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POLYNOMIALS RELATED TO GENERALIZED CHEBYSHEV POLYNOMIALS

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Abstract

We study several classes of polynomials, which are related to the Chebyshev, Morgan-Voyce, Horadam and Jacobsthal polynomials. Thus, we unify some of well-known results.

1 Introduction

Classes of Chebyshev polynomials are well-known. There are many classes of polynomials which are related to the Chebyshev polynomials. In this paper we are motivated by some recent results in this topic, such as [5] and [7]. In this section we first define polynomials which will be investigated in the rest of the paper. The main aim is to define classes of polynomials which include, as special cases, some well-known classes of polynomials. Then, we prove some properties of new polynomials, and thus justify the motivation for introducing them.

Throughout this paper we use $\mathbb{N}$ to denote the set of all nonnegative integers.

The generalized Chebyshev polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$ we introduce here as follows (x is a real variable):

$$\Omega_{n,m}(x) = x\Omega_{n-1,m}(x) - \Omega_{n-m,m}(x), \quad n \geq m, \; n,m \in \mathbb{N}, \tag{1.1}$$

with $\Omega_{n,m}(x) = x^n, \quad n = 1,2,\dots,m-1, \; \Omega_{m,m}(x) = x^m - 2$. Moreover,

$$V_{n,m}(x) = xV_{n-1,m}(x) - V_{n-m,m}(x), \quad n \geq m, \; n,m \in \mathbb{N}, \tag{1.2}$$

with $V_{n,m}(x) = x^n, \; n = 1,2,\dots,m-1, \; V_{m,m}(x) = x^m - 1$.

Using standard methods, we find that

$$F^m(t) = (1-t^m)(1-xt+t^m)^{-1} = \sum_{n=1}^{\infty} \Omega_{n,m}(x)t^n \tag{1.3}$$

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and

$$G^m(t) = (1 - xt + t^m)^{-1} = \sum_{n=1}^{\infty} V_{n,m}(x)t^n \tag{1.4}$$

are generating functions of polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$, respectively.

By (1.3) and (1.4), we get the following explicit formulas:

$$\Omega_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk}, \tag{1.5}$$

$$V_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \binom{n-(m-1)k}{k} x^{n-mk}. \tag{1.6}$$

For $m = 2$, these polynomials become the modified Chebyshev polynomials [7].

Next, we introduce the family of polynomials $P_{n,m}(x)$ by:

$$P_{n,m}(x) = xP_{n-1,m}(x) + 2P_{n-m,m}(x) - P_{n-2m,m}(x), \;\; n \geq 2m, \;\; n, m \in \mathbb{N}. \tag{1.7}$$

For $m = 1$, (1.7) becomes (and this is considered in [7])

$$P_n(x) = (x+2)P_{n-1}(x) - P_{n-2}(x), \;\; n \geq 2,$$

for every $P \in \{b, B, c, C\}$, where:

$b_0 = 1, \; b_1 = x + 1,$ (Morgan–Voyce polynomials);
$B_0 = 1, \; B_1 = x + 2$ (Morgan–Voyce polynomials);
$c_0 = 1, \; c_1 = x + 3$ (Horadam polynomials);
$C_0 = 2, \; C_1 = x + 2$ (Horadam polynomials).

The generalized Jacobsthal $J_{n,m}(x)$ and the Jacobsthal–Lucas $j_{n,m}(x)$ polynomials (see [2], [3], [4]) are given by recurrence relations, respectively:

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \;\; n \geq m, \;\; m, n \in \mathbb{N}; \tag{1.8}$$

with initial values

$$J_{0,m}(x) = 0, \;\; J_{n,m}(x) = 1, \;\; n = 1, 2, \ldots, m-1;$$

and by

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \;\; n \geq m, \;\; n, m \in \mathbb{N}, \tag{1.9}$$

with initial values

$$j_{0,m} = 2, \;\; j_{n,m}(x) = 1, \;\; n = 1, 2, \ldots, m-1.$$

By (1.8) and (1.9), we find the following explicit formulas ([2], [3], [4]):

$$J_{n+1,m}(x) = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} (2x)^k; \tag{1.10}$$

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k. \tag{1.11}$$

The rest of the paper is organized as follows. In Section 2 we investigate the relationship between the Chebyshev polynomials and polynomials $P_{n,3}(x)$. In Section 3 we consider a general class of polynomials that include polynomials $b_{n,m}(x)$, $B_{n,m}(x)$, $c_{n,m}(x)$ and $C_{n,m}(x)$. In Section 4 we investigate the relationship between the Chebyshev and the Jacobsthal polynomials. Finally, in Section 5 we consider mixed convolutions of the Chebyshev type.

2 Polynomials $P_{n,3}(x)$ and Chebyshev polynomials

In this section we consider polynomials $P_{n,m}(x)$ for $m = 3$. Thus, we get $P_{n,3}(x)$, which generalize polynomials $P_n(x)$, and they satisfy the following recurrence relation (by (1.7)):

$$P_{n,3}(x) = xP_{n-1,3}(x) + 2P_{n-3,3}(x) - P_{n-6,3}(x),\ n \geq 6, \tag{2.1}$$

where $P_{n,3} \in \{b_{n,3}, B_{n,3}, c_{n,3}, C_{n,3}\}$, with the following sets of initial values, respectively:

$$b_{0,3}(x) = 1,\ b_{1,3}(x) = x,\ b_{2,3}(x) = x^2,\ b_{3,3}(x) = x^3 + 1,$$
$$b_{4,3}(x) = x^4 + 3x,\ b_{5,3}(x) = x^5 + 5x^2;$$

$$B_{0,3}(x) = 1,\ B_{1,3}(x) = x,\ B_{2,3}(x) = x^2,\ B_{3,3}(x) = x^3 + 2,$$
$$B_{4,3}(x) = x^4 + 4x,\ B_{5,3}(x) = x^5 + 6x;$$

$$c_{0,3} = 1,\ c_{1,3} = x,\ c_{2,3} = x^2,\ c_{3,3} = x^3 + 3,\ c_{4,3} = x^4 + 5x,$$
$$c_{5,3} = x^5 + 7x^2;$$

$$C_{0,3}(x) = 2,\ C_{1,3}(x) = x,\ C_{2,3}(x) = x^2,\ C_{3,3}(x) = x^3 + 2,$$
$$C_{4,3}(x) = x^4 + 4x,\ C_{5,4}(x) = x^5 + 6x^2.$$

Now, we prove the following result.

Theorem 2.1. *Using previous notations, the following identities are fulfilled:*

$$(-1)^n x c_{n,3}(-x^2) = \Omega_{2n+1,6}(x), \quad n \geq 0; \tag{2.2}$$

$$(-1)^n C_{n,3}(-x^2) = \Omega_{2n,6}(x), \quad n \geq 0; \tag{2.3}$$

$$(-1)^n b_{n,3}(-x^2) = V_{2n,6}(x), \quad n \geq 0; \tag{2.4}$$

$$(-1)^n x B_{n,3}(-x^2) = V_{2n+1,6}(x), \quad n \geq 0; \tag{2.5}$$

$$c_{n+3,3}(x) - c_{n,3}(x) = C_{n+3,3}(x), \quad n \geq 0; \tag{2.6}$$

$$b_{n+3,3}(x) + b_{n,3(x)} = C_{n+3,3}(x), \quad n \geq 0; \tag{2.7}$$

$$C_{n+3,3}(x) - C_{n,3}(x) = x c_{n+2,3}(x), \quad n \geq 0; \tag{2.8}$$

$$B_{n,3}(x) + B_{n-3,3}(x) = c_{n,3}(x), \quad n \geq 3; \tag{2.9}$$

$$B_{n,3}(x) - B_{n-6,3}(x) = C_{n,3}(x), \quad n \geq 6. \tag{2.10}$$

Proof. We prove theorem using the induction on n. The equality (2.3) is satisfied for $n = 1$, by (1.1). Suppose that (2.3) holds for $n - 1$ instead of n ($n \geq 1$). Then, using (2.1), we get:

$$\begin{aligned}
&(-1)^n C_{n,3}(-x^2) = (-1)^n(-x^2 C_{n-1,3}(-x^2) + 2C_{n-3,3}(-x^2) - C_{n-6,3}(-x^2)) \\
&= (-1)^{n-1} x^2 C_{n-1,3}(-x^2) - 2(-1)^{n-3} C_{n-3,3}(-x^2) - (-1)^{n-6} C_{n-6,3}(-x^2) \\
&= x^2 \Omega_{2n-2,6}(x) - 2\Omega_{2n-6,6}(x) - \Omega_{2n-12,6}(x) \\
&= x(\Omega_{2n-1,6}(x) + \Omega_{2n-7,6}(x)) + 2(\Omega_{2n,6}(x) - x\Omega_{2n-1,6}(x)) \\
&+\Omega_{2n-6,6}(x) - x\Omega_{2n-7,6}(x) \\
&= 3\Omega_{2n,6}(x) + 2\Omega_{2n-6,6}(x) - 2x\Omega_{2n-1,6}(x) \\
&= \Omega_{2n,6}(x) + 2(\Omega_{2n,6}(x) + \Omega_{2n-6,6}(x) - x\Omega_{2n-1,6}(x)) \\
&= \Omega_{2n,6}(x).
\end{aligned}$$

It is easy to verify the equality (2.4) for $n = 1$ and $n = 2$, from initial values. Suppose that (2.4) holds for n ($n \geq 2$). Then, from (2.1), we have:

$$\begin{aligned}
&(-1)^{n+1} b_{n+1,3}(-x^2) = (-1)^{n+1}\left(-x^2 b_{n,3}(-x^2) + 2b_{n-2,3}(-x^2) - b_{n-5,3}(-x^2)\right) \\
&= x^2((-1)^n b_{n,3}(-x^2)) - 2(-1)^{n-2} b_{n-2,3}(-x^2) - (-1)^{n-5} b_{n-5,3}(-x^2) \\
&= x^2 V_{2n,6}(x) - 2V_{2n-4,6}(x) - V_{2n-10,6}(x) \\
&= x\left(V_{2n+1,6}(x) + V_{2n-5,6}(x)\right) - 2V_{2n-4,6}(x) + V_{2n-4,6}(x) - xV_{2n-5,6}(x) \\
&= xV_{2n+1,6}(x) - V_{2n-4,6}(x) = V_{2n+2,6}(x).
\end{aligned}$$

We immediately prove the equality (2.6) for $n = 1$ i $n = 2$. Suppose that (2.6) holds for n ($n \geq 2$). So, for $n + 1$ instead of n, it follows that

$$\begin{aligned}
C_{n+4,3} &= xC_{n+3,3} + 2C_{n+1,3} - C_{n-2,3} \\
&= x(c_{n+3,3} - c_{n,3}) + 2(c_{n+1,3} - c_{n-2,3}) - c_{n-2,3} + c_{n-5,3} \\
&= xc_{n+3,3} - xc_{n,3} + 2c_{n+1,3} - 3c_{n-2,3} + c_{n-5,3} \\
&= xc_{n+3,3} + 2c_{n+1,3} - (xc_{n,3} + 2c_{n-2,3}) - c_{n-2,3} + c_{n-5,3} \\
&= xc_{n+3,3} + 2c_{n+1,3} - c_{n-2,3} - c_{n+1,3} = c_{n+4,3} - c_{n+1,3}.
\end{aligned}$$

In a similar way, we also can prove equalities (2.2), (2.5), (2.7)– (2.10). □

Now we consider two sequences of numbers $\{\Omega_{n,3}(2) \equiv a_n\}$ and $\{V_{n,3}(2) \equiv b_n\}$. For these numbers we find the corresponding explicit formulas.

For $x = 2$ in (1.1), we get the following difference equation

$$a_n = 2a_{n-1} - a_{n-3}, \;\; a_1 = 2, \;\; a_2 = 4, \;\; a_3 = 6. \tag{2.11}$$

The solution of the difference equation (2.11) is given by

$$a_n = \frac{2\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right). \tag{2.12}$$

For $x = 2$ in (1.2), we get the following difference equation

$$b_n = 2b_{n-1} - b_{n-3}, \;\; b_1 = 2, \;\; b_2 = 4, \;\; b_3 = 7. \tag{2.13}$$

The solution of (2.13) is given by

$$b_n = -1 + \frac{2+\sqrt{5}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{2-\sqrt{5}}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}. \tag{2.14}$$

The following result can be proved.

Theorem 2.2. *The sequences of numbers* $\{a_n\})$ *and* $\{(b_n\})$ *satisfy the following relations:*

$$a_{n+1} = a_n + a_{n-1}; \tag{2.15}$$

$$b_{n+1} = b_n + b_{n-1} + 1. \tag{2.16}$$

3 Generalized polynomials

The family of polynomials $P_{n,m}(x)$, which is given by (1.7), for different initial values produces special polynomials: $b_{n,m}(x)$, $B_{n,m}(x)$, $c_{n,m}(x)$, $C_{n,m}(x)$. These special polynomials obey the following properties.

Theorem 3.1. *Using previos notations, for all* $n \geq m$ $(n \in \mathbb{N},\; m \in 2\mathbb{N}+1)$ *the following hold:*

$$(-1)^n x c_{n,m}(-x^2) = \Omega_{2n+1,2m}(x), \tag{3.1}$$

$$(-1)^n C_{n,m}(-x^2) = \Omega_{2n,2m}(x), \tag{3.2}$$

$$(-1)^n b_{n,m}(-x^2) = V_{2n,2m}(x), \tag{3.3}$$

$$(-1)^n x B_{n,m}(-x^2) = V_{2n+1,2m}(x), \tag{3.4}$$

$$c_{n+m,m}(x) - c_{n,m}(x) = C_{n+m,m}(x), \tag{3.5}$$

$$b_{n+m,m}(x) + b_{n,m}(x) = C_{n+m,m}(x), \tag{3.6}$$

$$C_{n+m,m}(x) - C_{n,m}(x) = x c_{n+m-1,m}(x), \tag{3.7}$$

$$B_{n+m,m}(x) + B_{n,m}(x) = c_{n+m,m}(x), \tag{3.8}$$

$$B_{n,m}(x) - B_{n-2m,m}(x) = C_{n,m}(x), \quad n \geq 2m. \tag{3.9}$$

Proof. Suppose that the equality (3.1) holds for $n-1$ instead of n $(n \geq 1)$. Then, by (1.7) we get that the following is satisfied:

$$\begin{aligned}
&(-1)^n x c_{n,m}(-x^2) = \\
&(-1)^n x \left(-x^2 c_{n-1,m}(-x^2) + 2c_{n-m,m}(-x^2) - c_{n-2m,m}(-x^2)\right) \\
&= x^2 \left((-1)^{n-1} x c_{n-1,m}(-x^2)\right) + 2(-1)^m(-1)^{n-m} x c_{n-m,m}(-x^2) \\
&-(-1)^{2m}(-1)^{n-2m} c_{n-2m,m}(-x^2) \\
&= x^2 \Omega_{2n-1,2m}(x) - 2\Omega_{2n-2m+1,2m}(x) - \Omega_{2n-4m+1,2m}(x) \\
&= x(\Omega_{2n,2m}(x) + \Omega_{2n-2m,2m}(x)) - 2(x\Omega_{2n-2m,2m}(x) \\
&-\Omega_{2n-4m+1,2m}(x)) - \Omega_{2n-4m+1,2m}(x) \\
&= x\Omega_{2n,2m}(x) - x\Omega_{2n-2m,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x) + \Omega_{2n-2m+1,2m}(x) - x\Omega_{2n-2m,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x) - \Omega_{2n-4m+1,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x).
\end{aligned}$$

Next, suppose that (3.9) holds for $n-1$ instead of n $(n \geq 1)$. Then, by (1.7), we get:

$$\begin{aligned}
&C_{n,m}(x) = xC_{n-1,m}(x) + 2C_{n-m,m}(x) - C_{n-2m,m}(x) \\
&= x(B_{n-1,m}(x) - B_{n-1-2m,m}(x)) + 2(B_{n-m,m}(x) - B_{n-3m,m}(x)) \\
&- B_{n-2m,m}(x) + B_{n-4m,m}(x) = xB_{n-1,m}(x) \\
&- xB_{n-1-2m,m}(x) + 2B_{n-m,m}(x) - 2B_{n-3m,m}(x) - B_{n-2m,m}(x) + B_{n-4m,m}(x) \\
&= B_{n,m}(x) - 2B_{n-m,m}(x) + B_{n-2m,m}(x) - B_{n-2m,m}(x) + 2B_{n-3m,m}(x) \\
&- B_{n-4m,m}(x) + 2B_{n-m,m}(x) - 2B_{n-3m,m}(x) - B_{n-2m,m}(x) + B_{n-4m,m}(x) \\
&= B_{n,m}(x) - B_{n-2m,m}(x).
\end{aligned}$$

In a similar way, equalities (3.2)–(3.8) can be proved. □

Corollary 3.1. *If we exchange x by ix in Theorem 3.1 $(i^2 = -1)$, then we obtain the following identities:*

$$\begin{aligned}
&(-1)^n (ix) c_{n,m}(x^2) = \Omega_{2n+1,2m}(ix); \ (by\ (3.1)) \\
&(-1)^n C_{n,m}(x^2) = \Omega_{2n,2m}(ix); \ (by\ (3.2)) \\
&(-1)^n b_{n,m}(x^2) = V_{2n,2m}(ix); \ (by\ (3.3)) \\
&(-1)^n (ix) B_{n,m}(x^2) = V_{2n+1,2m}(ix), \ (by\ (3.4)).
\end{aligned}$$

4 Chebyshev and Jacobsthal polynomials

In this section we discover connections between polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$ on one side, and polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$ on the other side.

Theorem 4.1. *For all* $n \geq m \quad (n, m \in \mathbb{N})$, *the following hold:*

$$V_{n,m}(x) = x^n J_{n+1,m}(-(2x^m)^{-1}); \tag{4.1}$$

$$\Omega_{n,m}(x) = x^n j_{n,m}(-(2x^m)^{-1}). \tag{4.2}$$

Proof. By (1.6) and (1.10), we have:

$$\begin{aligned} x^n J_{n+1,m}(-(2x^m)^{-1}) &= x^n \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} (-2(2x^m)^{-1})^k \\ &= \sum_{k=0}^{[n/m]} (-1)^k \binom{n-(m-1)k}{k} x^{n-mk} \\ &= V_{n,m}(x). \end{aligned}$$

Hence, (4.1) is proved. Next, from (1.5) and (1.11), we obtain (4.2) as follows:

$$\begin{aligned} x^n j_{n,m}(-(2x^m)^{-1}) &= x^n \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (-2(2x^m)^{-1})^k \\ &= \sum_{k=0}^{[n/m]} (-1)^k \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk} \\ &= \Omega_{n,m}(x). \end{aligned}$$

□

We also prove the following result.

Theorem 4.2. *For all* $n \geq m \quad (n, m \in \mathbb{N})$ *the following hold:*

$$\Omega_{2n+1,2m}(x) = x^{2n+1} j_{2n+1,2m}(-(2x^{2m})^{-1}); \tag{4.3}$$

$$\Omega_{2n,2m}(x) = x^{2n} j_{2n,2m}(-(2x^{2m})^{-1}); \tag{4.4}$$

$$V_{2n,2m}(x) = x^{2n} J_{2n+1,2m}(-(2x^{2m})^{-1}); \tag{4.5}$$

$$V_{2n+1,2m}(x) = x^{2n+1} J_{2n+2,2m}(-(2x^{2m})^{-1}). \tag{4.6}$$

Proof. By (1.11) and (1.5), we obtain:

$$\begin{aligned} & x^{2n} j_{2n,2m}(-(2x^{2m})^{-1}) = \\ & \quad = x^{2n} \sum_{k=0}^{[n/m]} \frac{2n-(2m-2)k}{2n-(2m-1)k} \binom{2n-(2m-1)k}{k} (-2(2x^{2m})^{-1})^k \\ & \quad = \sum_{k=0}^{[n/m]} (-1)^k \frac{2n-(2m-2)k}{2n-(2m-1)k} \binom{2n-(2m-1)k}{k} x^{2n-2mk} \\ & \quad = \Omega_{2n,2m}(x). \end{aligned}$$

So, the equality (4.4) is proved.

Next, by (1.6) and (1.10), we get:

$$\begin{aligned} x^{2n+1} J_{2n+2,2m}(-(2x^{2m})^{-1}) = \\ &= x^{2n+1} \sum_{k=0}^{[n/m]} \binom{2n+1-(2m-1)k}{k} (-2(2x^{2m})^{-1})^k \\ &= \sum_{k=0}^{[n/m]} (-1)^k \binom{2n+1-(2m-1)k}{k} x^{2n+1-2mk} \\ &= V_{2n+1,2m}(x). \end{aligned}$$

Hence, (4.6) holds. Equalities (4.3) and (4.5) can be proved similarly. □

The following result is an exercise for a reader.

Theorem 4.3. *For all* $n \geq 2m \quad (n \in \mathbb{N},\ m \in 2\mathbb{N}+1)$ *the following hold:*

$$c_{n,m}(-x^2) = (-1)^n x^{2n} j_{2n+1,2m}(-(2x^{2m})^{-1}); \tag{4.7}$$

$$C_{n,m}(-x^2) = (-1)^n x^{2n} j_{2n,2m}(-(2x^{2m})^{-1}); \tag{4.8}$$

$$b_{n,m}(-x^2) = (-1)^n x^{2n} J_{2n+1,2m}(-(2x^{2m})^{-1}); \tag{4.9}$$

$$B_{n,m}(-x^2) = (-1)^n x^{2n} J_{2n+2,2m}(-(2x^{2m})^{-1}). \tag{4.10}$$

Corollary 4.1. *Taking* x *instead of* $-x^2$ *in Theorem 4.3, we get the following relations:*

$$\begin{aligned} c_{n,m}(x) &= x^n j_{2n+1,2m}((2x^m)^{-1}); \\ C_{n,m}(x) &= x^n j_{2n,2m}((2x^m)^{-1}); \\ b_{n,m}(x) &= x^n J_{2n+1,2m}((2x^m)^{-1}); \\ B_{n,m}(x) &= x^n J_{2n+2,2m}((2x^m)^{-1}). \end{aligned}$$

In this section we prove one more result.

Theorem 4.4. *For* $r \geq 1$ *and* $n \geq m \quad (n, m, r \in \mathbb{N})$, *the following hold:*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+(m-1)i,m}(x) = (-1)^n h_{r+mn,m}(x); \tag{4.11}$$

$$\sum_{i=0}^{n} \binom{n}{i} h_{r+mi,m}(x) = x^n h_{r+(m-1)n,m}(x), \tag{4.12}$$

where $h_{n,m}(x) = \Omega_{n,m}(x)$ *or* $h_{n,m}(x) = V_{n,m}$.

Proof. It is easy to prove the equality (4.11) for $n = 1$. Suppose that (4.11) holds for $n \geq 1$. Then, from (1.1) and (1.2), we get:

$$h_{r+m(n+1),m}(x) = xh_{r+m-1+mn,m}(x) - h_{r+mn,m}(x)$$
$$= (-1)^n x \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+m-1+(m-1)i,m}(x) -$$
$$(-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+(m-1)i,m}(x)$$
$$= (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{i+1} h_{r+m-1+(m-1)i,m}(x)$$
$$+(-1)^n \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} x^i h_{r+(m-1)i,m}(x)$$
$$= (-1)^n \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} x^i h_{r+(m-1)i,m}(x)$$
$$+(-1)^n \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} x^i h_{r+(m-1)i,m}(x)$$
$$= (-1)^n \sum_{i=1}^{n} (-1)^{i-1} \binom{n+1}{i} x^i h_{r+(m-1)i,m}(x) + x^{n+1} h_{r+(m-1)(n+1),m}(x)$$
$$-(-1)^n h_{r+(m-1)0,m}(x) = (-1)^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} x^i h_{r+(m-1)i,m}(x)$$
$$= (-1)^{n+1} h_{r+m(n+1),m}(x).$$

The relation (4.12) can be proved similarly. □

5 Mixed convolutions of the Chebyshev type

In this section we introduce and study polynomials $V^r_{n,m}(x)$, which are the r^{th}–convolutions of polynomials $V_{n,m}(x)$. We also study polynomials $\Omega^s_{n,m}(x)$, which are the s^{th}–convolutions of $\Omega_{n,m}(x)$. Finally, we investigate polynomials $v^{r,s}_{n,m}(x)$, which are the mixed convolutions of the Chebyshev type, where r and s are nonnegative integers with $r + s \geq 1$.

In the rest of the section we have $m, n, r, s \in \mathbb{N}$ and $r + s \geq 1$.

Polynomials $V^r_{n,m}(x)$ are defined by

$$G^m_r(t) = (1 - xt + t^m)^{-(r+1)} = \sum_{n=1}^{\infty} V^r_{n,m}(x) t^n. \tag{5.1}$$

Hence, using standard methods, we get the following recurrence relation

$$nV^r_{n,m}(x) = x(r+n)V^r_{n-1,m}(x) - (n+mr)V^r_{n-m,m}(x). \tag{5.2}$$

By expanding $G_r^m(t)$ (see (5.1)) in a power series of t, we get:

$$(1-xt+t^m)^{-(r+1)} = \sum_{n=1}^{\infty} \binom{-(r+1)}{n} (-t)^n (x-t^{m-1})^n \tag{5.3}$$

$$= \sum_{n=1}^{\infty} \frac{(r+1)!}{n!} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (-t^{m-1})^k t^k \tag{5.4}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{[n/m]} (-1)^k \frac{(r+1)_{n-(m-1)k}}{k!(n-mk)!} x^{n-mk} t^n. \tag{5.5}$$

Now, using the following equalities (see [4]):

$$\frac{(-1)^k (r+1)_{n-(m-1)k}}{(n-mk)!} \cdot \frac{(x^{-m})^k}{k!} =$$

$$\frac{(-1)^k (-1)^{(m-1)k} (r+1)_n (-n)_{mk}}{(-r-n)_{(m-1)k} (-1)^{mk} n!} \cdot \frac{(x^{-m})^k}{k!} =$$

$$\frac{(r+1)_n m^{mk} \left(\frac{-n}{m}\right)_k \left(\frac{1-n}{m}\right)_k \cdots \left(\frac{m-1-n}{m}\right)_k}{n!(m-1)^{(m-1)k} \left(\frac{-r-n}{m-1}\right)_k \left(\frac{1-r-n}{m-1}\right)_k \cdots \left(\frac{m-2-r-n}{m-1}\right)_k} \cdot \frac{(x^{-m})^k}{k!} =$$

$$\frac{(r+1)_n}{n!} \cdot \frac{\left(\frac{-n}{m}\right)_k \left(\frac{1-n}{m}\right)_k \cdots \left(\frac{m-1-n}{m}\right)_k}{\left(\frac{-r-n}{m-1}\right)_k \left(\frac{1-r-n}{m-1}\right)_k \cdots \left(\frac{m-2-r}{m-1}\right)_k} \cdot \frac{\frac{m^m}{(m-1)^{(m-1)}} x^{-m}}{k!},$$

in (5.3)–(5.5), we get the following formula

$$V_{n,m}^r(x) = \frac{x^n (r+1)_n}{n!} {}_mF_{m-1} \left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}, \dots \frac{m-1-n}{m}; \frac{x^{-m} m^m}{(m-1)^{(m-1)}} \\ \frac{-r-n}{m-1}, \frac{1-r-n}{m-1}, \dots \frac{m-2-r-n}{m-1}; \end{matrix} \right] \tag{5.6}$$

So, for $m=2$ and $r=0$, we obtain the following formula

$$V_n(x) = x^n {}_2F_1 \left[\begin{matrix} \frac{-n}{2}, \frac{1-n}{2}; \frac{4}{x^2} \\ -n; \end{matrix} \right] \tag{5.7}$$

since $V_{n,2}^0(x) \equiv V_n(x)$.

The s^{th}– convolutions $\Omega_{n,m}^s(x)$ we define by

$$F_s^m(t) = (1-t^m)(1-xt+t^m)^{-(s+1)} = \sum_{n=1}^{\infty} \Omega_{n,m}^s(x) t^n. \tag{5.8}$$

Hence, we find that polynomials $\Omega_{n,m}^s(x)$ satisfy the recurrence relation

$$n\Omega_{n,m}^s(x) = x(n+s)\Omega_{n-1,m}^s(x) - 2m(s+1)\Omega_{n-m,m}^s(x) \tag{5.9}$$

$$= x(s+1)(2m-n)\Omega_{n-1-m,m}^s(x) + (n-2m)\Omega_{n-2m,m}^s(x). \tag{5.10}$$

Also, from (5.8), we find the following formula

$$\Omega^s_{n,m}(x) = \sum_{j=0}^{[n/m]} \binom{s+1}{j} V^s_{n-mj,m}(x). \tag{5.11}$$

Mixed convolutions $v^{r,s}_{n,m}(x)$ are defined by

$$S^m(t) = \frac{(1-t^m)^{s+1}}{(1-xt+t^m)^{r+s+2}} = \sum_{n=1}^{\infty} v^{r,s}_{n,m}(x)t^n. \tag{5.12}$$

From (5.12) we get the following formulas:

$$\begin{aligned} S^m(t) &= \frac{(1-t^m)^{s+1}}{(1-xt+t^m)^{s+1}} \cdot \frac{1}{(1-xt+t^m)^{r+1}} \\ &= \left(\sum_{n=1}^{\infty} \Omega^s_{n,m}(x)t^n\right)\left(\sum_{n=1}^{\infty} V^r_{n,m}(x)t^n\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \Omega^s_{n-k,m}(x)V^r_{k,m}(x)\right)t^n. \end{aligned}$$

By (5.12), using the well–known manner, we obtain the recurrence relation

$$\begin{aligned} &nv^{r,s}_{n,m}(x) = x(r+s+2)v^{r+1,s}_{n-1,m}(x) \\ &-m(s+1)v^{r+1,s-1}_{n-m,m}(x) - m(r+s+2)v^{r+1,s}_{n-m,m}(x). \end{aligned}$$

Again from (5.12) we find that

$$\sum_{n=1}^{\infty} v^{r,s}_{n,m}(x)t^n = (1-t^m)^{s+1}\sum_{n=1}^{\infty} V^{r+s+1}_{n,m}(x)t^n. \tag{5.13}$$

Furthermore, for $r = s$ in (5.13), we get the following representation

$$v^{s,s}_{n,m}(x) = \sum_{k=1}^{[n/m]} (-1)^k \binom{s+1}{k} V^{2s+1}_{n-mk,m}(x). \tag{5.14}$$

Hence, for $s = 0$ (and respectively for $r = 0$), we have:

$$v^{r,0}_{n,m}(x) = V^r_{n,m}(x), \text{ the rth –convolutions of } V_{n,m}(x); \tag{5.15}$$

$$v^{0,s}_{n,m}(x) = \Omega^s_{n,m}(x), \text{ the sth–convolutions of } \Omega_{n,m}(x). \tag{5.16}$$

Thus, for $m = 2$ in (5.15) and (5.16), we get, repsectively: $v^{r,0}_{n,2}(x) = V^r_n(x)$, the r^{th}–convolutions of $V_n(x)$; and $v^{0,s}_{n,2}(x) = \Omega^s_n(x)$, the s^{th}–convolutions of $\Omega_n(x)$.

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