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POLYNOMIALS ON AFFINE MANIFOLDS

BY

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ABSTRACT. For a closed affine manifold M of dimension m the developing map defines an open subset $D(\tilde{M}) \subset \mathbf{R}^m$. We show that $D(\tilde{M})$ cannot lie between parallel hyperplanes. When $m \leqslant 3$ we show that any nonconstant polynomial p: $\mathbf{R}^m \to \mathbf{R}$ is unbounded on $D(\tilde{M})$. If $D(\tilde{M})$ lies in a half-space we show M has zero Euler characteristic. Under various special conditions on M we show that M has no nonconstant functions given by polynomials in affine coordinates.

An *affine n-manifold* is a manifold modelled on affine n-space A^n (see **[FGH, NY** or **Th]** for more details). From the viewpoint of algebraic geometry, one should study those real valued functions on M which are polynomials in local coordinates, which we shall call polynomial maps on M. Since bounded polynomials on A^n are constant, the only polynomial maps on a *complete* closed affine manifold are constants.

The main consequence of our investigation (Theorem 4) is that the developing image $D(\tilde{M})$ of a closed affine three-manifold is large, in the sense that it isn't confined between the level surfaces of a polynomial on A^n. Here $\tilde{M}$ denotes the universal cover of M and D: $\tilde{M} \to A^n$ is the developing map (D is characterized up to an affine automorphism of A^n as a local diffeomorphism preserving the affine structure **[FGH**, §2**]**). The method of proof is to pass from nonconstant polynomials bounded on $D(\tilde{M})$ to nonconstant polynomials on M (Theorem 1) and then to show that closed affine manifolds of dimension $\leqslant 3$ admit only constant polynomial maps. At the end of the paper we extend this result somewhat in the setting of affine foliations and syndetic actions.

In higher dimensions the corresponding question is unresolved. We show (Theorem 2) that the developing image of a closed affine manifold cannot lie between parallel hyperplanes, but we don't know whether polynomial maps of high degree can exist on some high dimensional affine manifolds.

In a different vein, we use Theorem 2 to prove that the Euler characteristic of a closed affine manifold vanishes whenever the developing image lies in a half-space (Theorem 3).

We thank Moe Hirsch for some helpful conversations and particularly for drawing attention to Proposition 1 below.

1. General results. We collect here a number of results concerning polynomials on closed affine manifolds that are valid in all dimensions.

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For some geometrical insight into the general arguments that follows, we describe an interesting special case. Suppose the developing image of a closed affine three-manifold M is an unbounded region contained in an infinite cylinder, $D(\tilde{M}) \subset D^2 \times \mathbf{R}$. Then the direction of the axis of this cylinder is well defined, since cylinders with nonparallel axes have bounded intersection. Passing to the quotient affine space by $\Pi: A^3 \to A^2$, where Π collapses lines parallel to the axis to a point, we obtain a bounded open region $\Pi D(\tilde{M}) \subset A^2$. This region is mapped to itself by an affine action Γ' induced from the affine holomomy Γ of M (Γ is the image of the holonomy representation $\pi_1(M) \to \mathrm{Aff}(3)$, as in **[FGH, §2]**), since Γ preserves the family of lines parallel to the axis. We may find an inner product on A^2 preserved by Γ', since Γ' preserves the bounded open region $\Pi D(\tilde{M})$. The barycenter $b \in A^2$ of the convex hull is also preserved by Γ'. The function $\|x - b\|^2$ on A^2 induces a Γ-equivariant function on $D(\tilde{M})$ and hence a map $f\colon M \to \mathbf{R}$. But, using local coordinates, f has no local maximum. As M is closed, this is a contradiction and gives a special case of Theorem 2 below.

Certain arguments of the preceding paragraph are generalized in our first theorem.

THEOREM 1. *Suppose M is a closed affine manifold with developing image $D(\tilde{M}) \subset A^n$. Then there exists a nonconstant polynomial with bounded values on $D(\tilde{M})$ if and only if there is a nonconstant polynomial on M.*

PROOF. If we let Γ denote the affine holonomy of M and X denote $D(\tilde{M})$, we reduce to

LEMMA 1. *Let Γ be a subgroup of* $\mathrm{Aff}(n)$ *and X a Zariski-dense Γ-invariant subset of A^n. If there is a nonconstant polynomial bounded on X then some such polynomial is Γ-invariant as well.*

PROOF OF LEMMA 1. Let V_d be the finite dimensional real vector space consisting of all polynomials on A^n of degree $\leqslant d$ that are bounded on X. By assumption, we may choose d so that $V_d \neq V_0$.

Let B be the affine mapping from A^n to the dual vector space V_d^* given by $B(x)(p) = p(x)$, $x \in A^n$, $p \in V_d$. Then B takes values in the affine subspace $W \subset V_d^*$ consisting of all L with $L(1) = 1$. Note that Γ acts by affine transformations on V_d, V_d^* and W and that B is Γ-equivariant.

Since X is Zariski-dense, the image $B(X)$ doesn't lie in a proper affine subspace of W. The subgroup H of $\mathrm{Aff}(W)$ determined by the action of Γ preserves the bounded set $B(X) \subset W$. If we denote by C the interior of the convex hull of $B(X)$, we see that C is bounded, convex, open and nonempty and C is preserved by $H \subset \mathrm{Aff}(W)$.

It follows that H preserves the barycenter w_0 of C and that H is relatively compact in $\mathrm{Aff}(W)$. Regarding W as a vector space with origin w_0, we can find a Γ-invariant inner product on W. The map $\langle B(x), B(x)\rangle$ is a nonconstant Γ-invariant polynomial which is bounded on X. Q.E.D.

We remark that the nonconstant polynomial just constructed has degree $\leqslant 2d$, where d is the degree of a nonconstant polynomial bounded on $D(\tilde{M})$.

We will use the following lemma and its corollary in both propositions of this section.

LEMMA 2. *Suppose M is decomposable, that is there is a Γ-equivariant affine retraction of A^n onto an affine subspace $F \neq A^n$, $r\colon A^n \to F$. Let $V = \ker(Dr)$ be the vector space complementary to F and write $A^n = F \times V$. If p is a polynomial map on M then, in local coordinates, $p(f, v)$ depends only on f.*

PROOF. As in [**FGH**, Theorem 3.3] there is a flow on M induced by the flow $(f, e^t v)$ on A^n. Since M is closed, p must be bounded on $D(\tilde{M})$, hence $p(f, e^t v)$ is bounded in t for fixed $(f, v) \in D(M)$.

Thus the polynomial $p(f, sv)$ in the variable s is constant. Since $D(\tilde{M})$ is open, we see that p is independent of v. Q.E.D.

In case F has dimension zero, we obtain

COROLLARY 1. *If M is a radiant closed affine manifold, that is Γ fixes a point $a_0 \in A^n$, then M has no nonconstant polynomial maps.*

We can now show

PROPOSITION 1 (HIRSCH). *If p is a nonconstant polynomial on a closed affine manifold M then p must have a nonisolated local maximum and a nonisolated local minimum. In particular, p cannot have degree* 1 *or* 2.

PROOF OF PROPOSITION 1. As M is closed, p attains its maximum and minimum values somewhere on M. Assume some such extremum is isolated. Then, in $D(\tilde{M})$, p has a level set C for which some component is a point. Let $c_1, \ldots, c_k \in C$ be all the point components of C (note that k is finite because algebraic sets have only finitely many components). Then Γ permutes $\{c_1, \ldots, c_k\}$ and preserves its barycenter a_0, contradicting Corollary 1.

For polynomials of degree 1 there are no critical points whereas for polynomials of degree 2 one can't have both local maxima and local minima. Q.E.D.

Note. We have actually shown that p cannot have an isolated local extremum. More generally, if p determines a bounded set in affine space in some manner then affine maps preserving p will fix some point. Then p cannot be the local coordinate expression of a polynomial map on a closed affine manifold.

We now have the background results needed to extend the "special case" discussed at the beginning of this section.

THEOREM 2. *If M is a closed affine manifold then the developing image $D(\tilde{M})$ does not lie between parallel hyperplanes.*

PROOF. Otherwise one could construct a nonconstant polynomial p on M of degree $\leqslant 2d = 2$, using Theorem 1. This p would contradict Proposition 1. Q.E.D.

We show next that polynomial maps must be constant for the class of affine manifolds considered in [**FGH**].

PROPOSITION 2. *If the affine holonomy Γ of the closed affine manifold M has a nilpotent subgroup of finite index then all polynomials on M are constant.*

PROOF. Passing to a finite cover, we may assume that Γ is nilpotent. Then the Fitting Splitting [**FGH**] shows that $A^n = F \oplus V$, F an invariant affine subspace, V an invariant vector complement and where the restriction $\Gamma \mid F$ consists of unipotent transformations.

We know by Lemma 2 that $p(f, v)$ depends only on F. Since bounded polynomials on F are constant, we need only to show that the projection of $D(\tilde{M}) \subset F \oplus V$ to F is onto.

This was shown in [**FGH**, Theorem 6.8] for the case $V = 0$. Carrying through the inductive argument given there as far as possible, we obtain the required surjectivity for arbitrary v. Q.E.D.

We note that by [**T**] we may rephrase Proposition 2 as follows: *a closed affine manifold with a nonconstant polynomial map must have a fundamental group with exponential growth.*

We conclude this section with a demonstration of a special case of well-known conjecture that the Euler characteristic of a closed affine manifold must vanish.

THEOREM 3. *If M is a closed affine manifold and the convex hull of $D(\tilde{M})$ is a proper subset of A^n (that is, if $D(\tilde{M})$ is contained in a half-space) then $\chi(M) = 0$.*

PROOF. By choosing an origin we may identify A^n with $\mathbf{R}^n$. By assumption, the convex cone C of linear functionals on $\mathbf{R}^n$ bounded below on the developing image $D(\tilde{M})$ contains a nonzero element L. By Theorem 2, $L \in C - \{0\}$ implies $-L \notin C$. Thus $C - \{0\}$ is a proper nonempty convex cone.

Clearly $C - \{0\}$ is invariant under the linear holonomy. It thus determines a nontrivial continuous cone field in the cotangent bundle T^*M. Using the partition of unity one may construct a nonvanishing section to T^*M, showing $\chi(M) = 0$. Q.E.D.

2. Low dimensional results. We will show that no closed affine manifold of dimension $\leqslant 3$ has a nonconstant polynomial map. One could show this more directly in dimensions 1 and 2 by using Proposition 2 and Benzecri's result that the torus and Klein bottle are the only closed surfaces that admit affine structures [**B**]. As we need more general tools in dimension 3 anyway, we give a proof independent of [**B**].

THEOREM 4. *If M is a closed affine manifold of dimension $n \leqslant 3$ then the developing image $D(\tilde{M})$ does not lie between the level curves of any polynomial on A^n. In particular, M does not have any nonconstant polynomial maps.*

PROOF. By Theorem 1, we need only show the second statement of the theorem. By multiplying in factors of S^1, we may assume henceforward that $n = 3$.

Suppose p is a nonconstant polynomial map on M. Then the affine holonomy Γ is a subgroup of $G(p) = \{\gamma \in \mathrm{Aff}(3) \mid p \circ \gamma = p\}$. Since $G(p)$ is an algebraic group it has only finitely many components and we may pass to a finite cover of M to assume Γ lies in the identity component $G_0(p)$ of G. We will let $\mathcal{G}_0(p)$ denote the Lie algebra of $G_0(p)$ which we will identify with the associated Lie algebra of affine vector fields on A^3. We summarize the first main step of our argument in

LEMMA 3. *$G_0(p)$ contains no one-parameter groups of translations. Equivalently, p cannot be expressed as a function of $\leqslant 2$ linear variables.*

PROOF OF LEMMA 3. Suppose $\mathcal{G}_0$ contains $d > 0$ independent infinitesimal translations. If $d = 3$, p is constant. If $d = 2$ then p depends only on one variable, say x. Thus $G_0(p) = G_0(x)$ and x induces a nonconstant polynomial on M, in violation of Proposition 1. Hence we must have $d = 1, p = p(x, y)$.

We consider the highest homogeneous part $p_N(x, y)$ of $p(x, y)$, $N = \deg p$, and factor p_N over $\mathbf{R}$ into irreducible linear and quadratic terms $p_N = \Pi l_i \cdot \Pi q_j$. Projecting $G_0(p)$ into Gl(3) gives a connected linear group $H_0(p)$ that preserves p_N. It follows that the action of $H_0(p)$ on the factors l_i and q_j is to multiply each factor by a positive scalar; for $h \in H_0(p)$,

$$l_i \circ h = \lambda_i l_i \quad \text{and} \quad q_j \circ h = \mu_j^2 q_j, \quad \text{where } \lambda_i, \mu_j > 0.$$

Suppose a quadratic factor q_1 occurs in p_N. In appropriate coordinates, we have $q_1 = x^2 + y^2$. It follows easily that $\mu_j = \mu_1$ for all j and $\lambda_i = \mu_1$ for all i. Finally, since $\Pi\lambda_i \cdot \Pi\mu_j = 1$, we find that $\mu_1 = 1$.

This shows that the map $(x, y) \circ D\colon \tilde{M} \to \mathbf{R}^2$ is equivariant from the action of $\pi_1 M$ on $\tilde{M}$ by covering translations to a group of isometries of $\mathbf{R}^2$. The level curves of this map determine a Euclidean foliation of M. By [**Th**, 4.8.1] the map $(x, y) \circ D$ is onto. But as p is bounded on im D, we must have p constant. This contradiction shows that no quadratic factors appear in p_N.

If there are 3 or more nonproportional linear factors in p_N, then we again find that all λ_i equal to 1 and we obtain the same Euclidean foliation and contradiction as in the preceding paragraph.

Suppose p_N contains exactly two nonproportional linear terms. We may normalize so $p_N = x^i y^{N-i}$, $0 < i < N$. As in the preceding paragraphs, λ_x and λ_y are not identically 1. Thus $G_0(p)$ contains a one-parameter group of the form $(e^{t\alpha}x, e^{t\beta}y, f_t(x, y, z))$ with $\alpha i + \beta(N - i) = 0$, $\alpha \neq 0$, $\beta \neq 0$, and where we have translated the origin to eliminate the translational terms in the x and y coordinates. It follows that $p(x, y)$ is constant along the level curves of p_N. Thus $G_0(p_N) = G_0(p)$. But now p_N determines a polynomial map on M that is incompatible with Proposition 1.

Finally we must have all l_i's proportional to one another. We may normalize so $p_N = y^N$. We have a nontrivial affine relation between the partial derivatives p_x and p_y, since $\mathcal{G}_0(p)$ is nontrivial. As p is not a function of a single linear variable and p_x has lower degree than p_y, we find a relation $(ax + by + c)p_x + p_y = 0$, $(a, b) \neq (0, 0)$. Absorbing c into the linear terms by a translation of coordinates, we find that p is constant along the trajectories of $dx/dy = ax + by$.

If $a \neq 0$ then these trajectories are nonalgebraic curves of the form $x = -by/a + ke^{ay} - b/a^2$. Hence $a = 0$, the trajectories are given by $x = by^2/2 + k$ and p is functionally dependent on $x - by^2/2 = q$. It follows that q is preserved by $G_0(p)$ and so defines a polynomial on M. Since $\deg q = 2$ this violates Proposition 1. Q.E.D.

Lemma 3 implies that the natural map from $G_0(p)$ into $\mathrm{Gl}(3, \mathbf{R})$ is locally injective. We will continue to denote the image group by $H_0(p)$. We have $H_0(p) \cong G_0(p) \supset \Gamma$ so that Proposition 2 implies $G_0(p)$ is not nilpotent. Thus $\dim H_0(p) \geqslant 2$

and by Engel's theorem the Lie algebra $\mathcal{H}_0(p)$ of $H_0(p)$ contains an element A which isn't nilpotent.

Since $G(p)$ is an algebraic group, $\mathcal{H}_0(p)$ is invariant under the Jordan decomposition [**Bo**, (4.4)], so the semisimple part S of A belongs to $\mathcal{H}_0(p)$. Thus p_N is invariant under e^{tS}, where $S \in \mathrm{Hom}(\mathbf{R}^3)$ is semisimple, $S \neq 0$.

We first suppose S has a pair of nonreal conjugate eigenvalues. Putting S in the form

$$\begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad \beta \neq 0,$$

we find $p_N = Q(x^2 + y^2, z^2) \cdot z^e$ where $e = 0$ or 1 and Q is a homogeneous polynomial in 2 variables. We factor Q into linear and quadratic irreducible factors, $Q = \Pi l_i \cdot \Pi q_j$. If any quadratic factor of q_j occurs then $q_j(x^2 + y^2, z^2)$ is a positive definite homogeneous function of x, y and z. It follows as in Lemma 3 that $H_0(p)$ preserves $q_j(x^2 + y^2, z^2)$. But this implies that $H_0(p)$ is compact and so M is complete, p constant. Thus only l_i's occur. Again no $l_i(x^2 + y^2, z^2)$ can be definite. Also if two l_i aren't proportional then $H_0(p)$ transforms both $x^2 + y^2$ and z^2 by constants, implying $H_0(p)$ is abelian. Thus we may assume all the l_i's are equal to $a(x^2 + y^2) - bz^2$, $a, b \geqslant 0$. If $b = 0$, then M has a Euclidean foliation and we obtain a contradiction, as in proof of Lemma 3. Thus, by a linear change of variables we may put p_N in the form $(x^2 + yz)^b$ or z^N.

We next consider the alternative case when S has only real eigenvalues. We may assume

$$S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

is in diagonal form. Setting $p_N = \Sigma\, a_{ijk} x^i y^j z^k$ we obtain $0 = Dp_N(v)(Sv) = \Sigma(\lambda i + \mu j + \nu k) a_{ijk} x^i y^j z^k$, for all $v = (x, y, z) \in \mathbf{R}^3$. Thus $(\lambda i + \mu j + \nu k) = 0$ whenever $a_{ijk} \neq 0$, $(\lambda, \mu, \nu) \neq (0,0,0)$. By permuting the coordinates we may put p_N in the form

$$(*) \qquad p_N = x^\alpha y^\beta z^\gamma q\left(x^{b+c}, y^b z^c\right)$$

where q is homogeneous polynomial in 2 variables.

The analysis of S shows that in good coordinates p_N may always be put in the form $(*)$. We may make our choices so that either q is constant or $q(u, v) = a_0 u^m + a_1 u^{m-1} v + \cdots + a_m v^m$, $a_0 a_m \neq 0$, b relatively prime to c.

We proceed to use the fact that the nonabelian group $H_0(p)$ cannot consist exclusively of diagonal matrices. If q is constant we see that α, β and γ cannot all be nonzero, and we may put p_N in the form $x^\alpha y^\beta$. In case q isn't constant, we have

$$\begin{aligned} x(p_N)_x &= \alpha p_N + x^\alpha y^\beta z^\gamma (b + c) x^{b+c} q_u\left(x^{b+c}, y^b z^c\right), \\ y(p_N)_y &= \beta p_N + x^\alpha y^\beta z^\gamma b y^b z^c q_v\left(x^{b+c}, y^b z^c\right), \\ z(p_N)_z &= \gamma p_N + x^\alpha y^\beta z^\gamma c y^b z^c q_v\left(x^{b+c}, y^b z^c\right). \end{aligned}$$

Considering the relative position of the nonvanishing coefficients of these formulas in the triangular region $i + j + k = N$, $i, j, k \geqslant 0$ (see Figure 1), we find that either the only degree 1 equations amongst $(p_N)_x$, $(p_N)_y$ and $(p_N)_z$ are diagonal or $b = c$ or one of b and c is zero. Switching y and z coordinates if necessary, we obtain $b = 1$ and $c = 0$ or 1.

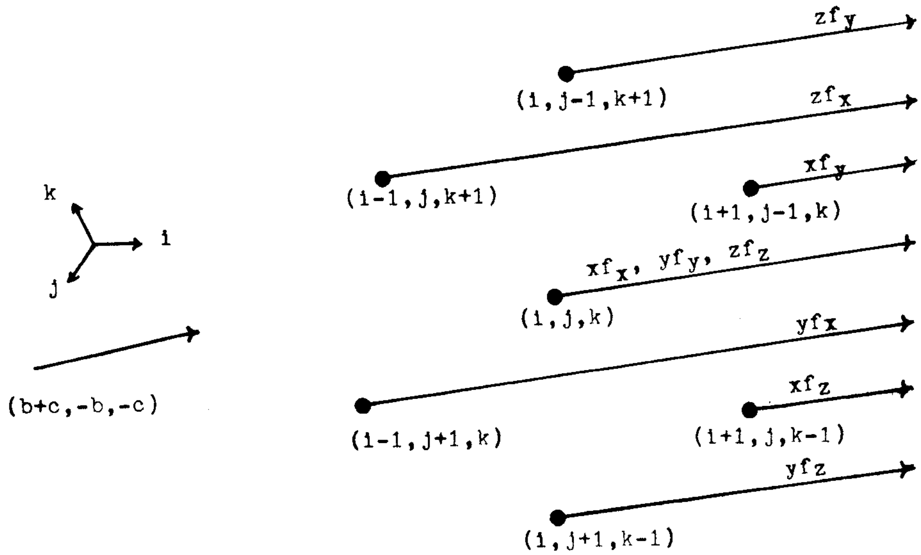

FIGURE 1. The partial derivatives of $f = p_N$

Thus we may replace the general form (∗) for the top homogeneous part p_N of p by one of 3 simpler forms,

$$\left.\begin{array}{ll} \text{I} & p_N = x^\alpha y^\beta z^\gamma q(x, y) \\ \text{II} & p_N = x^\alpha y^\beta z^\gamma q(x^2, yz) \end{array}\right\}, \quad q = a_0 u^m + \cdots + a_m v^m,\ m a_0 a_m \neq 0,$$
$$\text{III} \quad p_N = x^\alpha y^\beta,$$

which we shall dispose of separately.

Case I. $p_N = x^\alpha y^\beta z^\gamma q(x, y)$.

First suppose $\gamma = 0$. Then p_N is a product of linear and definite quadratic terms in x and y. The Euclidean foliation arguments from the proof of Lemma 3 show that p_N has no definite factors and at most 2 nonproportional linear factors, so we reduce to Case III.

For $\gamma \neq 0$, we again cannot have quadratic irreducible factors in q lest the holonomy be abelian (see the arguments above when S had nonreal eigenvalues). Again, if p_N has two or more nonproportional linear factors then $\mathcal{H}_0(p)$ is diagonalizable, hence abelian.

Thus by linear coordinate changes, Case I reduces to Case III.

Case II. $p_N = x^\alpha y^\beta z^\gamma q(x^2, yz)$.

From a nondiagonal first degree relation in the partial derivatives of p_N, we obtain a relation of the form $x(p_N)_z = ky(p_N)_x$, $k \neq 0$, after possibly switching y and z.

Rescaling the y coordinate so $k = \frac{1}{2}$, we find p_N depends only on y and $(x^2 + yz)$, and we may take $p_N = y^\beta(x^2 + yz)^n$, $n \neq 0$.

First suppose $\beta = 0$. Then $H_0(p) \subset SO(2,1)$. If we have $H_0(p) = SO(2,1)$, then $G_0(p) \cong SO(2,1)$ is semisimple and connected. Then $G_0(p)$ would fix a point in A^3 by **[M]**, contradicting Proposition 2. Thus $\dim H_0(p) = 2$. It follows that $H_0(p)$ is a parabolic subgroup of $SO(2,1)$.

If $\beta \neq 0$ then $H_0(p) \subset \mathbf{R}^+ \cdot SO(2,1)$. It is easily seen that the projection of $H_0(p)$ into $SO(2,1)$ is a 1-1 map onto a parabolic subgroup of $SO(2,1)$.

Regardless of whether $\beta = 0$ or $\beta \neq 0$, we have a linear basis for $\mathcal{G}_0(p)$ of the form

$$A = \left(\begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right), \quad B = \left(\begin{pmatrix} \beta & 0 & 0 \\ 0 & -2n & 0 \\ 0 & 0 & 2n + 2\beta \end{pmatrix} \begin{pmatrix} -\delta \\ 0 \\ 0 \end{pmatrix} \right)$$

where we have made the $y - z$ translational part of B vanish by proper choice of the origin (using the fact that $-2n$ and $2n + 2\beta$ are nonzero). Since A generates the commutator subgroup of $\mathcal{G}_0(p)$, we obtain by computation that $[A, B] = (\beta + 2n)A$, giving the equations

$$\begin{aligned} -\beta u &= (\beta + 2n)u, \\ 2nv + 2\delta &= (\beta + 2n)v, \\ -(2n + 2\beta)w &= (\beta + 2n)w \end{aligned}$$

from which follows $u = w = 0$, $\beta v = 2\delta$.

If $\beta \neq 0$ we may choose the origin so $\delta = 0$ and thus $v = 0$, contradicting Corollary 1.

Thus $\beta = 0$ and $\delta = \beta v/2 = 0$. For $a = (x, y, z)$ we have $Dp(a)(Aa) = 0$ and $Dp(a)(Ba) = 0$, that is

$$zp_x + (v - 2x)p_y = yp_y - zp_z = 0.$$

It follows that p is functionally dependent on $yz + x^2 - vx = q$. Thus q is invariant under Γ and determines a polynomial on M that violates Proposition 1.

Case III. $p_N = x^\alpha y^\beta$.

Our argument splits into two cases, according to whether α, β are both positive (we will restrict to this case for now) or whether one of α, β vanishes (which we shall consider below in Case IV).

$H_0(p)$ cannot preserve x and y by the Euclidean foliation argument in the proof of Lemma 3. Setting $x \circ h = \lambda x$ we obtain a homomorphism λ: $H_0(p) \to \mathbf{R}^+$ which is onto. The kernel K of λ has dimension $\geqslant 1$ and determines a Lie algebra $\mathcal{K}$.

Elements of $\mathcal{K} - 0$ give relations $\xi p_x = \eta p_y + (ax + by + cz + \delta)p_z = 0$. We must have $(\xi, \eta) \neq (0,0)$ and $(a, b, c) \neq (0,0,0)$ by Lemma 3. Let $\bar{x} = \xi x + \eta y$ and choose a complementary coordinate $\bar{y}$ for the $x - y$ plane. Then we have $p_{\bar{x}} + (a\bar{x} + b\bar{y} + cz + \delta)p_z$. In order to have algebraic level curves in the plane $\bar{y} =$ constant, we must have $c = 0$.

Since $(\xi, \eta) \neq (0,0)$ on $\mathcal{K} - 0$ we see $\dim \mathcal{K} = 1$ or 2. If $\dim \mathcal{K} = 2$, then we may translate x and y so $\delta \equiv 0$ on $\mathcal{K}$. Choosing $(\xi, \eta) = (1,0)$ and $(0,1)$ gives the pair of equations $p_x + (\sigma x + \tau y)p_z = p_y + (\mu x + \nu y)p_z = 0$. Thus p is constant along the families of curves

$$y = \text{constant}, \quad dz/dx = \sigma x + \tau y,$$
$$x = \text{constant}, \quad dz/dy = \mu x + \nu y.$$

It follows that p is functionally related to a polynomial of the form $z + q(x, y)$, and we again violate Proposition 1.

Thus $\dim \mathcal{K} = 1$. We may choose a basis for $\mathcal{G}_0(p)$ of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & b & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \\ \delta \end{pmatrix}, \quad B = \begin{pmatrix} -\beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix}$$

where we've used multiples of A to eliminate the lower left entry in B and then translated the origin to eliminate the $x - y$ translational component of B (α, β are nonzero). We find $[B, A] = (\beta + d)A$ and obtain the equations

$$(\beta + d)b = db - b\alpha, \quad (\beta + d)\eta = \alpha\eta,$$
$$(\beta + d)\delta = c\eta + \delta\delta, \quad (\beta + d)\xi = -\beta\xi.$$

Hence $b = 0$ and precisely one of ξ and η must vanish.

(a) $\xi = 0$, $\eta \neq 0$. We find $d = \alpha - \beta$, $\beta\delta = c\eta$. From $\eta p_y + (x + \delta)p_z = 0$ we obtain $p = q(x, yx + \delta y - \eta z)$. Setting $y = 0$ shows $q = q(x, v)$ is a polynomial. Using $\beta \in \mathcal{G}_0(p)$ gives an equation in p equivalent to

$$xq_x + \big(v(1 - \alpha/\beta) + e\eta/\beta\big)q_v = 0.$$

If $k = 1 - \alpha/\beta$ isn't zero, we may add a constant ε to v and obtain $xq_x + kvq_v = 0$. It follows that p is functionally dependent on $vx^{-k} = (yx + \delta y - \eta z + \varepsilon)x^{-k}$. This induces a function on M which has no local maximum, which is impossible. Thus k must vanish and we obtain $xq_x + lq_v = 0$, $l \in \mathbf{R}$. It follows that p is functionally dependent on $v - l\log x$. As p is algebraic, we must have $l = 0$. But v is quadratic and induces a map on M that violates Proposition 1.

(b) $\xi \neq 0$, $\eta = 0$. Then $\delta = 0$, $d = -2\beta$. From $p_x + xp_z = 0$ we find $p = q(y, z - x^2/2\xi)$ where, setting $x = 0$, q is a polynomial $q(y, w)$. From $-\beta xp_x + \alpha yp_y + (cy - 2\beta z + e)p_z = 0$ we obtain $\alpha yq_y + (cy - 2\beta w + e)q_w = 0$. Thus q is functionally dependent on $(w - cy/(\alpha + 2\beta) + e/2\beta)^\alpha y^{2\beta}$ and p is functionally dependent on $(z - x^2/2\xi - cy/(\alpha + 2\beta) + e/2\beta)^\alpha y^{2\beta}$. This last polynomial induces a polynomial map on M that violates Proposition 1.

At last we reduce to the final case.

Case IV. $p_N = z^N$.

Here $G_0(p)$ preserves the 1-form dz, although it cannot preserve z by Proposition 1. Therefore the map $z \circ q - z\colon G_0(p) \to \mathbf{R}$ is a homomorphism onto. This map has a nontrivial kernel K with associated Lie algebra $\mathcal{K} \neq 0$.

Suppose $\left(\begin{smallmatrix} A_1 & A_2 \\ 00 & 00 \end{smallmatrix}\right)$ is a nonzero element of $\mathcal{K}$.

(a) Assume A_1 is invertible. Then for proper choice of coordinates we have $A_2 = 0$.

(1) If A_1 has nonreal eigenvalues then the $x - y$ coordinates can be chosen so that for each z, p depends only on $x^2 + y^2$. Say $p(x, y, z) = q(z, x^2 + y^2)$. It is easy to see that q is a polynomial $q(z, s)$. Choosing an element of $\mathcal{G}_0(p) - \mathcal{K}$ expresses p_z as

an affine combination of p_x and p_y, that is,

$$q_z = (\alpha x + \beta y + \gamma z + \delta)\cdot 2yq_s + (\alpha' x + \beta' y + \gamma' z + \delta')\cdot 2xq_s.$$

The coefficient of q_s must depend only on z and x^2+y^2, so $\gamma = \delta = \gamma' = \delta' = 0$ and $2y(\alpha + \beta y) + 2x(\alpha' x + \beta' y) = k(x^2+y^2)$. It follows that $q_z = ksq_s$. Considering the terms of degree $N-1$ in this equation, we see $k=0$, and p is independent of z, contrary to Lemma 3.

(2) If A_1 is real and semisimple, then we may choose coordinates so A_1 is diagonal. We find $p(x, y, z) = q(z, x^m y^n)$ with m, n positive and relatively prime. Reasoning as in the preceding paragraph, we obtain again that p is independent of z and contradicts Lemma 3.

(3) If A_1 is real and not semisimple we may assume $A_1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Then for each z, p is functionally dependent on $x/y - \log y$. Since p is a polynomial, p must actually be dependent on z alone, contradicting Lemma 3.

(b) Thus we may assume A_1 isn't invertible and put our nonzero element of $\mathcal{K}$ in the form

$$\begin{pmatrix} 0 & b & c\\ 0 & \beta & \gamma\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} d\\ \delta\\ 0\end{pmatrix}.$$

For fixed z, p is constant along

$$\frac{dx}{dy} = \frac{by + cz + d}{\beta y + \gamma z + \delta}.$$

If $\beta \neq 0$, we may take $\beta = 1$. Then

$$\frac{dx}{dy} = b + \frac{c'z + d'}{y + \gamma z + \delta}$$

and $p(-,-,z)$ is functionally dependent on $-x + by + (c'z + d')\log(y + \gamma z + \delta)$. For this to be algebraic, we must have $(c'z + d') \equiv 0$. But then p depends only on $(x - by)$ and z, in contradiction to Lemma 3.

Thus $\beta = 0$. We may translate the origin to make precisely one of γ or δ vanish.

(1) $\gamma = 1, \delta = 0$.

$$p(x, y, z) = q\left(-zx + \frac{b}{2}y^2 + czy + dy, z\right) = q(w, z).$$

$p_z = q_z - xq_w + cyq_w$ is an affine combination of

$$p_x = -zq_w \quad\text{and}\quad p_y = (by + cz + d)q_w.$$

It follows that $q_z = (\alpha w_1 + r(z))q_w$, with $\deg r \leqslant 2$. Since q is a polynomial, we must have $\alpha = 0$. It follows that q is functionally dependent on $w + s(z)$, $\deg s \leqslant 3$. Thus $w + s = -zx + by^2/2 + czy + dy + s(z)$ defines a polynomial preserved by $G_0(p)$. Since $w + s$ has no local maxima if $b \geqslant 0$ and no local minima if $b \leqslant 0$, we contradict Proposition 1.

(2) $\gamma = 0, \delta = 1$.

$$p = q\left(-x + \frac{b}{2}y^2 + czy + dy, z\right).$$

The argument here is almost the same as in (1). One finds that $G_0(p)$ preserves a polynomial of the form $x + r(y, z)$ and contradicts Proposition 1. Q.E.D.

While we are primarily interested in Theorem 4 as it was stated we may generalize it in several ways without any change in the proof. We have used no special features of 3-manifold topology, so we have the following results.

THEOREM 5. *If M is a closed affine manifold and p a nonconstant polynomial on M then p is not expressible in terms of* $\leqslant 3$ *linear coordinates.*

Next we recall the notion of affine foliation for a manifold M from **[Th]**. The codimension-i foliation $\mathfrak{F}$ is *affine* if the changes of chart are the restrictions of affine transformations of $\mathbf{R}^i$. Then a map $p: \tilde{M} \to \mathbf{R}$ is *polynomial* if it is constant on leaves of $\mathfrak{F}$ and given by a polynomial in the transverse affine coordinates. This generalizes the notion of an affine structure, when $i = \dim M$.

THEOREM 6. *Let M be a closed manifold with an affine foliation $\mathfrak{F}$ of codimension i. If $i \leqslant 3$ then the only polynomials on M are constants.*

This is proved by using the developing map of the foliation $D: \tilde{M} \to \mathbf{R}^i$ and arguing as before.

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