POLYNOMIALS OF THE BEST UNIFORM APPROXIMATION TO SGN(x) ON TWO INTERVALS

By

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Abstract. We describe polynomials of best uniform approximation to sgn(x) on the union of two intervals $[-A, -1] \cup [1, B]$ in terms of special conformal mappings. This permits us to find the exact asymptotic behavior of the error in this approximation.

1 Introduction

In [5], we obtained precise asymptotics of the error of the best polynomial approximation of sgn(x) on two symmetric intervals, $[-A, -1] \cup [1, A]$. Paper [11] contains a somewhat simplified proof, together with generalizations. In this paper, we generalize the result to the case of two arbitrary intervals. Related problems on the asymptotics of the error of the best uniform approximation by polynomials of degree at most n to the functions x^{n+1} and $1/(x-c)$, $c \notin I$, defined on the union I of two intervals, were completely solved by N. I. Akhiezer in [2].

Fuchs [6, 7, 8] studied general problems of uniform polynomial approximation of piecewise analytic functions on finite systems of intervals. For the case of sgn(x) on two intervals, $I = [-A, -1] \cup [1, B]$, the result in [6] is

$$C_1 n^{-1/2} e^{-\eta n} \leq L_n \leq C_2 n^{-1/2} e^{-\eta n}.$$

Here

$$L_n = \inf_{p \in \mathcal{P}_n} \sup_{x \in I} |\mathrm{sgn}(x) - p(x)|,$$

where $\mathcal{P}_n$ is the set of polynomials of degree at most n, C_1 and C_2 are postive constants that depend on A and B, and η is the critical value of the Green's function G of the region $\overline{\mathbb{C}} \backslash I$ with pole at infinity. The arguments in [6] do not give optimal values of C_1, C_2.

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In the case $A = B$, we have $e^{-\eta} = \sqrt{(A-1)/(A+1)}$, and the result obtained in [5] is

$$L_{2m+2} = L_{2m+1} \sim \frac{\sqrt{2}(A-1)}{\sqrt{\pi A}}(2m+1)^{-1/2}\left(\frac{A-1}{A+1}\right)^m. \tag{1.1}$$

In this paper, we obtain a result of the same precision for arbitrary A and B. In the case $A \neq B$, the ratio $\sqrt{n}e^{n\eta}L_n$ oscillates. Similar oscillating asymptotic behavior was found by Akhiezer for the polynomials of least deviation from 0, that is, for the error of the best uniform approximation of x^{n+1} by polynomials of degree at most n on two intervals.

To state our main asymptotic result, we introduce certain characteristics of the region $\mathbb{C}\backslash I$. Let

$$G(x) = G(x,\infty) = \int_{-1}^{x} \frac{C-x}{\sqrt{(1-x^2)(x+A)(B-x)}}dx, \quad -1 < x < 1,$$

be the Green's function of $\overline{\mathbb{C}}\backslash I$ with pole at infinity (see, for example, [3]), where $C \in (-1,1)$ is the unique critical point and is given by

$$C = \frac{\int_{-1}^{1}((1-x^2)(x+A)(B-x))^{-1/2}x\,dx}{\int_{-1}^{1}((1-x^2)(x+A)(B-x))^{-1/2}dx}.$$

We introduce positive constants $\eta = G(C,\infty)$ and

$$\eta_1 = -\frac{1}{2}G''(C) = \frac{1}{2\sqrt{(1-C^2)(C+A)(B-C)}}.$$

The Green's function $G(z,C)$ satisfies

$$G(z,C) = -\ln|z-C| + \eta_2 + O(z-C), \quad z \to C,$$

and this relation defines the Robin constant η_2.

Let $\omega(x) = \omega(x,[-A,-1],\overline{\mathbb{C}}\backslash I)$ be the harmonic measure of the interval $[-A,-1]$. An explicit formula for ω is

$$\omega(z) = \operatorname{Im} \frac{\int_{-1}^{z}((x^2-1)(x+A)(B-x))^{-1/2}dx}{\int_{-1}^{1}((x^2-1)(x+A)(B-x))^{-1/2}dx}. \tag{1.2}$$

In our notation related to theta-functions, we follow Akhiezer's book [3].

Our main result is the following.

Theorem 1.1. *The error L_n of the best polynomial approximation of* $\operatorname{sgn}(x)$ *on $I = [-A,-1] \cup [1,B]$ satisfies*

$$L_n = (c+o(1))n^{-1/2}e^{-n\eta}\left|\frac{\vartheta_0\left(\frac{1}{2}(\{n\omega(\infty)+\omega(C)\}-\omega(C))|\,\tau\right)}{\vartheta_0\left(\frac{1}{2}(\{n\omega(\infty)+\omega(C)\}+\omega(C))|\,\tau\right)}\right|, \tag{1.3}$$

where

$$c = 2(\pi\eta_1)^{-1/2}e^{-\eta_2},$$

$$\tau = i\,\frac{\int_1^B((t^2-1)(B-t)(A+t))^{-1/2}dt}{\int_{-1}^1((1-t^2)(B-t)(A+t))^{-1/2}dt}, \tag{1.4}$$

and

$$\vartheta_0(t|\tau) = 1 - 2h\cos 2\pi t + 2h^4\cos 4\pi t - 2h^9\cos 6\pi t + \dots, \quad h = e^{\pi i\tau}$$

is the theta-function.

In (1.3), $\{x\}$ denotes the fractional part of x. Our method of proof is somewhat different from the methods of previous authors. It is based on an exact representation of the extremal polynomial as a composition of conformal maps of explicitly described regions. This can be considered as a development of the arguments in [4, 5]. Our representation of extremal polynomials allows us to find their asymptotic behavior in various regimes and their zero distribution.

Actually, the main asymptotic result of this paper is Theorem 7.1, which has a somewhat technical statement; Theorem 1.1 is a simple corollary. For example, according to [8], the numbers n_1 and n_2 of zeros of the extremal polynomial P_n on $[-A,-1]$ and $[1,B]$ satisfy

$$\lim_{n\to\infty} n_1/n = \omega(\infty) \quad \text{and} \quad \lim_{n\to\infty} n_2/n = 1-\omega(\infty),$$

respectively, while Theorem 7.1 implies a stronger conclusion: $n_1 = n\omega(\infty)+O(1)$.

However, in this paper we focus on the error term of the polynomial approximation and do not explore other corollaries of Theorem 7.1.

A representation of extremal polynomials is described in Sections 2 and 3, where we use an entire function introduced in [5]. In Section 4, we find an integral representation of the principal conformal map involved, and then study its asymptotics in Sections 5-7. We derive (1.1) as a special case of Theorem 1.1 in Section 8. Finally, in Section 9, we sketch, without proof, the limit case $B = \infty$. In this case, instead of approximation by polynomials, one has to consider approximation by entire functions of order $1/2$, normal type.

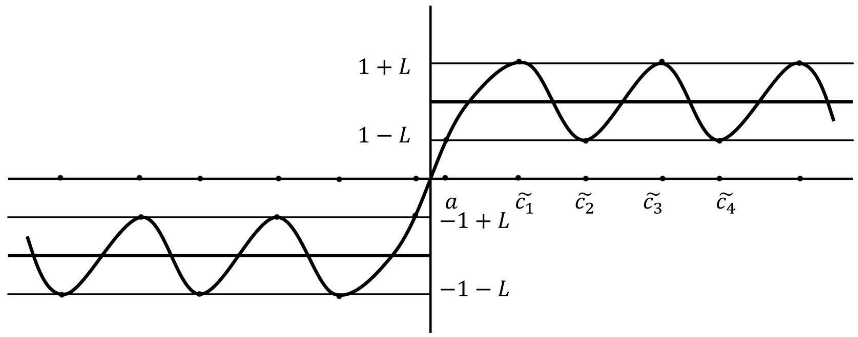

Figure 1. Graph of the function $\tilde{S}(z, a)$ on the real axis.

2 Preliminaries

We begin by recalling the construction of the entire function $\tilde{S}(z, a)$ of exponential type 1 that gives the best uniform approximation of $\operatorname{sgn}(x)$ on the set $(-\infty, -a] \cup [a, \infty)$, where $a > 0$. There is a unique such function for every $a > 0$; it is odd and satisfies

$$\tilde{S}(a, a) = 1 - L(a),\ \tilde{S}(\tilde{c}_k, a) = 1 - (-1)^k L(a), \tag{2.1}$$

where $L(a)$ is the approximation error, and $\tilde{c}_1 < \tilde{c}_2 < \cdots$ is the sequence of positive critical points. The graph of this function is shown in Figure 1. We define the positive number $b = b(a)$ by $\cosh b = 1/L(a)$. It is easy to see that b is a continuous increasing function of a, and the correspondence $a \mapsto b$ is a homeomorphism of the positive ray onto itself.

For every $b > 0$, we consider the region

$$\Omega = \{x + iy : x > 0, y > 0, x > \arccos(\cosh b/\cosh y) \text{ for } y > b\}.$$

This region is shown in Figure 2; it consists of the points in the first quadrant to the right of the curve

$$\gamma_b := \{\arccos(\cosh b/\cosh t) + it : b \le t < \infty\}.$$

Let $\tilde{\psi}$ be the conformal map of the first quadrant $\mathbb{C}_{++}$ onto Ω, normalized by

$$\tilde{\psi}(z) = z + \dots,\ z \to \infty \text{ and } \tilde{\psi}(0) = b. \tag{2.2}$$

Set $a = \tilde{\psi}^{-1}(0)$.

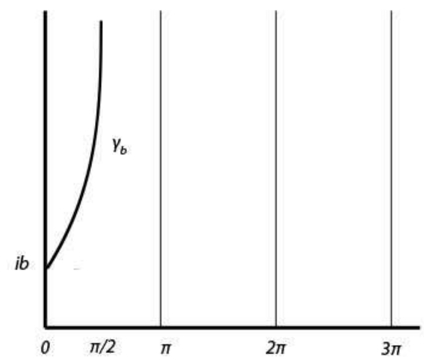

Figure 2. Domain Ω such that $\tilde{S}(z) = 1 - L(a)\cos\tilde{\psi}(z)$, $\tilde{\psi} : \mathbb{C}_{++} \to \Omega$.

In [5], we proved that

$$\tilde{S}(z, a) = 1 - L(a)\cos\tilde{\psi}(z), \quad z \in \mathbb{C}_{++} := \{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}. \tag{2.3}$$

As the right hand side of (2.3) takes real values on the positive ray and imaginary values on the positive imaginary ray, $\tilde{S}$ extends to the whole plane as an odd entire function.

The following asymptotics hold:

$$\lim_{a\to\infty} \sqrt{a}e^{a}L(a) = \sqrt{\frac{2}{\pi}}. \tag{2.4}$$

Notice that all critical points $\{\pm\tilde{c}_k\}$ of $\tilde{S}(z, a)$ are real, and $\tilde{\psi}(\tilde{c}_k) = \pi k$.

It is convenient to modify this conformal mapping slightly. We write $S(z, a) := \tilde{S}(\sqrt{z^2 + a^2}, a)$, where z belongs to the upper half-plane $\mathbb{C}_+$ with the slit $\{it : 0 < t \leq a\}$; see Figure 3. The function S is not entire; it is defined only in the upper half-plane.

Again we have the conformal mapping $\psi : \mathbb{C}_{++} \to \Omega$; but in contrast to (2.2),

$$\psi(z) = z + \dots,\ z \to \infty, \quad \psi(0) = 0, \tag{2.5}$$

and therefore $\psi(ia) = ib$. The full preimage of the real axis under S in the upper half-plane consists of the curves

$$\delta_k = \psi^{-1}(\{\pi k + it : t > 0\}), \quad k = \pm 1, \pm 2, \dots \tag{2.6}$$

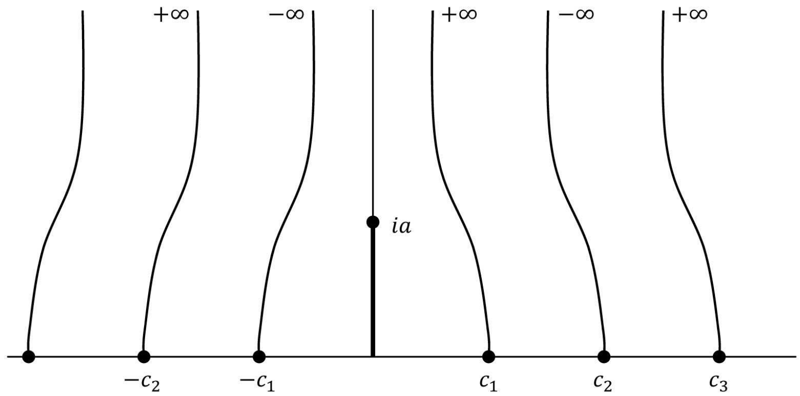

Figure 3. Preimage $S^{-1}(\mathbb{R}, a)$ in the upper half-plane.

shown in Figure 3. These curves have vertical asymptotes $\{\operatorname{Re} z = \pi k - \pi/2\}$.

Now let $P_n(z)$ be the best approximation of $\operatorname{sgn}(x)$ by polynomials of degree at most n on two intervals $I = I_- \cup I_+$, $I_\pm \subset \mathbb{R}_\pm$. Using a linear transformation, we may always assume that $I = [-A, -1] \cup [1, B]$. Our goal is to obtain a representation of the extremal polynomial in the form of the composition

$$P_n(z) = S(\Theta_n(z), a_n), \tag{2.7}$$

where Θ_n is the conformal mapping[1] of the upper half-plane on a suitable "curved" comb-like region, and a_n is an appropriate value of the parameter a.

First we give typical examples of the representation (2.7) and then show that these examples exhaust all possibilities.

First example. For $n = 4$, consider the region $\Pi_{2,3}$; see Figure 4. Its boundary consists of the vertical segment $[0, ia]$, the horizontal segment $[-c_2, c_3]$, and the curves δ_{-2} and δ_3.

Let $\Theta(z) = \Theta_4(z)$ be the conformal mapping

$$\Theta : \mathbb{C}_+ = \{z \in \mathbb{C},\ \operatorname{Im} z > 0\} \to \Pi_{2,3}, \quad \Theta(\pm 1) = 0,\ \Theta(\infty) = \infty. \tag{2.8}$$

The function $P(z) = S(\Theta(z), a)$ can be extended to the lower half-plane because of the symmetry principle. Therefore it is an entire function which is, in fact, a

[1]In what follows, the letters Θ and θ are used to denote conformal maps that have no relation to theta-functions ϑ.

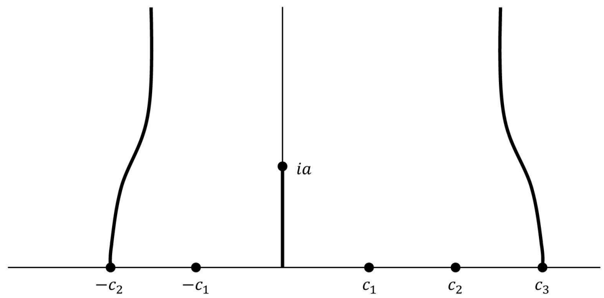

Figure 4. Domain $\Pi_{2,3}$.

polynomial of degree 4, because of its asymptotics at infinity. The graph of this polynomial on the real axis is of the form given in Figure 5, where $\phi = \Theta^{-1}$. By the Chebyshev Theorem (for the two interval version of this theorem, see [1], [2] [3]), $P(z)$ is the extremal polynomial on the set I with $A = -\phi(-c_2)$ and $B = \phi(c_3)$.

Second example. Let us point out that the above polynomial $P(z)$ has 7 points of alternance instead of 6, as required by the Chebyshev Theorem for a polynomial of degree 4. Therefore the same polynomial is extremal on two kinds of sets:

$$I = [-A, -1] \cup [1, \phi(c_3)], \quad \phi(-c_2) < -A \le \phi(-c_1) \tag{2.9}$$

and

$$I = [\phi(-c_2), -1] \cup [1, B], \quad \phi(c_2) \le B \le \phi(c_3). \tag{2.10}$$

Third example. From the position $I = [\phi(-c_2), -1] \cup [1, \phi(c_2)]$, we can start a deformation of the set I and of the extremal polynomial. Namely, consider the region $\Pi^+_{2,3}(h)$; see Figure 6. Here, we have added to the boundary a segment of the curve δ_2 that starts at the critical point c_2 and has length $h \in (0, \infty)$. In this case,

$$\Theta : \mathbb{C}_+ = \{z \in \mathbb{C},\ \operatorname{Im} z > 0\} \to \Pi^+_{2,3}(h), \quad \Theta(\pm 1) = 0,\ \Theta(\infty) = \infty, \tag{2.11}$$

and $P(z) := S(\Theta(z), a)$ is of the form given in Figure 7. For this new family of regions, which we parametrized by positive h, the polynomial is extremal on the set $I = [\phi(-c_2), -1] \cup [1, \phi(c_2 - 0)]$.

In the next section, we show that these examples exhaust all possibilities for the extremal polynomials.

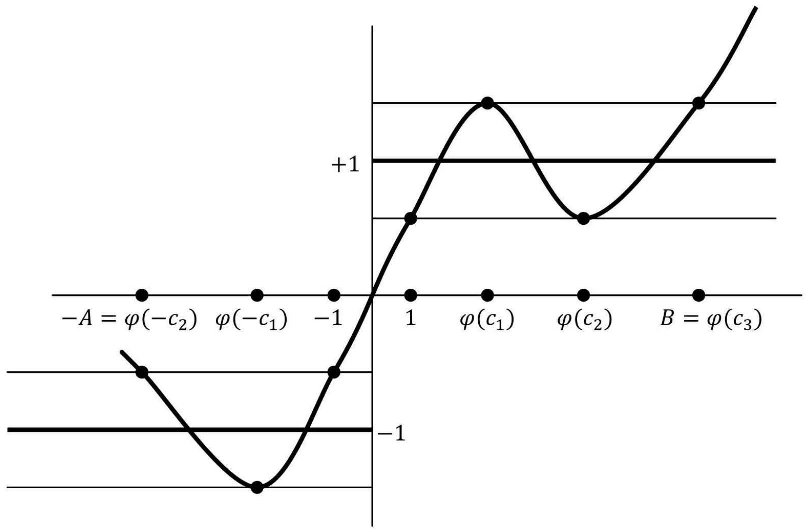

Figure 5. Extremal polynomial corresponding to the region $\Pi_{2,3}$.

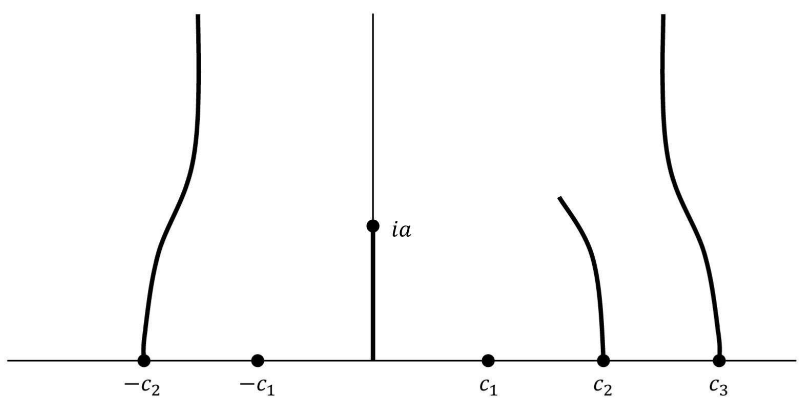

Figure 6. Domain $\Pi^+_{2,3}(h)$.

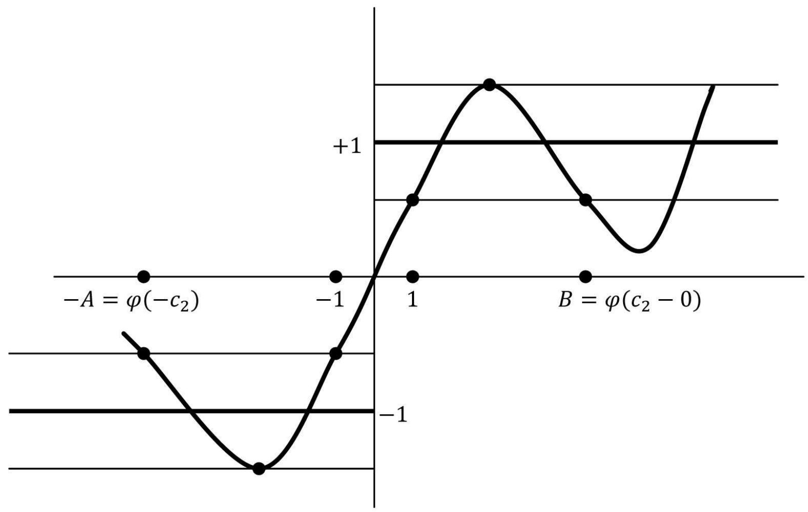

Figure 7. ...and the corresponding extremal polynomial.

3 Parametrization

We begin with a general description of extremal polynomials. Fix $a > 0$. This defines the number $L = L(a)$ and the function $S(\cdot, a)$. Let k_1 and k_2 be two positive integers. Consider the region Π_{k_1,k_2} in the upper half-plane bounded by the curves δ_{-k_1} and δ_{k_2}. Then for $h \geq 0$ and $k_2 \geq 2$, we define the region $\Pi^+_{k_1,k_2}(h)$ by making a slit along δ_{k_2-1} in Π_{k_1,k_2} starting from c_{k_2-1} and such that the length of its image under ψ is h. Thus $\Pi^{\pm}_{k_1,k_2}(0) = \Pi_{k_1,k_2}$. Similarly, we define $\Pi^-_{k_1,k_2}(h)$ for $k_1 \geq 2$ by making a slit along δ_{-k_1+1}.

Let Θ be the conformal map of the upper half-plane onto $\Pi^+_{k_1,k_2}(h)$, normalized by $\Theta(\pm 1) = 0,\ \Theta(\infty) = \infty$. Consider the function

$$P(z) = S(\Theta(z), a). \tag{3.1}$$

By construction, P is real on the real line, so the symmetry principle implies that P extends to an entire function. By looking at the asymptotic behavior as $z \to \infty$, we conclude that P is a polynomial of degree $k_1 + k_2 - 1$. All critical points of this polynomial are real. If $h = 0$, then all critical values are $-1 \pm L$ and $1 \pm L$ on the negative and positive rays, respectively. If $h > 0$, either the extreme left critical value is changed to $-1 \pm L\cosh h$, or the extreme right critical value is changed to $1 \pm L\cosh h$.

We have seen in the previous section that each of these polynomials P is the extremal polynomial for some A and B. Now we prove that for every A and B, one of these polynomials is extremal.

Proposition 3.1. *All extremal polynomials are of the form* (3.1) *with* Θ *defined as above for some* k_1, k_2, a *and* h.

We give an elementary proof of this proposition, which is based on counting critical points and alternance points. This proof does not extend to the case of entire functions. We give another proof, which is less elementary but avoids counting, in Section 9.

Proof. We use the following fact which is well known and easy to prove.

Fact. *Let* P_1 *and* P_2 *be two real polynomials having only real and simple critical points, and suppose that their critical points are listed in increasing order, respectively, as* $c_1 < c_2 < \ldots < c_{n-1}$ *and* $c_1' < c_2' < \ldots < c_{n-1}'$. *If* $P_1(c_j) = P_2(c_j')$ *for* $1 \leq j \leq n-1$. *Then* $P_1(z) = P_2(cz+b)$ *for some* $c > 0$ *and real* b.

For a discussion and generalizations of this fact to entire functions, see [10], [12].

Suppose $A > 1$, $B > 1$, and a positive integer n are given. (We deal with the degenerate case $A = 1$ or $B = 1$ later.) Let P be the extremal polynomial of degree n. By Chebyshev's Theorem, P exists and is unique. According to Chebyshev's "Alternance Theorem", P is characterized by the following properties. Let $Q(x) = P(x) - \operatorname{sgn}(x)$. Then

$$|Q(x)| \leq L, \quad x \in [-A, -1] \cup [1, B], \tag{3.2}$$

and there exist

$$m \geq n+2 \tag{3.3}$$

points $x_1 < x_2 < \ldots < x_m$ in $[-A, -1] \cup [1, B]$ such that

$$|Q(x_j)| = L, \quad 1 \leq j \leq m, \quad \text{and} \quad Q(x_j)Q(x_{j+1}) < 0, \quad 1 \leq j \leq m-1. \tag{3.4}$$

These points x_j are called the **alternance points**. Evidently, all alternance points in $(-A, -1) \cup (1, B)$ are critical, that is, $P'(x) = 0$ at all such points. Let K be the number of critical alternance points and N be the number of non-critical alternance points. We have the evident inequalities

$$K \leq n-1 \quad \text{and} \quad N \leq 4.$$

Combined with (3.3), this gives

$$n+2 \leq m = K+N \leq n+3.$$

So we have three possibilities.

a): $m = n+3$, $N = 4$, $K = n-1$. The last two equalities imply that all critical points of P are real and simple, and each of them is an alternance point which belongs to $(-A,-1) \cup (1,B)$. All 4 points $-A$, -1, 1, and B are non-critical alternance points. So the graph has the shape shown in Figure 5.

b): $m = n+2$, $N = 3$, $K = n-1$. Again, all critical points are real, simple, and belong to $[-A,-1] \cup [1,B]$. Moreover, each critical point is an alternance point. All endpoints $-A$, -1, 1, and B, except possibly one, are alternance points. Let us show that -1 and 1 *are* alternance points.

The proof is by contradiction. Suppose, for example, that -1 is not an alternance point. Then 1 cannot be a critical point because $N = 3$. Thus $P'(x) \neq 0$ on $[-1, 1]$, and $P(1) \geq 1-L > -1+L \geq P(-1)$. We conclude that P is strictly increasing on an interval $(-1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. This implies that 1 is also not an alternance point, a contradiction.

Thus P is of the type described in (2.9), (2.10).

c): $m = n+2$, $K = n-2$, $N = 4$. In this case, we have exactly one simple critical point z that is not an alternance point. Evidently, this exceptional critical point is real. We claim that it belongs to $\mathbb{R}\backslash[-A,B]$.

First of all, $z \notin \{-A,-1,1,B\}$ because $N = 4$ implies that all 4 of these points are non-critical.

Secondly, z cannot be in the interior of one of the intervals $(-A,-1)$ or $(1,B)$. Indeed, if it is in the interior of one of these intervals, consider the adjacent alternance points x_j and x_{j+1} on the same interval that satisfy $x_j < z < x_{j+1}$. Such x_j and x_{j+1} exist because all endpoints of each interval are alternance points, and z is not an alternance point. Since z is the unique critical point on (x_j, x_{j+1}), we obtain a contradiction with the alternance condition (3.4).

Finally, we prove that $z \notin (-1, 1)$, Again the proof is by contradiction. Suppose that $z \in (-1, 1)$. Since -1 is an alternance point, $P(-1) = -1 \pm L$. Suppose first that

$$P(-1) = -1-L. \tag{3.5}$$

Then $P'(-1) < 0$ because -1 is not critical ($N = 4$ in the case we consider now), and (3.2) implies that $P'(-1) \leq 0$. Since P' changes sign exactly once on $(-1, 1)$ and the point 1 is also non-critical, we conclude that $P'(1) > 0$. Since $P(1) = 1 \pm L$, (3.2) implies $P(1) = 1 - L$. This equality and (3.5) contradict the alternance condition (3.4). The case $P(-1) = -1 + L$ is considered similarly.

Let c be the critical point which is outside $[A, B]$. It is easy to see that $|P(c) + 1| > L$ if $c < 0$ and $|P(c) - 1| > L$ if $c > 0$. This is because $N = 4$, and thus A and B are non-critical alternance points.

So we have the graph of the type shown in Figure 7.

To summarize, we proved that in all cases, the critical points are real and simple; all critical values, with at most one exception at $-1\pm L$ on the negative ray and $1\pm L$ on the positive ray; and the exceptional critical value, if it exists, corresponds to an extreme (left or right) critical point. If the exceptional critical point c is positive, then $|P(c) - 1| > L$, and if c is negative, then $|P(c) + 1| > L$.

The polynomials $S(\Theta, a)$ constructed above allow us to match any such critical value pattern; so we conclude that $P(z) = S(\Theta(cz + b), a)$ with $c > 0$ and $b \in \mathbb{R}$. Finally, the points -1 and 1 are always non-critical alternance points, and this implies that $c = 1$ and $b = 0$.

It remains to consider the degenerate case. Suppose, for example, that $B = 1$. Then only 3 alternance points can be non-critical, so we are in the case b). The extremal polynomial in this degenerate case can be easily written explicitly as

$$P_n(x) = L_n T_n\left(\frac{2x + A + 1}{A - 1}\right) - 1,$$

where $T_n(x) = \cos n \arccos x$, and

$$L_n = \frac{2}{T_n(1 + 4/(A - 1)) + 1} \sim 4 \exp\left(-n \operatorname{ch}^{-1}\left(1 + \frac{4}{A - 1}\right)\right)$$

is the approximation error. □

Remarks. It is easy to see that our polynomials depend continuously on h. As $h \to \infty$, we have $\Pi^+_{k_1,k_2}(h) \to \Pi_{k_1,k_2-1}$ and $\Pi^-_{k_1,k_2}(h) \to \Pi_{k_1-1,k_2}$ in the sense of Carathéodory, and the corresponding polynomials converge uniformly on compact subsets of the plane.

Let us show that $A(h)$ and $B(h)$ depend monotonically on h. Let $h > h_1$, $\Theta(z) = \Theta(z, h)$, and $\Theta_1(z) = \Theta(z, h_1)$. Then the function $w(z) = \Theta_1^{-1} \circ \Theta(z)$ maps the upper half-plane into itself, and has the properties $w(\pm 1) = \pm 1$, $w(\infty) = \infty$. Thus it has a representation

$$w(z) = \rho_\infty(z - 1) + 1 + \int \frac{z - 1}{(x - 1)(x - z)} d\rho(x),$$

where $d\rho$ is positive and supported on a compact set I such that $x > B$ for all $x \in I$ and $\rho_\infty > 0$. Here, we have used $w(1) = 1$ and $w(\infty) = \infty$. We now use the condition $w(-1) = -1$ and obtain

$$\rho_\infty = 1 - \int \frac{1}{(x-1)(x+1)} d\rho(x). \tag{3.6}$$

Since $w(B) = B_1$, we have

$$B_1 - B = (\rho_\infty - 1)(B-1) + (B-1) \int \frac{1}{(x-1)(x-B)} d\rho(x).$$

Using (3.6), we get

$$B_1 - B = (B-1) \int \frac{B+1}{(x-1)(x-B)(x+1)} d\rho(x) > 0.$$

Similarly,

$$-A_1 + A = (A+1) \int \frac{A-1}{(x-1)(x+A)(x+1)} d\rho(x) > 0.$$

4 Integral representations

Asymptotic relations for the extremal polynomials are based on an integral representation of the conformal map Θ.

Consider the conformal map of $\overline{\mathbb{C}} \backslash I$ onto the annulus in Figure 8. Here we assume that the upper half-plane is mapped onto the upper part of the annulus with the boundary correspondence

$$B \mapsto -1, \ -A \mapsto -\lambda, \ -1 \mapsto \lambda, \ 1 \mapsto 1. \tag{4.1}$$

We denote by $G(z, z_0)$ the (real) Green function of the region $\bar{\mathbb{C}} \setminus I$ with pole at z_0, where $I = [-A, -1] \cup [1, B]$. Define $G(z) := G(z, \infty)$. Recall that in the upper half-plane, $G(z)$ can be represented as the imaginary part of the conformal mapping $\Phi(z)$ of the upper half-plane onto the region Π of Figure 9;

$$G(z) = \operatorname{Im} \Phi(z), \ \operatorname{Im} z > 0, \ \Phi(\pm 1) = 0, \ \Phi(\infty) = \infty. \tag{4.2}$$

The map $\Phi(z)$ defines certain important characteristics of the region: the critical value

$$\eta = G(C), \quad C \in [-1, 1] \quad \text{such that} \quad \nabla G(C) = 0, \tag{4.3}$$

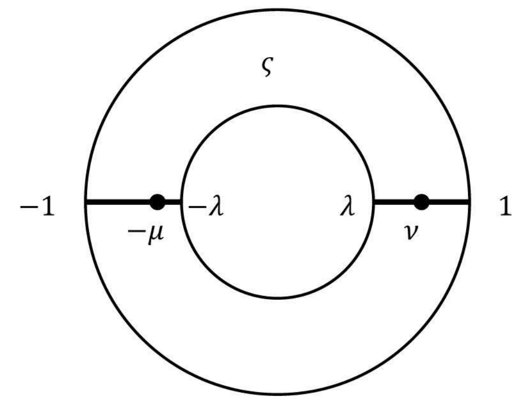

Figure 8. The conformally equivalent annulus.

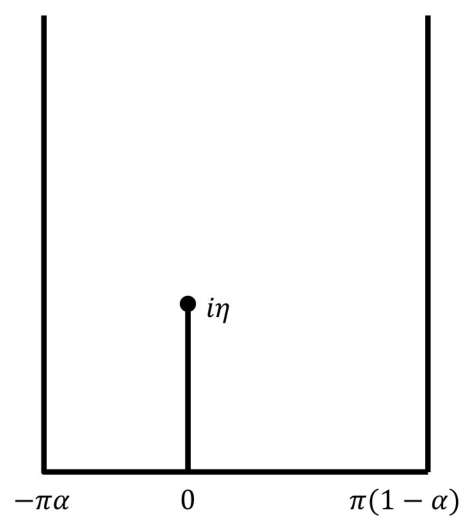

Figure 9. The image Π of the map Φ.

and the harmonic measure $\omega(z)$ of the interval $[-A,-1]$. We have

$$\Phi(-A)=-\pi\alpha, \quad \text{where} \quad \alpha=\omega(\infty). \tag{4.4}$$

Now we associate to $\Phi(z)$ the function

$$g(\zeta)=-i\Phi(z(\zeta)), \tag{4.5}$$

where ζ belongs to the upper half of the annulus; see Figure 8. This function can be extended to the upper half-plane by the symmetry principle. We have $G(z(\zeta))=\operatorname{Re} g(\zeta)$. We call g the **complex Green's function**.

Let $-\mu\in[-1,-\lambda]$ correspond to infinity in the z-plane, $-\mu=\zeta(\infty)$ (see Figure 8). We define the jump function

$$j(\xi)=\begin{cases}1, & \xi\in(-1,-\mu),\\ 0, & \xi\in(-\mu,-\lambda),\end{cases} \tag{4.6}$$

and extend it symmetry to the whole negative ray by $j(1/\xi)=j(\xi)$, $j(\lambda^2\xi)=j(\xi)$. Since

$$\operatorname{Im} g(\xi)=\begin{cases}0, & \xi>0,\\ \pi\alpha, & \xi\in(-\mu,-\lambda),\\ \pi(\alpha-1), & \xi\in(-1,-\mu),\end{cases}$$

we obtain the following integral representation for $g(\zeta)$ in the upper half-plane:

$$g(\zeta)=\int_{-\infty}^{0}\left\{\frac{1}{\xi-\zeta}-\frac{1}{\xi-1}\right\}(\alpha-j(\xi))d\xi. \tag{4.7}$$

Remark. In the representation (4.7), the normalization condition $\Phi(1)=0$ was used. The second normalization condition, $\Phi(-1)=0$, gives

$$\alpha=\overline{j}:=\frac{\int_{-\infty}^{0}\left\{\frac{1}{\xi-\lambda}-\frac{1}{\xi-1}\right\}j(\xi)d\xi}{\int_{-\infty}^{0}\left\{\frac{1}{\xi-\lambda}-\frac{1}{\xi-1}\right\}d\xi}.$$

In what follows, we use the bar over a function to denote similar averages.

Naturally, we can simplify (4.7); but the point is that we can write a similar representation for the conformal mapping $\Theta_n(z)$. Recall that for a given n, there exists a unique region $\Pi(n)=\Pi^{\pm}_{k_1(n),k_2(n)}(h_n)$ (see Figure 6) such that the conformal mapping $\Theta_n:\mathbb{C}_+\to\Pi(n)$ represents the extremal polynomial (2.7). We define the function

$$\theta_n(\zeta)=-i\Theta_n(z(\zeta)), \quad \theta_n(\lambda^2\zeta)=\theta_n(\zeta). \tag{4.8}$$

We write the imaginary part of $\theta_n(\xi)$, $\xi < 0$ as a sum

$$\operatorname{Im}\theta_n(\xi) = \pi k_1(n) - \frac{\pi}{2} + \pi n j(\xi) + \chi_n(\xi), \tag{4.9}$$

so that $\chi_n(\xi)$ is a continuous function, which is normalized by the condition $\chi_n(-\mu) = 0$. Then

$$\theta_n(\zeta) = \frac{1}{\pi}\int_{-\infty}^{0}\left\{\frac{1}{\xi-\zeta} - \frac{1}{\xi-1}\right\}(\pi k_1(n) - \frac{\pi}{2} - \pi n j(\xi) + \chi_n(\xi))d\xi. \tag{4.10}$$

Theorem 4.1. *In the above notation,*

$$\theta_n(\zeta) - ng(\zeta) = \frac{1}{\pi}\int_{-\infty}^{0}\left\{\frac{1}{\xi-\zeta} - \frac{1}{\xi-1}\right\}(\chi_n(\xi) - \bar{\chi}_n)d\xi, \tag{4.11}$$

where

$$\bar{\chi}_n = \frac{\int_{-\infty}^{0}\left\{\frac{1}{\xi-\lambda} - \frac{1}{\xi-1}\right\}\chi_n(\xi)d\xi}{\int_{-\infty}^{0}\left\{\frac{1}{\xi-\lambda} - \frac{1}{\xi-1}\right\}d\xi} = \pi n\alpha - \pi k_1(n) + \frac{\pi}{2}. \tag{4.12}$$

Proof. We subtract $ng(\zeta)$ in the form (4.7) from (4.11). Then we use the second normalization condition, $g(\lambda) = \theta_n(\lambda) = 0$. □

This representation implies Fuchs' asymptotics once it is shown that $\chi_n(\xi)$ is uniformly bounded.

5 Fuchs' asymptotics

Let us begin with a simple remark.

Lemma 5.1. *Let $w(z)$ be a conformal mapping of the upper half-plane onto a sub-region of the upper half-plane which contains the half-plane* $\operatorname{Im} w > \tau_0$. *Assume the normalization $w(z) \sim z$, $z \to \infty$. Then*

$$\operatorname{Im} w(z) - \operatorname{Im} z \in (0, \tau_0). \tag{5.1}$$

Proof. Consider the integral representation of $\operatorname{Im} w(z)$,

$$\operatorname{Im} w(z) = \operatorname{Im} z + \frac{1}{\pi}\int_{-\infty}^{\infty} P(z,t)v(t)dt, \tag{5.2}$$

where $P(z,t)$ is the Poisson kernel. Since $0 \le v(t) \le \tau_0$, we obtain the desired inequality. □

Proposition 5.2. *There exist constants C_1 and C_2 (depending on the given set of intervals I) such that*

$$C_1 \le a_n - n\eta \le C_2. \tag{5.3}$$

Proof. Recall that the curves in Figure 3 were defined as preimages of vertical lines in the region Ω of Figure 2 under a conformal mapping which maps the right half-plane into a sub-region of the right half-plane. Thus, we can apply Lemma 5.1 to conclude that $|\chi_n(\xi)| \le 2\pi$. Therefore, $|\chi_n(\xi) - \bar{\chi}_n| < 2\pi$ also. Now a_n is the maximum of $\theta_n(\xi)$ on the interval $(\lambda, 1)$, and η is the maximum of $g(\xi)$ on the same interval. By the integral representation (4.11), the difference between these two functions is uniformly bounded in this interval. □

Corollary 5.3.

$$\lim_{n\to\infty} \frac{\theta_n(\zeta)}{n} = g(\zeta). \tag{5.4}$$

In particular,

$$\lim_{n\to\infty} \frac{k_1(n)}{n} = \alpha, \quad \lim_{n\to\infty} \frac{\ln(1/L_n)}{n} = \lim_{n\to\infty} \frac{a_n}{n} = \eta. \tag{5.5}$$

Proof. Divide (4.11) by n and pass to the limit as $n \to \infty$. □

Corollary 5.3 has the following geometric interpretation. Rescaling by $1/n$ so that $\Pi(n) \to \Pi(n)/n$, we obtain the limit conformal mapping onto the region shown in Figure 9. Let us look more carefully at the limit procedure; see Figure 10. The distance between the additional cut and one of the infinite cuts (left or right one) approaches zero; however, the position d_n of the end point of the additional cut influences the asymptotic behavior along various subsequences $\{n_l\}$. We define the subsequences by the following condition: there exists a $d = d(\{n_l\}) = \lim_{l\to\infty} d_{n_l}$. Taking the point d into account in the next section, we describe the asymptotic behavior of L_n more precisely.

6 The limit density $\chi(\xi)$

We have fixed a subsequence $\{n_l\}$ such that the limit $d = \lim_{l\to\infty} d_{n_l}$ exists. Our main goal in this section is to show that the limit density

$$\chi(\xi) = \lim_{l\to\infty} \chi_{n_l}(\xi) \tag{6.1}$$

exists and to find this limit.

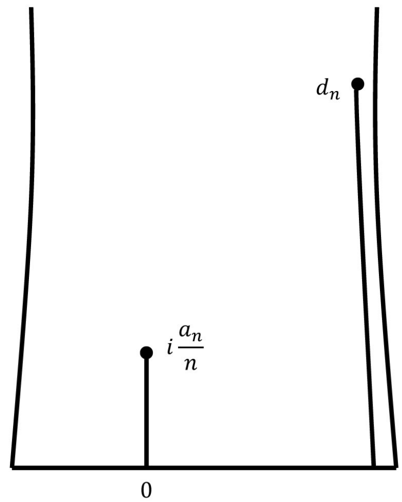

Figure 10. The rescaled region $\Pi(n)/n$ for a large n.

We start with the following general lemma. Let $f : (0, \infty) \to \mathbb{R}$ be a bounded increasing differentiable function, and suppose that $f(x) = 0$ in $(0, b]$ for some $b > 0$. We consider the region

$$\tilde{\Omega}_f = \{z = x + iy : x > 0,\ y > f(x)\} \tag{6.2}$$

(it looks like the region Ω in Figure 2 reflected through the line $x = y$). Let w be the conformal map from the first quadrant $\mathbb{C}_{++}$ onto $\tilde{\Omega}_f$ with the normalization

$$w(z) \sim z,\ z \to \infty, \quad w(0) = 0. \tag{6.3}$$

Let a be the point such that $w(a) = b$.

Lemma 6.1. *Let $w(x) = u(x) + iv(x)$, $x \geq a$. Then $f(x) \leq v(x)$.*

Proof. We extend w by the symmetry principle to the map of the upper half-plane into itself (the extended map is still denoted by w), and use the integral representation

$$w(z) = z + \frac{1}{\pi} \int_a^{\infty} \left\{ \frac{1}{t - z} - \frac{1}{t + z} \right\} v(t) dt. \tag{6.4}$$

For $x \geq a$, we have

$$w(x) = x + \frac{1}{\pi}\int_a^\infty \frac{2x}{t+x}\frac{\upsilon(t)-\upsilon(x)}{t-x}dt + \frac{\upsilon(x)}{\pi}\ln\frac{x+a}{x-a} + i\upsilon(x). \tag{6.5}$$

Therefore

$$u(x) = x + \frac{1}{\pi}\int_a^\infty \frac{2x}{t+x}\frac{\upsilon(t)-\upsilon(x)}{t-x}dt + \frac{\upsilon(x)}{\pi}\ln\frac{x+a}{x-a} > x. \tag{6.6}$$

Since $f(x)$ is increasing, we obtain

$$f(x) < f(u(x)) = \upsilon(x). \tag{6.7}$$

We apply Lemma 6.1 to obtain the limit density for the conformal map of the first quadrant onto the region Ω in Figure 2. Namely, as before, we consider the conformal map $w(z) = -i\psi(-iz)$, where ψ is defined in (2.5) and extended by symmetry to the right half-plane. We also consider the integral representation (6.4) for w. Observe that in our case, we have the exact formula

$$f(x) = \arccos\frac{\cosh b}{\cosh x}, \quad x \geq b. \tag{6.8}$$

Between the values a and b, f gives a one-to-one correspondence. Moreover, $b \sim a + 1/2\ln a$. Thus we have the density $\upsilon(x) = \upsilon(x, a)$ in (6.4) as a function of the parameter a. We are interested in the limit density $\tilde{\upsilon}(x) := \lim_{a\to\infty}\upsilon(ax, a)$.

Lemma 6.2. *The following limit exists:*

$$\tilde{\upsilon}(x) := \lim_{a\to\infty}\upsilon(ax, a) = \begin{cases} 0, & x < 1 \\ \frac{\pi}{2}, & x > 1. \end{cases} \tag{6.9}$$

Proof. It is evident that $\tilde{\upsilon}(x) = 0$ for $x \in (0, 1)$. For $x > 1$, we use Lemma 6.1 and the asymptotic relation between a and b:

$$\upsilon(ax, a) \geq \arccos\frac{\cosh b}{\cosh ax} \sim \arccos\frac{\sqrt{a}}{e^{(x-1)a}}. \tag{6.10}$$

On the other hand, $\upsilon(ax, a) \leq \pi/2$. □

Now we are in position to evaluate the limit density (6.1).

Theorem 6.1. *Let* $\{n_k\}$ *be a subsequence such that* $\lim d_{n_k} = d$. *Without loss of generality, we assume that* $\operatorname{Re} d = \pi(1-\alpha)$ *(alternatively,* $\operatorname{Re} d = -\pi\alpha$*). The relation*

$$d = \Phi(D) = ig(-\kappa) \tag{6.11}$$

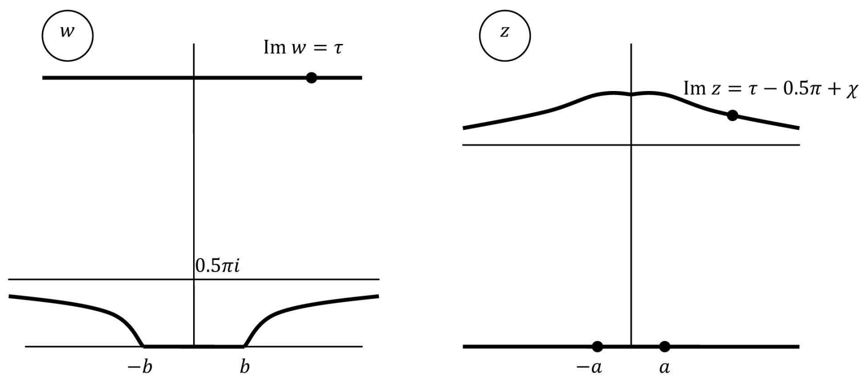

Figure 11. The preimage of the level line Im $w = \tau$.

uniquely defines $D \in [B, \infty]$ *and* $-\kappa \in [-1, -\mu]$. *Then*

(6.12)

$$\chi(\xi) = \lim_{l\to\infty} \chi_{n_l}(\xi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{|t|<\eta} \frac{\pi\alpha}{(t-y)^2+(\pi\alpha)^2}dt, & -\mu < \xi < -\lambda, \\ \dfrac{1}{2}\displaystyle\int_{|t|<\eta} \frac{\pi(\alpha-1)}{(t-y)^2+(\pi(\alpha-1))^2}dt, & -\kappa < \xi < -\mu, \\ \dfrac{1}{2}\displaystyle\int_{|t|<\eta} \frac{\pi(\alpha-1)}{(t-y)^2+(\pi(\alpha-1))^2}dt + \pi, & -1 < \xi < -\kappa, \end{cases}$$

where $y = \mathrm{Re}\, g(\xi)$.

Proof. First we assume that $-\mu < \xi < -\lambda$. Let $z_l = \theta_{n_l}(\xi)$. For sufficiently large l, we have, by (6.4),

$$\mathrm{Im}\, w_l = \mathrm{Im}\, z_l + \frac{1}{\pi}\int_{|t|>a_{n_l}} \frac{\mathrm{Im}\, z_l}{(t-\mathrm{Re}\, z_l)^2+\mathrm{Im}\, z_l^2} v(t, a_{n_l})dt. \tag{6.13}$$

Substituting $\mathrm{Im}\, z_l = \pi k_1(n_l) - \frac{\pi}{2} + \chi_{n_l}(\xi)$ and $\mathrm{Im}\, w_l = \pi k_1(n_l)$ into (6.13) (see Figure 11),we obtain

$$\begin{aligned} \pi/2 - \chi_{n_l}(\xi) &= \frac{1}{\pi}\int_{|t|>a_{n_l}} \frac{(\pi k_1(n_l) - \pi/2 + \chi_{n_l}(\xi))v(t, a_{n_l})}{(t-y_l(\xi))^2 + (\pi k_1(n_l) - \pi/2 + \chi_{n_l}(\xi))^2}dt \\ &= \frac{1}{\pi}\int_{|t|>1} \frac{(\pi k_1(n_l) - \pi/2 + \chi_{n_l}(\xi))a_{n_l}v(a_{n_l}t, a_{n_l})}{(a_{n_l}t - y_l(\xi))^2 + (\pi k_1(n_l) - \pi/2 + \chi_{n_l}(\xi))^2}dt. \end{aligned} \tag{6.14}$$

By the leading term asymptotics, Corollary 5.3, we have

$$\lim_{l\to\infty}\frac{k_1(n_l)}{n_l}=\alpha,\ \lim_{l\to\infty}\frac{a_{n_l}}{n_l}=\eta,\ \lim_{l\to\infty}\frac{y_l(\xi)}{n_l}=y(\xi). \tag{6.15}$$

Passing to the limit as $l\to\infty$ in (6.14), we get

$$\pi/2-\chi(\xi)=\frac{1}{\pi}\int_{|t|>1}\frac{\eta\pi\alpha}{(\eta t-y)^2+(\pi\alpha)^2}\tilde{v}(t)dt. \tag{6.16}$$

By Lemma 6.2, after trivial simplifications, we obtain the first equation in (6.12).

In the second case, $-\kappa<\xi<-\mu$, for sufficiently large l, the point z_l corresponds to a point w_l on the line $\operatorname{Im} w_l=\pi k_2(n_l)$. Thus we can repeat the previous arguments with α replaced with $1-\alpha$ (let us mention that $\chi_{n_l}(\xi)$ is negative here).

In the last case, $\operatorname{Im} w_l=\pi(k_2(n_l)-1)$. This leads to the shift of the limit value by π. □

7 Simplifying the result

In this section, we prove the following theorem, which is our main result, and which implies Theorem 1.1.

Theorem 7.1. *Let ν be the point in the interval $(\lambda,1)$ such that $g(\nu)=\eta$. Fix a subsequence $\{n_l\}$ such that $\lim_{l\to\infty}d_{n_l}=d=ig(-\kappa)$. Let $g(\zeta,\nu)$ and $g(\zeta,-\kappa)$ be the corresponding complex Green's functions. Then*

$$\lim_{l\to\infty}\{\theta_{n_l}(\zeta)-n_lg(\zeta)\}=\frac{1}{2}\ln\frac{\eta-g(\zeta)}{\eta+g(\zeta)}+g(\zeta,\nu)-g(\zeta,-\kappa). \tag{7.1}$$

Proof. First of all, we split $\chi(\xi)$ into the sum of a continuous function $\chi_c(\xi)$ and the jump function

$$j_1(\xi)=\begin{cases}1, & \xi\in(-1,-\kappa),\\ 0, & \xi\in(-\kappa,-\lambda).\end{cases} \tag{7.2}$$

As usual, the jump function is extended to the negative ray by the reflections $j_1(1/\xi)=j_1(\xi)$, $j_1(\lambda^2\xi)=j_1(\xi)$.

Note that the jump function is related to the Green's function $G(z,D)$; compare (4.6) and (4.7). Since

$$\operatorname{Im} g(\xi,-\kappa)=\begin{cases}0, & \xi>0,\\ \pi\omega(D), & \xi\in(-\kappa,-\lambda),\\ \pi(\omega(D)-1), & \xi\in(-1,-\kappa),\end{cases}$$

we obtain for $g(\zeta, -\kappa)$ in the upper half-plane, the integral representation

$$g(\zeta, -\kappa) = \int_{-\infty}^{0} \left\{ \frac{1}{\xi - \zeta} - \frac{1}{\xi - 1} \right\} (\bar{j}_1 - j_1(\xi)) d\xi, \tag{7.3}$$

where we have used the notation $\bar{j}_1$ introduced in the Remark in Section 4. Notice that $G(z(\zeta), D) = \operatorname{Re} g(\zeta, -\kappa)$, and

$$\bar{j}_1 = \omega(D). \tag{7.4}$$

Because of the chosen normalization $\chi(-\mu) = 0$, we have

$$\chi(\xi) = \begin{cases} \chi_c(\xi) + \pi j_1(\xi), & -\kappa < -\mu, \\ \chi_c(\xi) + \pi j_1(\xi) - \pi, & -\mu < -\kappa. \end{cases} \tag{7.5}$$

The main point is to evaluate the Cauchy transform of the continuous part, $\chi_c(\xi)$.

Lemma 7.1. *Let $\nu \in (\lambda, 1)$ be such that $g(\nu) = \eta$, that is, ν corresponds to the critical point $C \in (-1, 1)$. Let $g(\zeta, \nu)$ be the corresponding complex Green's function. Then*

$$\frac{1}{\pi} \int_{-\infty}^{0} \left\{ \frac{1}{\xi - \zeta} - \frac{1}{\xi - 1} \right\} (\chi_c(\xi) - \bar{\chi}_c) d\xi = \frac{1}{2} \ln \frac{\eta - g(\zeta)}{\eta + g(\zeta)} + g(\zeta, \nu). \tag{7.6}$$

Proof. Using (6.12), we have for $z = y(\xi) + i\alpha = g(\xi)$, $\xi \in (-\mu, -\lambda)$,

$$\chi_c(\xi) = \frac{1}{2} \operatorname{Im} \int_{-\eta}^{\eta} \frac{1}{t - z} dt = \frac{1}{2} \operatorname{Im} \ \ln \frac{\eta - z}{-\eta - z}. \tag{7.7}$$

Consider the function

$$f(\zeta) := \frac{1}{2} \ln \frac{\eta - g(\zeta)}{-\eta - g(\zeta)}. \tag{7.8}$$

The image of the function $g(\zeta)$ is shown in Figure 12; the image of the fractional linear transformation is shown in Figure 13. Let us point out that for ζ in the upper half of the ring in Figure 8, we obtain the values of $g(\zeta)$ in the right half-plane and for $(\eta - g(\zeta))/(-\eta - g(\zeta))$ in the unit disk. Thus

$$\rho(\xi) := \operatorname{Im} f(\xi) = \begin{cases} \pi/2, & \lambda < \xi < \nu, \\ -\pi/2, & \nu < \xi < 1. \end{cases} \tag{7.9}$$

Here, ν is such that $g(\nu) = \eta$.

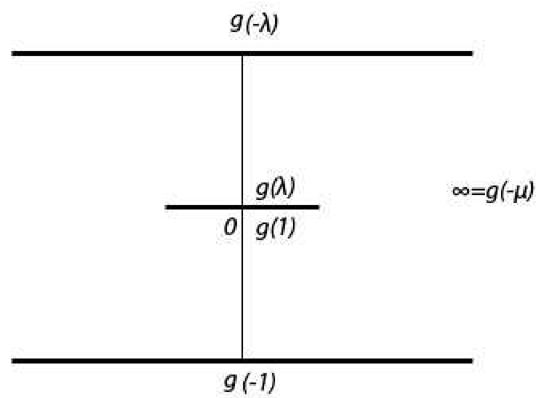

Figure 12. The image of the Green's function.

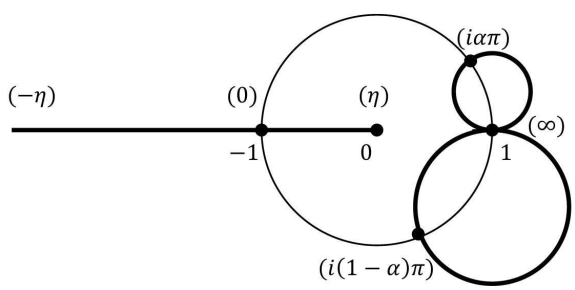

Figure 13. The image of $(\eta - g(\zeta))/(-\eta - g(\zeta))$.

We use the integral representation of $f(\zeta)+i\pi/2$,

$$\frac{1}{2}\ln\frac{\eta-g(\zeta)}{\eta+g(\zeta)}=f(\zeta)+\frac{\pi}{2}i=\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{\xi-\zeta}-\frac{1}{\xi-1}\right\}\left(\rho(\xi)+\frac{\pi}{2}\right)d\xi. \tag{7.10}$$

We still normalize the complex Green's function related to the critical point $C\in(-1,1)$ by the condition $g(1,\nu)=0$. Therefore,

$$\operatorname{Im} g(\xi,\nu)=\begin{cases}-\pi, & \xi\in(\lambda,\nu),\\ -\pi(1-\omega(C)), & \xi<0,\\ 0, & \xi\in(\nu,1).\end{cases}$$

According to (7.9), we can represent $g(\zeta,\nu)$ as

$$\begin{aligned}g(\zeta,\nu)=&-(1-\omega(C))\int_{-\infty}^{0}\left\{\frac{1}{\xi-\zeta}-\frac{1}{\xi-1}\right\}d\xi\\ &-\frac{1}{\pi}\int_{0}^{\infty}\left\{\frac{1}{\xi-\zeta}-\frac{1}{\xi-1}\right\}\left(\rho(\xi)+\frac{\pi}{2}\right)d\xi.\end{aligned} \tag{7.11}$$

Recall that on the negative ray, $\rho(\xi)=\chi_c(\xi)$; see (7.7). Adding (7.11) and (7.10), we obtain (7.6); moreover,

$$\bar{\chi}_c=\pi(\frac{1}{2}-\omega(C)). \tag{7.12}$$

□

Theorem 7.1 follows from Lemma 7.1, (4.11) and (7.3).

Completion of the proof of Theorem1.1 The error term L_n satisfies

$$L_n\sim\sqrt{2/\pi}\,a_n^{-1/2}e^{-a_n}, \tag{7.13}$$

where $a_n=\max\{\theta_n(\xi):\lambda<\xi<1\}$. This follows from (2.4) and our explicit representation of the extremal polynomial, (2.7).

The necessary constants C, η, η_1, η_2, and α, which depend only on A and B, and the harmonic measure $\omega(x)=\omega(x,[-A,-1],\overline{\mathbb{C}}\backslash I)$ were defined in the Introduction.

According to (7.4), (7.12), and (7.5), we have

$$\frac{\bar{\chi}}{\pi}=\begin{cases}1/2-\omega(C)+\omega(D), & -\kappa<-\mu,\\ -1/2-\omega(C)+\omega(D), & -\mu<-\kappa.\end{cases} \tag{7.14}$$

Notice that ω is a strictly increasing function on $\mathbb{R}\backslash(-A,B)$ and the image of this set (together with the infinite point) equals $[0,1]$. Therefore for every integer n, there exists a unique solution D_n of the equation

$$\omega(D_n)=\{\alpha n+\omega(C)\}, \tag{7.15}$$

where $\{\cdot\}$ stands for the fractional part.

Equations (4.12) and (7.14), and the fact that $d_{n_l} \to d$ imply that

$$\omega(D_{n_l}) \to \omega(D). \tag{7.16}$$

Let $-\kappa_n \in (-1, -\lambda)$ be the point in ζ-plane that corresponds to D_n in z-plane; see Figure 8. Then $g(\zeta, -\kappa_{n_l}) \to g(\zeta, -\kappa)$. Now (7.1) gives

$$\lim_{n_l \to \infty} (\theta_{n_l}(\zeta) - n_l g(\zeta) + g(\zeta, -\kappa_{n_l})) = \frac{1}{2} \ln \frac{\eta - g(\zeta)}{\eta + g(\zeta)} + g(\zeta, \nu). \tag{7.17}$$

The right hand side is independent of the subsequence $\{n_l\}$, so the limit as $n \to \infty$ on the left hand side exists as $n \to \infty$. In the resulting formula, we let ζ tend to ν and obtain

$$\lim_{n \to \infty} (\theta_n(\nu) - ng(\nu) + g(\nu, -\kappa_n)) = -\frac{1}{2} \ln \frac{2\eta}{\eta_1} + \eta_2. \tag{7.18}$$

The functions $g(\zeta, -\kappa_n)$ are uniformly bounded and have bounded derivatives on $(\lambda, 1)$. Therefore,

$$a_n = \max\{\theta_n(\xi) : \lambda < \xi < 1\} = \theta_n(\nu) + o(1), \quad n \to \infty.$$

Thus

$$a_n = \eta n - G(D_n, C) - \frac{1}{2} \ln \frac{2\eta}{\eta_1} + \eta_2 + o(1). \tag{7.19}$$

To obtain the final result, this expression for a_n has to be substituted into (7.13).

We can simplify the expression $e^{G(D_n, C)}$ in the resulting formula in the following way and thus avoid having to solve equation (7.15). Let F be the conformal map of the upper half-plane onto a rectangle $(0, p, p+i, i)$, where $p > 0$ and the vertices of the rectangle correspond to $(1, B, -A, -1)$ in that order. It is easy to see that $\omega = \operatorname{Im} F$. Thus

$$F(C) = i\omega(C), \tag{7.20}$$

and, in view of (7.15),

$$F(D_n) = p + i\omega(D_n) = p + i\{\alpha n + \omega(C)\}. \tag{7.21}$$

The Christoffel–Schwarz formula gives (1.2), and $p = \tau/i$, where τ is defined by (1.4). We reflect our rectangle with respect to the imaginary axis and apply the map $z \mapsto (i\pi/p)z$ to obtain the new rectangle

$$R = \{x + iy : -\pi/p < x < 0, \ |y| < \pi\}.$$

The map $z \to e^z$ then maps R onto a ring $e^{-\pi/p} < |w| < 1$. We use the expression for the Green's function of this ring [3, §55 (4)], substituting into this formula[2] $\ln w = i\pi - (\pi/p)\omega(D)$, $\ln c = (\pi/p)\omega(C)$ and using τ instead of $-1/\tau$. The result is simplified using Table VIII in [3]. We obtain

$$e^{G(D_n,C)} = \left| \frac{\vartheta_0\left(\frac{1}{2}(\{n\omega(\infty) + \omega(C)\} - \omega(C))|\,\tau\right)}{\vartheta_0\left(\frac{1}{2}(\{n\omega(\infty) + \omega(C)\} + \omega(C))|\,\tau\right)} \right|,$$

where $\tau = ip$ is given by (1.4). Combining this with (7.13) and (7.19) yields Theorem 1.1. □

8 Example: the symmetric case

We consider the case $I = [-A, -1] \cup [1, A]$. In this case,

$$G(z, \infty) = \int_A^z \frac{xdx}{\sqrt{(x^2 - 1)(x^2 - A^2)}}. \tag{8.1}$$

Therefore,

$$\eta = \int_0^1 \frac{xdx}{\sqrt{(x^2 - 1)(x^2 - A^2)}} = \frac{1}{2} \int_1^{\frac{A^2+1}{A^2-1}} \frac{dt}{\sqrt{t^2 - 1}} = \frac{1}{2} \ln \frac{A + 1}{A - 1} \tag{8.2}$$

and

$$\eta_1 = -\frac{1}{2} G''(0, \infty) = \frac{1}{2A}. \tag{8.3}$$

Also,

$$G(z, 0) = \int_{-1}^z \frac{Adx}{x\sqrt{(x^2 - 1)(x^2 - A^2)}} \sim \ln \frac{1}{z} + \ln \frac{2A}{\sqrt{A^2 - 1}}. \tag{8.4}$$

Notice that $\omega(\infty) = 1/2$, $C = 0$, and $\omega(C) = 1/2$. Therefore, for $n = 2m + 2$, we have $D_n = \infty$, so $L_{2m+2} = L_{2m+1}$.

For $n = 2m + 1$, we get

$$\begin{aligned} &\sqrt{(2m + 1)\eta} \sqrt{\frac{\eta_1}{2\eta}} e^{(2m+1)\eta + \eta_2} L_{2m+1} \\ &= \sqrt{\frac{2m + 1}{4A}} \left(\frac{A + 1}{A - 1}\right)^m \sqrt{\frac{A + 1}{A - 1}} \frac{2A}{\sqrt{A^2 - 1}} L_{2m+1}. \end{aligned} \tag{8.5}$$

[2] In the English edition of 1990, this formula contains two misprints: an extra vertical line and missing subscript 1 in the theta-function in the denominator.

Finally,

$$\lim_{m\to\infty} \sqrt{2m+1}\left(\frac{A+1}{A-1}\right)^m \frac{\sqrt{A}}{A-1} L_{2m+1} = \sqrt{\frac{2}{\pi}}, \tag{8.6}$$

as is proved in [5].

9 Approximation of sgn(x) by entire functions on $[-A, -1] \cup [1, +\infty)$

Only a minor variation of our method is needed to investigate the following problem.

Problem. *Minimize*

$$\sup\{\,|f(x) - \mathrm{sgn}(x)| : x \in [-A, -1] \cup [1, +\infty)\} \tag{9.1}$$

among all entire functions f of order $1/2$, type σ.

Let $E(\sigma)$ be the infimum (9.1). Using normal families arguments, it is easy to prove the existence of an extremal function.

We now describe a construction of extremal functions. We take the error E as an independent parameter. Let $a > 0$ be the unique solution of the equation $L(a) = E$, where $L(a)$ is defined in the beginning of Section 2. For $h \geq 0$ and an integer $k \geq 2$, let $\Pi_k(h) = \Pi^-_{k,\infty}(h)$, that is, the region in the upper half-plane whose boundary with respect to the upper half-plane consists of the segment $[0, ia]$ and the curve δ_{-k}, as in (2.6). Let $\Theta_{k,h} : \mathbb{C}_+ \to \Pi_k(h)$ be the conformal map normalized by $\Theta_{k,h}(\pm 1) = 0$, $\Theta_{k,h}(\infty) = \infty$.

Proposition 9.1. *If $h = 0$, then $S(\Theta_{k,0}, a)$ is the unique extremal function of*

$$A \in [\Theta^{-1}_{k,0}(-c_k), \Theta^{-1}_{k,0}(-c_{k-1})]. \tag{9.2}$$

If $h > 0$, then $S(\Theta_{k,h}, a)$ is the unique extremal function of

$$A = \Theta_{k,h}(-c_{k-1} + 0).$$

The proof of this theorem is similar to the proof of Theorem 3 in [5]. We recall the argument for the reader's convenience.

Proof. Let $f(z) = S(\Theta(z), a)$. Let $x_1 < x_2, \ldots \to +\infty$ be the sequence of all alternance points. Let $\sigma > 0$ be the same type as f with respect to the order $1/2$. Let g be an entire function of the same type σ, order $1/2$. Without loss of generality, we may assume that g is real. Then there exists a sequence $\{y_k\}$ *interlaced* with $\{x_k\}$, that is,

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \dots,$$

such that $f(y_k) = g(y_k)$. Consider the product

$$F(z) = \prod_{k=1}^{\infty} \frac{1 - z/x_k}{1 - z/y_k}.$$

This product converges uniformly on compact subsets of the plane and has imaginary part of a constant sign in the upper half-plane and of the opposite sign in the lower half-plane [9, VII, Thm1]. This implies that

$$F(re^{it}) = O(r), \quad r \to \infty \tag{9.3}$$

uniformly with respect to t in $\epsilon < t < 2\pi - \epsilon$ for every $\epsilon > 0$. Since $f(y_k) = g(y_k)$, we have

$$\frac{f(z) - g(z)}{f'(z)} = \frac{P(z)}{(z - c)F(z)}, \tag{9.4}$$

where c is the critical point of f which is outside the set $[-A, -1] \cup [1, \infty)$. If there is no such point c, then the factor $(z - c)$ in (9.4) has to be omitted. Thus P is an entire function of order $1/2$.

We now observe that the left hand side of (9.4) is bounded for $|\mathrm{Im}\, z| > 1$. Indeed, g and $f - g$ are at most of type σ, order $1/2$, while f' has indicator $\sigma \sin(t/2)$, $0 < t < 2\pi$; so the ratio has zero type in $\mathbb{C} \backslash \mathbb{R}_+$ and thus is bounded, by the Phragmén–Lindelöf Theorem.

Combining this with (9.3), we conclude that P is a polynomial, and

$$P(z)/(z - c) = O(z), \ z \to \infty \tag{9.5}$$

if the point c exists, and

$$P(z) = O(z), \ z \to \infty \tag{9.6}$$

if the point c does not exist.

On the other hand, $P(x) = 0$ for each non-critical alternance point x. From our construction of $f = S(\Theta, a)$, it follows that when c exists, there are three non-critical alternance points, namely, $-A$, -1, and 1; while when c is absent, there are at least two non-critical alternance points, namely, -1 and 1. Together with (9.5) and (9.6), this implies that $P = 0$, that is, $f = g$. □

Proposition 9.2. *For every E and A, there exist k, h and a such that $S(\Theta_{k,h}, a)$ is an extremal function for the set* $[-A, -1] \cup [1, +\infty)$.

Proof. For given E, we can choose a such that $L(a) = E$. To prove the existence of k and h, we use a monotonicity argument as in the Remarks in Section 3. Namely, we introduce the following order relation on the pairs (k, h): $(k, h) \prec (k', h')$ if $k < k'$, or $k = k'$ and $h > h'$. With this order, the set of pairs (k, h) becomes isomorphic to the positive ray, and the correspondence $(k, h) \mapsto A$ becomes monotone increasing. This function is continuous for $h \neq 0$ and has a jump at each point $(k, 0)$ (this jump is seen in the right hand side of (9.2)). Thus we can obtain any $A > 1$ from some pair (k, h). □

Theorem 9.1. *For every A and σ, there exists a unique extremal function f of type σ, and $f = S(\Theta_{k,h}, a)$ for some positive integer k, $h \geq 0$ and $a > 0$.*

Proof. Let $\sigma(A, E)$ be the type (with respect to order $1/2$) of the extremal function defined in Proposition 9.2. Then Proposition 9.1 implies that for every A, the function $E \mapsto \sigma(A, E)$ is strictly decreasing. It is easy to check that $\sigma(A, 1) = 0$ and $\sigma(A, 0+) = +\infty$. Moreover, $E \mapsto \sigma(A, E)$ is continuous. Thus there exists a unique $E = E(A, \sigma)$ that is the error of the best approximation for given A and σ. From this E and A, we define k and h using Proposition 9.2. □

To state the asymptotic result, we introduce the Martin function $M(x)$ of the region $\mathbb{C}\backslash I$, where $I = [-A, -1] \cup [1, +\infty)$, replacing the Green's function which we used before. The Martin function is characterized by being positive and harmonic in $\mathbb{C}\backslash I$, vanishing on I, and having the asymptotic behavior

$$M(-x) \sim \sqrt{x}, \ x \to +\infty.$$

We have $M(z) = \operatorname{Im} \mathcal{M}(z)$, where $\mathcal{M}$ is the conformal map of the upper half-plane onto the region $\{x + iy : x > -\pi\alpha, \ y > 0\}\backslash[0, i\eta]$ such that

$$\mathcal{M}(\pm 1) = 0, \ \mathcal{M}(-A) = -\pi\alpha, \ \mathcal{M}(-x) \sim i\sqrt{x}, \ x \to +\infty.$$

These relations define α and η uniquely.

The Martin function has a single critical point $C \in (-1, 1)$ and we write $\eta = M(C)$ and $\eta_1 = -M''(C)/2$, as before. The Green's function $G(x, C)$ satisfies

$$G(x, C) = -\ln|x - C| + \eta_2 + O(x - C), \quad x \to C,$$

and this defines η_2. We also introduce the harmonic measure

$$\omega(z) = \omega(z, [-A, -1], \mathbb{C}\backslash I).$$

Then $\omega(x)$ is continuous and strictly increasing on $[-\infty, -A)$, and maps this ray onto $[0, 1)$. Thus the equation

$$\omega(D_\sigma) = \{\alpha\sigma + \omega(C)\},$$

where $\{x\}$ is the fractional part of x, has a unique solution for every $\sigma > 0$.

Theorem 9.2. *The error of the best uniform approximation of the function* $\mathrm{sgn}(x)$ *on* $[-A, -1] \cup [1, +\infty)$ *by entire functions of order* $1/2$*, type* σ*, satisfies*

$$E(\sigma) \sim \sqrt{\frac{2}{\pi}}(a(\sigma))^{-1/2} e^{-a(\sigma)},$$

where

$$a(\sigma) = \eta\sigma - G(D_\sigma, C) - \frac{1}{2}\ln\frac{2\eta}{\eta_1} + \eta_2. \tag{9.7}$$

Equation (9.7) is analogous to (7.19). One can simplify $e^{G(D_\sigma, C)}$, as was done in Section 7, by using an expression for the Green's function in terms of theta-functions.

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