM signals [1] in the sense that it yields a continuum of delta functions along the signal's instantaneous frequency (IF) law. It is this property that makes the WVD useful for IF estimation [1], [2]. However, for nonlinear FM signals, optimal concentration is not obtained, and smeared spectral representations result. We show in this paper that it is possible to design *Polynomial WVD (PWVD's)*, which exhibit a continuum of delta functions along the IF law for arbitrary order polynomial FM signals [3]. To explain how this is achieved, one needs to look closely at the mechanism by which the WVD attains optimal concentration for linear FM signals. Consider a unit amplitude analytic signal $z(t) = e^{j\phi(t)}$. The WVD of this signal is defined by

$$W_z(t, f) = \underset{\tau \to f}{\mathcal{F}} [K_z(t, \tau)] \tag{1}$$

where $\underset{\tau \to f}{\mathcal{F}}$ denotes the Fourier transform with respect to τ, and the bilinear kernel $K_z(t, \tau)$ is

$$K_z(t, \tau) = z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right). \tag{2}$$

Substitution of $z(t) = e^{j\phi(t)}$ and (2) into (1) yields

$$W_z(t, f) = \underset{\tau \to f}{\mathcal{F}} \left[e^{j(\phi(t+\tau/2) - \phi(t-\tau/2))}\right]. \tag{3}$$

The term $\phi(t + \tau/2) - \phi(t - \tau/2)$ in (3) can be re-expressed as

$$\phi(t + \tau/2) - \phi(t - \tau/2) = 2\pi\tau \hat{f}_i(t, \tau) \tag{4}$$

where $\hat{f}_i(t, \tau)$ can be considered to be an IF estimate. This estimate is the difference between two phase values divided by $2\pi\tau$, where τ is the separation in time of the phase values. This estimator is simply a finite difference of phases centrally located about time instant t and is known as the *central finite difference* (CFD) estimator [4]. Equation (3) can therefore be rewritten as

$$W_z(t, f) = \underset{\tau \to f}{F} \left[e^{j(2\pi\tau \hat{f}_i(t, \tau))}\right]. \tag{5}$$

Thus, the WVD kernel is a function that is reconstructed from the CFD-derived IF estimate. It is now apparent why the WVD yields good energy concentration for linear FM signals since the CFD estimator is known to be unbiased for such signals and in the absence of noise $\hat{f}_i(t, \tau) = f_i(t)$. Thus, linear FM signals are transformed into sinusoids in the WVD kernel with the frequency of the sinusoid being equal to the IF of the signal $z(t)$ at time t. Fourier transformation of the kernel then becomes

$$W_z(t, f) = \delta(f - f_i(t)) \quad (6)$$

that is, a row of delta functions along the true IF of the signal. The above equation is valid only for unit amplitude linear FM signals of infinite duration in the absence of noise. For nonlinear FM signals, a different formulation of the WVD has to be introduced in order to satisfy (6) under the same conditions.

B. Polynomial WVD's

Nonlinear FM signals are common both in nature and in engineering applications. For example, the sonar system of some bats use *hyperbolic FM* and *quadratic FM* signals for echo location. In radar, certain pulse-compression schemes employ linear FM and *quadratic FM* signals [1].

One can generalize the WVD in order to effectively analyze nonlinear polynomial FM signals by replacing the central finite difference IF estimator (inherent in the WVD formulation) with an alternative IF estimator. The purpose of this corependence is to introduce alternative IF estimators that are unbiased for polynomial frequency (or phase) laws of arbitrary order and assume a phase model given by

$$\phi(t) = \sum_{i=0}^{p} a_i t^i. \quad (7)$$

The problem of designing these IF estimators in discrete-time form reduces to the design of FIR-differentiating filters for polynomial laws, which is described in [4]. Essentially, the filter coefficients are obtained by solving a system of linear equations that relate the filter output to the true phase derivative.

The estimators described in [4] weight the phases at equally spaced sample points and then sum these weighted phases. This process is defined by

$$\hat{f}_{i_q}(t, \tau) = \frac{1}{2\pi\tau} \sum_{k=-q/2}^{q/2} b_k \, \phi(t + k\tau) \quad q \in Z^+ \quad (8)$$

where

- t time at which the estimate is taken
- τ lag
- $\phi(t)$ phase of the signal $z(t)$
- b_k a constant controlling the weighting of the different phase values.

The "order" of the estimator is denoted by q and, in (8), is assumed to be an even number such that $q \geq p$. Coefficient b_0 in (8) is always zero [4].

An alternative way of implementing the required estimators is to weight the phases at unequally spaced samples and then to take the sum. This allows the weights b_k to be prespecified, which is a fact that will be seen to be important in Section II-C. This type of estimator [3], [5] is defined by introducing c_k into (8) as follows:

$$\hat{f}_{i_q}(t, \tau) = \frac{1}{2\pi\tau} \sum_{k=-q/2}^{q/2} b_k \, \phi(t + c_k\tau) \quad q \in Z^+. \quad (9)$$

Here, c_k is a constant that controls the separation of the different phase values used to construct the filter. The coefficients b_k and c_k may be varied to yield unbiased IF estimates for signals with an arbitrary polynomial FM law. The procedure for determining b_k and c_k for the case of $q = 4$ is given in Section II-C. The PWVD's that result from incorporating the phase difference estimators in (9) are defined analogously to (1) by

$$W_z^g(t, f) = \underset{\tau \to f}{\mathcal{F}} [K_z^g(t, \tau)] \quad (10)$$

where $K_z^g(t, \tau)$ is the *polynomial kernel* given by

$$K_z^g(t, \tau) = \prod_{k=-q/2}^{q/2} [z(t + c_k\tau)]^{b_k}. \quad (11)$$

The above expression for the kernel may be rewritten in symmetric form as

$$K_z^g(t, \tau) = \prod_{k=0}^{q/2} [z(t + c_k\tau)]^{b_k} \, [z^*(t + c_{-k}\tau)]^{-b_{-k}}. \quad (12)$$

The discrete-time version of the PWVD is given by the discrete Fourier transform of

$$K_z^g(n, m) = \prod_{k=0}^{q/2} [z(n + c_k m)]^{b_k} \, [z^*(n + c_{-k} m)]^{-b_{-k}}. \quad (13)$$

where $n = tf_s$, $m = \tau f_s$, and f_s is the sampling frequency.

The conventional WVD is a special case of the PWVD and may be recovered by setting $q = 2, b_{-1} = -1, b_0 = 0, b_1 = 1, c_{-1} = -1/2, c_0 = 0, c_1 = 1/2$. Note that if b_k is an integer, the PWVD kernel is *multi-linear*, as opposed to the *bilinear* kernel of the WVD. While the bilinear kernel transforms linear FM signals into sinusoids, the multi-linear kernel can be designed to transform higher order FM signals into sinusoids. These sinusoids are manifest as delta functions when Fourier transformed. Thus the WVD kernel may be interpreted as a first order approximation in a polynomial expansion of phase differences.

C. Example: Design of a Practical Polynomial Kernel

One of the simplest generalizations of the usual WVD kernel can be achieved by making $q = 4$. This selection of q enables high energy concentration in the $t - f$ plane for polynomial phase laws up to order $p = 4$ (i.e., for FM laws up to order 3). The set of coefficients b_k and c_k must be found to completely specify the new kernel. Although b_k may theoretically take any value [5], it is practically constrained to be an integer for two reasons. First, the use of noninteger b_k would lead to a kernel in which one must raise signal values to noninteger powers. This would lead to ambiguities in the case of complex signals, which is clearly undesirable. Second, the use of integer b_k enables the expected value of the PWVD to be interpreted as TVHOS (see Section III-B).

It is also desirable that the sum of the magnitude of b_k be as small as possible since the greater the sum of $|b_k|$, the greater the deviation of the kernel from nonlinearity. This phenomenon results from the multiplication of b_k with the phase terms, which in turn translate into powers of $z(t + c_k\tau)$. The extent of the nonlinearity in the kernel should therefore be limited to prevent excessively poor performance of the PWVD in noise.

In order to transform second-order phase law signals into sinusoids, b_k must assume the values $b_{-1} = 1$, $b_0 = 0$, and $b_1 = +1$. To transform third- and fourth-order phase law signals into sinusoids, it is necessary to incorporate two extra b_k terms since the phase differentiating filter must have two extra taps [4]. Simply assigning ± 1 to these two extra terms would be fraught as the IF estimator obtained would be biased. The simplest values that these terms could then assume are ± 2, and therefore, the simplest possible kernel satisfying the above criteria is

$$b_2 = -b_{-2} = 1, \; b_1 = -b_{-1} = 2, \; b_0 = 0. \quad (14)$$

The coefficients c_k must then be found so that the polynomial kernel transforms unit amplitude cubic, quadratic, or linear FM signals into sinewaves. The design procedure necessitates the construction of a system of equations that relate the polynomial IF of the signal to the IF estimates obtained from the polynomial phase differences and solving for c_k. The procedure is described below.

The construction of the design equations assume that the signal phase (in discrete-time form) is a pth-order polynomial given by

$$\phi[n] = \sum_{i=0}^{p} a_i n^i \tag{15}$$

where a_i are the polynomial coefficients. The estimate of the IF is then given by [1], [4]

$$\hat{f}_i[n] = \frac{1}{2\pi} \sum_{i=0}^{p} i a_i n^{i-1}. \tag{16}$$

A qth-order phase difference estimator (with order $q \geq p$) is applied to the signal. It is required that at any discrete-time index n, the output of the estimator yield the true IF. The system of equations required to ensure this is given by

$$\frac{1}{2\pi m} \sum_{k=-q/2}^{q/2} b_k \, \phi(n + c_k m) = f_i[n] \tag{17}$$

that is

$$\frac{1}{2\pi m} \sum_{k=-q/2}^{q/2} b_k \sum_{i=0}^{p} a_i \, (n + c_k m)^i = \frac{1}{2\pi} \sum_{i=1}^{p} i \, a_i \, n^{i-1}. \tag{18}$$

Note that because of the invariance of a polynomial's order to its origin, n may be set equal to zero without loss of generality. Setting n equal to zero in (18) then yields

$$\frac{1}{m} \sum_{i=0}^{p} a_i \, m^i \sum_{k=-q/2}^{q/2} b_k \, c_k^i = a_1. \tag{19}$$

The a_i coefficients on the left- and right-hand sides of (19) may be equated to yield a set of $p+1$ equations. Performing this operation for the values of b_k specified in (14) yields

$$a_0[1 - 1 + 2 - 2] = 0 \times a_0 \tag{20}$$

$$a_1[c_2 - c_{-2} + 2c_1 - 2c_{-1}] = 1 \times a_1 \tag{21}$$

$$a_2\left[c_2^2 - c_{-2}^2 + 2c_1^2 - 2c_{-1}^2\right] = 0 \times a_2 \tag{22}$$

$$a_3\left[c_2^3 - c_{-2}^3 + 2c_1^3 - 2c_{-1}^3\right] = 0 \times a_3 \tag{23}$$

$$a_4\left[c_2^4 - c_{-2}^4 + 2c_1^4 - 2c_{-1}^4\right] = 0 \times a_4. \tag{24}$$

It is obvious from inspection that if $c_1 = -c_{-1}$ and $c_2 = -c_{-2}$, then (20), (22), and (24) are verified. This condition amounts to verifying the symmetry property of the FIR filter. Solving for c_1, c_{-1}, c_2, and c_{-2} then becomes straightforward from (21) and (23), subject to the condition that $c_1 = -c_{-1}$ and $c_2 = -c_{-2}$. The solution is

$$c_1 = -c_{-1} = \frac{1}{2(2 - 2^{1/3})} = 0.675 \tag{25}$$

$$c_2 = -c_{-2} = -2^{1/3} c_1 = -0.85. \tag{26}$$

The resulting discrete-time kernel is then given by

$$K_z^g(n, m) = [z(n + 0.675m) \, z^*(n - 0.675m)]^2 \, z^*(n + 0.85m) \, z(n - 0.85m). \tag{27}$$

Fig. 1 shows the WVD and the PWVD based on the kernel in (27) for the same quadratic FM signal (noiseless case). The superior behavior of the later is indicated by the sharpness of the peaks in Fig. 1(b).

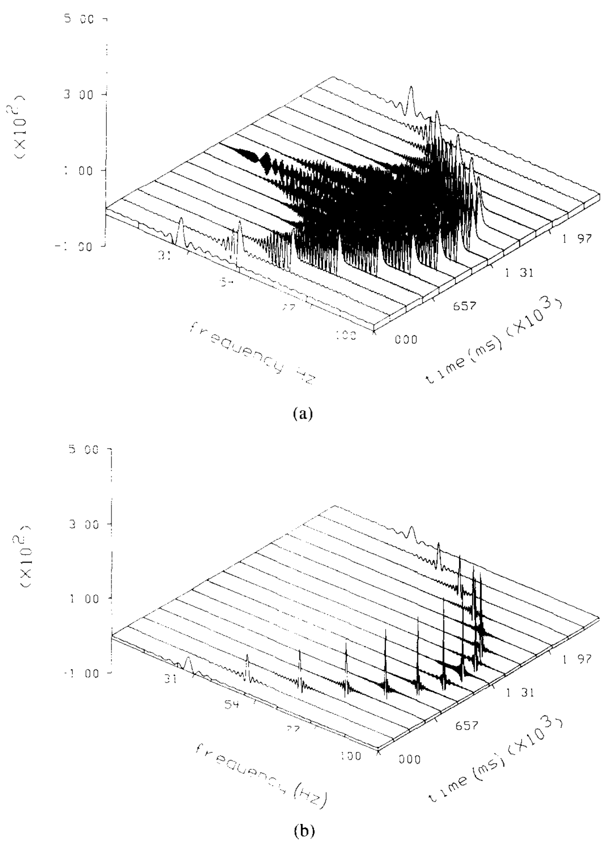

Fig. 1. (a) WVD and (b) PWVD of the same quadratic FM signal (noiseless case) (sampling frequency is 200 Hz; number of samples is 512; time resolution is 0.16 s; a full-size window was used).

The quadratic IF law can be easily recovered from the peaks of the PWVD, as opposed to the peaks of the WVD. Note that the WVD exhibits oscillations that have no physical interpretation.

Several important points need to be made concerning the practical implementation of the kernel in (27). First, to form the discrete kernel, one must have signal values at noninteger time intervals. The signal must therefore be sampled reasonably densely or interpolation used. (The interpolation can be performed by use of an FFT-based interpolation filter.) Second, it is crucial to use analytic signals so that the artifacts that arise due to the interaction between positive and negative frequencies are suppressed [1]. Third, the PWVD is best implemented by calculating a frequency-scaled version of the kernel in (27) and then accounting for the scaling in the Fourier transform operation, that is, the PWVD is best implemented as

$$W_z^g(n, f) = \underset{m \to \frac{1}{0.85} f}{DFT} \left\{ [z(n + 0.794m) \, z^*(n - 0.794m)]^2 \, z^*(n + 1.0m) \, z(n - 1.0m) \right\}. \tag{28}$$

This formulation reduces errors that may arise from the interpolation process since it causes some of the terms within the kernel to occur at integer lags. The properties of the PWVD are not listed in this correspondence due to space limitations. They appear in [5].

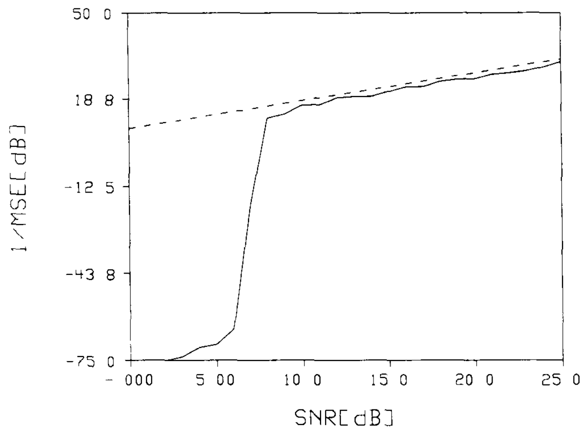

Fig. 2. Statistical performance of the PWVD-based instantaneous frequency estimate of quadratic FM signal in white Gaussian noise. Mean square error of the estimate versus the Cramer-Rao bound for $N = 64$ points (dashed line shows the CR bound). A curve for the mean square error is obtained by simulations over 250 trials.

III. Polynomial Time-Frequency Distributions, Instantaneous Frequency Estimation and Higher Order Spectra

A. Polynomial Time-Frequency Distributions and IF Estimation

In practical Wigner-Ville analysis, some form of windowing is usually applied since the observed signal is not infinite in extent. The generalized class of one or two dimensionally windowed WVD's is often referred to as *Cohen's class* of time frequency distributions [9]. Similarly, Cohen's class may be further generalized. Thus, we define a new class of polynomial time-frequency distributions (PTFD's) as

$$\rho_z^g(t,f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi(\nu,\tau) \times \left\{\prod_{k=0}^{q/2}[z(\lambda+c_k\tau)]^{b_k}\,[z^*(\lambda+c_{-k}\tau)]^{-b_{-k}}\right\} \times e^{j2\pi\nu(\lambda-t)}d\nu d\lambda e^{-j2\pi f\tau}d\tau \qquad (29)$$

where $\phi(\nu,\tau)$ is the 2-D smoothing (filtering) function. For practical purposes, it is desirable that the time-frequency distribution is real. This constraint requires that $b_k = -b_{-k}$ and $c_k = -c_{-k}$. Note that the smoothing that occurs as a result of the 2-D windowing becomes a very important consideration for multicomponent signals. This is due to the fact that the PWVD's multilinear kernel creates a multiplicity of artifacts, which must be reduced for practical analysis. The question of 2-D window design for GTFD's will be investigated in future work.

Another natural question that arises is whether the PWVD can be used for accurate estimation of the IF of nonlinear polynomial FM signals. The performance of the PWVD-based IF estimation for polynomial phase laws up to order $p = 4$ is described in [16] and [5] and is illustrated in Fig. 2. It shows that PWVD peak-based IF estimates meet the Cramer-Rao lower variance bounds at high SNR's and thus provide a very accurate means for IF estimation.

B. TVHOS Based on Polynomial WVD's

The vast majority of research in the area of HOS generally assume that the signals under consideration are stationary. However, various methods have been proposed in an attempt to deal with the nonstationary case. Some of the earlier techniques include the third-order Wigner distribution [12] and the third-order spectrogram ("running bispectrum") [15]. Recently, other extensions of these methods have been developed in [11], [14], [10], and [13].

In this section, another form of TVHOS is defined, based on a link with the PWVD's. It is of considerable interest to note that the expected values of the polynomial kernels (given by (11) and (12)) correspond to higher order moments and/or higher order cumulants evaluated at particular lags. Consequently, just as the expected value of the WVD is the time-varying power spectral density [1], one may form TVHOS by introducing the expectation operator into the PWVD formulation [3], [5]. Thus, if an ensemble of random processes are available, one can obtain a high-resolution time-varying higher order spectral representation. If only one realization is available, one may assume a local ergodicity and perform smoothing in the $t-f$ plane, that is, the use of one of the polynomial time-frequency distributions described in (29) as the polyspectral estimate.

Consider a random signal $z_r(t)$. The expectation of the polynomial kernel is

$$\xi\{K^g_{z_r}(t,\tau)\} = \xi\left\{\prod_{k=1}^{q/2} z_r^{b_k}(t+c_k\tau)\,[z_r^*(t+c_{-k}\tau)]^{-b_{-k}}\right\}. \qquad (30)$$

The resulting TVHOS relating to the fourth-order polynomial phase law is

$$\xi\{W^g_{z_r}(t,f)\} = \underset{\tau\to f}{\mathcal{F}}\,\xi\{z_r^2(t+.675\tau)\,[z_r^*(t-.675\tau)]^2\, z_r^*(t+.85\tau)\,z_r(t-.85\tau)\}. \qquad (31)$$

This expression is a spectrum of a higher order moment (sixth order) in which there is only one frequency variable. It is in fact a "reduced" TVHOS. It may be rewritten as

$$\xi\{W^g_{z_r}(t,f)\} = \underset{\tau\to f}{\mathcal{F}}[m_6'(t,\tau)] \qquad (32)$$

where $m_6'(t,\tau)$ is a time-varying sixth-order moment reduced to one lag variable. A full analysis of this class of TVHOS appears in [3] and [5]. Further results will appear in [7], [8], and [6], which describe the application of a member of this class of TVHOS to the analysis of FM signals in multiplicative noise.

Notice that as the time-varying power spectrum (expected value of the WVD) of a stationary random process reduces to the power spectrum, the TVHOS based on the PWVD of the same process reduces to the polyspectra. One possible application relates to its use as a discriminator for nonstationary random processes with differing higher order moments (e.g., between Gaussian and nonGaussian processes).

IV. Conclusion

In this corespondence, the conventional WVD has been generalized in order to effectively characterize signals with nonlinear polynomial FM laws. It is shown that the PWVD of such signals produce a row of delta functions along its IF law in the $t-f$ plane. In addition, a class of the PTFD's, which includes Cohen's class of TFD's, have been defined as a tool for time-frequency signal analysis. The expected values of these PWVD's are the Fourier transforms of some particular higher order moments and/or cumulants. The ensemble averaged PWVD's therefore have the interpretation as TVHOS.

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Maximum Likelihood Estimation of the Attenuated Ultrasound Pulse

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***Abstract*—The attenuated ultrasound pulse is divided into two parts: a stationary basic pulse and a nonstationary attenuation pulse. A standard ARMA model is used for the basic pulse, and a nonstandard ARMA model is derived for the attenuation pulse. The maximum likelihood estimator of the attenuated ultrasound pulse, which includes a maximum likelihood attenuation estimator, is derived. The results of this correspondence are of great importance for deconvolution and attenuation imaging in medical ultrasound.**

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I. Introduction

In medical ultrasound, a short pressure pulse is emitted from a transducer. The ultrasound pulse then propagates in a narrow beam in the tissue. When the pulse arrives at inhomogeneities in the tissue, a part of the pulse is scattered back and received by the transducer. By mechanically or electronically changing the beam direction, an image of the acoustical properties of the tissue can be formed. Usually, only the envelope of the received signal is displayed. The attenuation of the tissue is not displayed directly. As the attenuation of the tissue is a clinically relevant feature, several attenuation estimation methods have been developed, e.g., the spectral-shift method and the spectral-difference method [3], [4]. However, none of these attenuation estimation methods are based on the maximum likelihood principle. Attenuation estimation is of interest in medical ultrasound for another reason. The resolution of the envelope-detected image is poor because of the extent of the ultrasound pulse. The resolution can be improved by deconvolution, e.g., [6], but an estimate of the attenuated ultrasound pulse is needed by the deconvolution algorithm. This applies to both the axial and to the lateral direction, but only the axial (1-D) case is treated in this correspondence. As the maximum likelihood estimate of the attenuated ultrasound pulse includes a maximum likelihood attenuation estimate, it is seen that there is a close connection between attenuation estimation and pulse estimation.

This correspondence is organized as follows. In Section II, a nonstandard ARMA model of the attenuated ultrasound pulse is derived. The maximum likelihood estimator of the attenuated ultrasound pulse in a constant attenuating medium is derived in Section III. Section IV presents an example, and the conclusion is given in Section V.

II. A Parametric Model of the Attenuated Ultrasound Pulse

The propagation of ultrasound waves takes place in three dimensions, but we consider 1-D effects only. The attenuated ultrasound pulse can be divided into two parts: a stationary basic pulse and a nonstationary attenuation pulse. The basic pulse consists mainly of the electromechanical response of the transducer and the scattering function; see [7, ch. 8].

The signal $y(n)$ received by the transducer is given by

$$y(n) = H_1(z)H_2(z, n)u(n) \tag{1}$$

where $n = 1, \ldots, N$ denotes the discrete-time index, $z = e^{j\omega}$ the z-transform variable, $H_1(z)$ the stationary basic pulse, and $H_2(z, n)$ the nonstationary attenuation pulse. The radian frequency is denoted ω. The 1-D reflection sequence $u(n)$ is assumed Gaussian i.i.d. with zero mean and variance σ_0^2. A standard ARMA model is used for the basic pulse

$$H_1(z) = \frac{B(z)}{A(z)} \tag{2}$$

$$B(z) = 1 + \sum_{k=1}^{N_b} b_k z^{-k} \tag{3}$$

$$A(z) = 1 + \sum_{k=1}^{N_a} a_k z^{-k}. \tag{4}$$

Maximum likelihood estimation of the parameters of nonstationary models is possible in the time domain only. The following ARMA