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Pontryagin's Maximum Principle for the Loewner Equation in Higher Dimensions

Oliver Roth

Abstract. In this paper we develop a variational method for the Loewner equation in higher dimensions. As a result we obtain a version of Pontryagin's maximum principle from optimal control theory for the Loewner equation in several complex variables. Based on recent work of Arosio, Bracci, and Wold, we then apply our version of the Pontryagin maximum principle to obtain first-order necessary conditions for the extremal mappings for a wide class of extremal problems over the set of normalized biholomorphic mappings on the unit ball in $\mathbb{C}^n$.

1 Introduction

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$ denote the unit ball of $\mathbb{C}^n$ with respect to the euclidean norm $\|\cdot\|$ and let $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ be the vector space of all holomorphic maps from $\mathbb{B}^n$ into $\mathbb{C}^n$. The set

$$\mathcal{S}_n := \{f \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n) : f(0) = 0, df_0 = \mathrm{id}, f \text{ univalent}\}$$

of normalized biholomorphic mappings on $\mathbb{B}^n$ was introduced by H. Cartan [9]. One of the main problems when dealing with univalent mappings in the class $\mathcal{S}_n$ in dimensions $n > 1$ is the fact that there is no Riemann mapping theorem available. In particular, this makes it fairly difficult to construct variations of a given map in the class $\mathcal{S}_n$.

The aim of this paper is to develop a variational method that works effectively for univalent mappings that can be obtained as solutions of Loewner-type differential equations. We present the details only for the class $\mathcal{S}_n^0 \subset \mathcal{S}_n$ of all mappings that admit a so-called parametric representation by means of the Loewner equation. This class was introduced by I. Graham, G. Kohr *et al.* (see *e.g.*, [18, 20]) and is obtained in a most natural way by generalizing the classical one-dimensional Loewner equation(see [24]) to higher dimensions. We note that the approach of this paper can also be used for other more general Loewner-type equations, *e.g.*, for the class of mappings that have a so-called A-parametric representation (see [12, 18, 19]) and also for the various Loewner equations in the unit disk and complete hyperbolic manifolds, which have recently been studied intensively (see [1–7]).

We now give a short account of the results of this paper and start by introducing some notation.

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Definition 1.1 Let

$$\mathcal{M}_n := \left\{ h \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n) : h(0) = 0, dh_0 = -\operatorname{id}, \operatorname{Re}\langle h(z), z\rangle \leq 0 \text{ for all } z \in \mathbb{B}^n \right\}.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product of $\mathbb{C}^n$. Let

$$\mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}.$$

A *Herglotz vector field in the class* $\mathcal{M}_n$ is a mapping $G\colon \mathbb{B}^n \times \mathbb{R}^+ \to \mathbb{C}^n$ such that

(i) $G(z, \cdot)$ is measurable on $\mathbb{R}^+$ for every $z \in \mathbb{B}^n$, and
(ii) $G(\cdot, t) \in \mathcal{M}_n$ for almost every $t \in \mathbb{R}^+$.

It is not difficult to show that a mapping $h \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ satisfying $h(0) = 0$ and $dh_0 = -\operatorname{id}$ belongs to $\mathcal{M}_n$ if and only if $\operatorname{Re}\langle -h(z), z\rangle > 0$ for all $z \in \mathbb{B}^n \backslash \{0\}$; see [8, Remark 2.1]. In particular, the set $\mathcal{M}_n$ is exactly the class $-\mathcal{M}$ as defined *e.g.*, in [20, p. 203]. Hence, $\mathcal{M}_n$ is a compact subset of $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ (see [20, Theorem 6.1.39]). This fact will play an important role in this paper.

Definition 1.2 (The Loewner Equation on the unit ball $\mathbb{B}^n$) Let $G(z, t)$ be a Herglotz vector field in the class $\mathcal{M}_n$. We denote by φ_t^G the unique solution φ_t of the Loewner ODE

$$\begin{aligned} \dot{\varphi}_t(z) &= G(\varphi_t(z), t) \quad \text{for a.e. } t \geq 0, \\ \varphi_0(z) &= z \in \mathbb{B}^n. \end{aligned} \tag{1.1}$$

For any Herglotz vector field $G(z, t)$ in the class $\mathcal{M}_n$, the limit

$$f^G := \lim_{t\to\infty} e^t \varphi_t^G$$

exists locally uniformly in $\mathbb{B}^n$ and belongs to $\mathcal{S}_n$; see [20, Thm. 8.1.5]. We can therefore define

$$\mathcal{S}_n^0 := \left\{ f \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n) \mid f = f^G \text{ for some Herglotz vector field } G \text{ in the class } \mathcal{M}_n \right\}.$$

The class $\mathcal{S}_n^0$ is exactly the class of mappings in $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ which have a *parametric representation* as introduced by Graham, Hamada, and Kohr [17, Definition 1.5]; see also [20, 21]. It is known that the class $\mathcal{S}_n^0$ is compact (see [20, Corollary 8.3.11]) and that $e^t\varphi_t^G \in \mathcal{S}_n^0$ for all $t \in \mathbb{R}_0^+ := \{t \in \mathbb{R} : t \geq 0\}$ and every Herglotz vector field G in the class $\mathcal{M}_n$ (see [32, Lemma 2.6]). Hence one may think of $\mathcal{S}_n^0$ as the "reachable set" of the Loewner equation (1.1).

Theorem 1.3 (A variational formula in $\mathcal{S}_n^0$) *Let $f \in \mathcal{S}_n^0$. Suppose that $G(z, t)$ is a Herglotz vector field in the class $\mathcal{M}_n$ such that $f = f^G$. Then for almost every $t \geq 0$ and any $h \in \mathcal{M}_n$ there exists a family of mappings $f^\varepsilon \in \mathcal{S}_n^0$ such that*

$$f^\varepsilon(z) = f(z) + \varepsilon\, d(f)_z \cdot \left[d(\varphi_t^G)_z \right]^{-1} \left[h\big(\varphi_t^G(z)\big) - G\big(\varphi_t^G(z), t\big) \right] + r^\varepsilon(z).$$

Here, the error term $r^\varepsilon \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ has the property that $r^\varepsilon/\varepsilon \to 0$ locally uniformly in $\mathbb{B}^n$ as $\varepsilon \to 0+$.

The variations f^ε in Theorem 1.3 will be constructed with help of "spike variations". This is a well-known method in control theory and the calculus of variations that goes back at least to Weierstraß. In proving Theorem 1.3 we shall show that it is possible to modify this technique in such a way that it can be applied for the infinite-dimensional Fréchet space $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ (endowed with the standard compact-open topology). We note that a different variational technique in $\mathcal{S}_n^0$ has recently been developed by Bracci, Graham, Hamada, and Kohr [8]. Theorem 1.3 has the advantage that it works for any mapping $f \in \mathcal{S}_n^0$, while the method of [8] is restricted to those mappings in $\mathcal{S}_n^0$ which can be embedded in a so-called "geräumig" Loewner chain; see [8] for details.

One main field of application of the variational formula of Theorem 1.3 is the study of extremal problems in the class $\mathcal{S}_n^0$. We call a mapping $F \in \mathcal{S}_n^0$ an *extremal mapping* for a functional $\Phi\colon \mathcal{S}_n^0 \to \mathbb{C}$ if $\operatorname{Re}\Phi(f) \le \operatorname{Re}\Phi(F)$ for every $f \in \mathcal{S}_n^0$. Here and henceforth we assume that the functional $\Phi\colon \mathcal{S}_n^0 \to \mathbb{C}$ is complex differentiable in the sense of R. Hamilton's Fréchet space calculus as developed in [23] (see Definition 4.1 for details).

Theorem 1.4 *Let $F \in \mathcal{S}_n^0$ be an extremal mapping for a functional $\Phi\colon \mathcal{S}_n^0 \to \mathbb{C}$ with complex derivative L at F. Suppose that $G(z,t)$ is a Herglotz vector field in the class $\mathcal{M}_n$ such that $F = f^G$. For each $t \ge 0$ let L_t be the continuous linear functional on* $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ *defined by*

$$L_t(h) := L\Big(d(F)_z \cdot \big[d(\varphi_t^G)_z \big]^{-1} \cdot h(\varphi_t^G) \Big), \qquad h \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n).$$

Then for almost every $t \ge 0$,

$$\operatorname{Re} L_t(h) \le \operatorname{Re} L_t(G(\,\cdot\,,t)) \quad \text{for all } h \in \mathcal{M}_n.$$

Theorem 1.4 is in fact a version of Pontryagin's maximum principle for the case of the Loewner equation in higher dimensions. It generalizes earlier well-known work on control theory of the Loewner equation in one dimension which has been initiated by Goodman [16], Popov [27] and Friedland and Schiffer [14, 15], and which has been developed into a powerful theory by D. Prokhorov [28–30], see also [31].

At first sight it is not clear that Pontryagin's maximum principle (Theorem 1.4) carries any useful information about the Herglotz vector field $G(\,\cdot\,,t)$ at all, simply because the linear functionals L_t in Theorem 1.4 might be constant on the class $\mathcal{M}_n$. In particular, Theorem 1.4 alone is not sufficient to deduce that $G(\,\cdot\,,t)$ is a support point (see Definition 4.5) in the class $\mathcal{M}_n$. However, referring to a deep result of Docquier and Grauert [10], it has recently been observed by Arosio, Bracci, and Wold [5] that all domains $\varphi_t^G(\mathbb{B}^n)$ are *Runge domains*. Using this Runge property we will show in Proposition 4.6 that if L is not constant on $\mathcal{S}_n^0$, then L_t is never constant on $\mathcal{M}_n$. In combination with Pontryagin's maximum principle in the form of Theorem 1.4, we are therefore led to the following necessary condition for extremal problems in the class $\mathcal{S}_n^0$.

Theorem 1.5 *Let $F \in \mathcal{S}_n^0$ be an extremal mapping for a functional $\Phi\colon \mathcal{S}_n^0 \to \mathbb{C}$ with complex derivative L at F. Suppose that L is not constant on $\mathcal{S}_n^0$. If $G(z,t)$ is a Herglotz* *vector field in the class $\mathcal{M}_n$ such that $F = f^G$, then $G(\cdot, t)$ is a support point in the class $\mathcal{M}_n$ for almost every $t \geq 0$.*

Roughly speaking, Theorem 1.5 says that if a Herglotz vector field $G(z,t)$ generates an extremal mapping in the class $\mathcal{S}_n^0$ via the Loewner equation, then for almost every $t \geq 0$ the mapping $G(\cdot, t) \in \mathcal{M}_n$ itself has to be extremal in the class $\mathcal{M}_n$. As an illustration of the use of Theorem 1.5, we prove in Corollary 4.9 a generalization of a recent result due to Bracci, Graham, Hamada, and Kohr [8] about support points in $\mathcal{S}_n^0$.

Finally, we point out another consequence of Theorem 1.3.

Theorem 1.6 *Let $F \in \mathcal{S}_n^0$ be an extremal mapping for a functional $\Phi\colon \mathcal{S}_n^0 \to \mathbb{C}$ with complex derivative L at F. Then*

$$\max_{h \in \mathcal{M}_n} \operatorname{Re} L(d(F)_z \cdot h) = -\operatorname{Re} L(F).$$

Theorem 1.6 extends a result of Pommerenke (see [26, p. 185]), which deals with the case of dimension $n = 1$ (and functionals of finite degree), to the cases $n > 1$ and arbitrary complex differentiable functionals. We note that the case $n = 1$ allows a fairly elementary proof, which is based on the "lucky accident" (see [11, p. 231]) that the Koebe functions

$$k_\zeta(z) := \frac{z}{(1+\bar{\zeta}z)^2}, \qquad \zeta \in \partial\mathbb{B}^1,$$

generate the set $\operatorname{ext}\mathcal{M}_1$ of extreme points of $\mathcal{M}_1$ via

$$\operatorname{ext}\mathcal{M}_1 = \Big\{ -z\frac{\zeta+z}{\zeta-z} : \zeta \in \partial\mathbb{B}^1 \Big\} = \Big\{ -\big[d(k_\zeta)_z\big]^{-1} \cdot k_\zeta(z) : \zeta \in \partial\mathbb{B}^1 \Big\}.$$

For $n > 1$, however, the set $\operatorname{ext}\mathcal{M}_n$ of extreme points of $\mathcal{M}_n$ is not known (see [33] for recent results in this direction), so we employ a completely different approach for the proof of Theorem 1.6.

This paper is organized in the following way. We start in Section 2 by constructing variations of evolution families for the Loewner equation in higher dimensions. In Section 3 we generalize this result to produce variations in the class $\mathcal{S}_n^0$ and prove Theorem 1.3. We also produce variations of normal Loewner chains, which partly extend the recent results in [8]. In Section 4 we apply the results of Sections 2 and 3 to study extremal problems in the class $\mathcal{S}_n^0$, and we prove Theorems 1.4, 1.5, and 1.6.

2 Variations of Evolution Families

In this section, we construct variations of Loewner evolution families.

Definition 2.1 Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$. For fixed $s \geq 0$ denote by $\varphi_{s,t}^G$ the solution to

$$\dot{\varphi}_{s,t}(z) = G(\varphi_{s,t}(z), t) \quad \text{for a.e. } t \geq s, \qquad \varphi_{s,s}(z) = z \in \mathbb{B}^n. \tag{2.1}$$

We call $(\varphi_{s,t}^G)_{0 \leq s \leq t}$ *the evolution family generated by $G(z,t)$.*

Lemma 2.2 *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$. Then there exists a set $E_G \subseteq \mathbb{R}^+$ of zero measure such that for all $t \in (0,\infty)\backslash E_G$ the condition*

$$G(z,t) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} G(z,\tau)\, d\tau \tag{2.2}$$

holds locally uniformly with respect to $z \in \mathbb{B}^n$.

Proof We fix $z \in \mathbb{B}^n$. Since $G(\,\cdot\,,t) \in \mathcal{M}_n$ for almost every $t > 0$ and $\mathcal{M}_n$ is a compact subset of $\operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$, the measurable function $t \mapsto G(z,t)$ is (essentially) bounded on the interval $(0,\infty)$. Therefore, there exists a set $E_G(z) \subseteq \mathbb{R}^+$ of zero measure such that condition (2.2) holds for all $t \in \mathbb{R}^+\backslash E_G(z)$. Now choose a dense countable set $A \subseteq \mathbb{B}^n$ and set $E_G := \cup_{a\in A} E_G(a)$. Then E_G has zero measure and (2.2) holds for every $t \in \mathbb{R}^+\backslash E_G$ and every point z in the dense subset $A \subseteq \mathbb{B}^n$. Since $\mathcal{M}_n$ is a normal family and $G(\,\cdot\,,t) \in \mathcal{M}_n$ for a.e. $t \geq 0$, this implies that (2.2) holds locally uniformly in $\mathbb{B}^n$ for every fixed $t \in \mathbb{R}^+\backslash E_G$ by Vitali's theorem. ∎

Remark 2.3 We call the set $R_G := \mathbb{R}^+\backslash E_G$ the *regular set* of the Herglotz vector field $G(z,t)$, and every $T \in R_G$ is called a *regular point* for $G(z,t)$. Note that if T is a regular point for $G(z,t)$ and $\varphi_{s,t} := \varphi_{s,t}^G$, then Lemma 2.2 implies that for any $s < T$,

$$G(\varphi_{s,T}(z),T) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{T-\varepsilon}^{T} G(\varphi_{s,\tau}(z),\tau)\, d\tau$$

locally uniformly with respect to $z \in \mathbb{B}^n$, since $\varphi_{s,\,\cdot}(z)$ is absolutely continuous on compact intervals of $\mathbb{R}_0^+$ locally uniformly with respect to $z \in \mathbb{B}^n$.

We can now state the main result of this section.

Theorem 2.4 *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$ with associated evolution family $\varphi_{s,t} := \varphi_{s,t}^G$ and let $T \in R_G$ be a regular point. Then for any $h \in \mathcal{M}_n$ and any $\varepsilon \in (0,T)$ there exists an evolution family $(\varphi_{s,t}^\varepsilon)_{0\leq s\leq t}$ such that*

$$\varphi_{s,t}^\varepsilon = \varphi_{s,t} + \varepsilon\, \alpha_{s,t}^h + o_{s,t}^\varepsilon,$$

where

$$\alpha_{s,t}^h = \begin{cases} 0 & \text{if } s \leq t < T-\varepsilon \text{ or } T \leq s \leq t,\\ d(\varphi_t)_z \cdot [d(\varphi_T)_z]^{-1} \cdot [h(\varphi_T) - G(\varphi_T,T)] & \text{if } s < T \leq t. \end{cases}$$

Here, $o_{s,t}^\varepsilon \in \operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$ indicates a term such that

$$\lim_{\varepsilon\to 0+} \frac{o_{s,t}^\varepsilon}{\varepsilon} = 0 \quad \text{locally uniformly in } \mathbb{B}^n$$

for any fixed s,t such that $s < T \leq t$.

In order to prove Theorem 2.4 we are going to adapt the standard method of *needle or spike variations* for the particular case of the Loewner equation (2.1).

Definition 2.5 (Needle variations) Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$, $h \in \mathcal{M}_n$ and $T > 0$. For each $\varepsilon \in (0,T)$ let

$$G_\varepsilon(\,\cdot\,,t) := G_{\varepsilon,h,T}(\,\cdot\,,t) := \begin{cases} G(\,\cdot\,,t) & \text{if } t \in \mathbb{R}^+ \setminus (T-\varepsilon, T), \\ h & \text{if } t \in (T-\varepsilon, T). \end{cases}$$

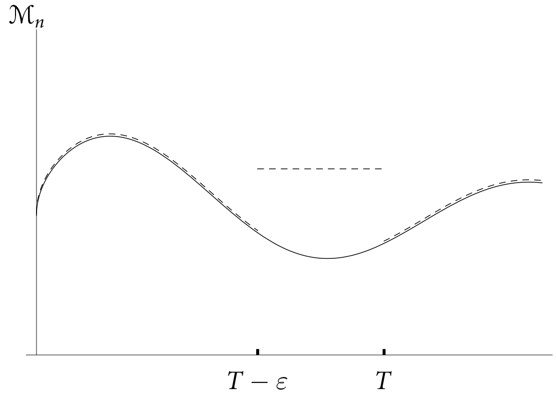

Figure 1: The graphs of $G(\,\cdot\,,t)$ (solid) and $G_\varepsilon(\,\cdot\,,t)$ (dashed).

We call the Herglotz vector fields $G_\varepsilon(z,t)$ in the class $\mathcal{M}_n$ the *needle variations of* $G(z,t)$ *with data* (T,h). We also call the evolution families $(\varphi^\varepsilon_{s,t}) := (\varphi^{G_\varepsilon}_{s,t})$ the *needle variations of the evolution family* $(\varphi^G_{s,t})$ *with data* (T,h).

Remark 2.6 Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$, $h \in \mathcal{M}_n$ and let $T > 0$. Since $G_\varepsilon(\,\cdot\,,t) = G(\,\cdot\,,t)$ for any $t \notin (T-\varepsilon, T)$, we immediately get that

$$\varphi^\varepsilon_{s,t} = \varphi_{s,t} \quad \text{if} \quad s \le t \le T-\varepsilon \quad \text{or} \quad T \le s \le t.$$

In particular, we have

$$\varphi^\varepsilon_{s,t}(z) = \varphi^\varepsilon_{s,T-\varepsilon}(z) + \int_{T-\varepsilon}^t G_\varepsilon(\varphi^\varepsilon_{s,\tau}(z),\tau)\,d\tau = \varphi_{s,T-\varepsilon}(z) + \int_{T-\varepsilon}^t h\left(\varphi^\varepsilon_{s,\tau}(z)\right)\,d\tau \tag{2.3}$$

if $s \le T-\varepsilon \le t \le T$.

In what follows we use the notation $\overline{\mathbb{B}}^n_r := \{z \in \mathbb{C}^n : \|z\| \le r\}$.

Lemma 2.7 (Convergence of needle variations) *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$, $h \in \mathcal{M}_n$ and $T > 0$. Denote by $(\varphi^\varepsilon_{s,t})$ the needle variations of $(\varphi_{s,t}) := (\varphi^G_{s,t})$ with data (T,h). Then for fixed $s \ge 0$, we have*

$$\lim_{\varepsilon \to 0+} \varphi^\varepsilon_{s,t}(z) = \varphi_{s,t}(z)$$

uniformly for $(z,t) \in \overline{\mathbb{B}}^n_r \times [s,\infty)$ for any $r \in (0,1)$.

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Proof In view of Remark 2.6, we may assume $s < T$. Fix $r \in (0,1)$. Since $\mathcal{M}_n$ is compact, there is a constant $L_r > 0$ such that $\|g(z) - g(z')\| \le L_r\|z - z'\|$ for any $g \in \mathcal{M}_n$ and every $z, z' \in \overline{\mathbb{B}}^n_r$, see [20, p. 298]. For every $t \in [s,T]$ we have $\|\varphi_{s,t}(z)\| \le \|z\|$, and therefore we get the following estimate from identity (2.3) and the fact that $\varphi_{s,t}$ is a solution to the evolution equation (2.1):

$$\begin{aligned}
&\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\| \\
&\quad = \Big\| \varphi_{s,T-\varepsilon}(z) + \int_{T-\varepsilon}^{t} h(\varphi^{\varepsilon}_{s,\tau}(z))\,d\tau - \varphi_{s,T-\varepsilon}(z) - \int_{T-\varepsilon}^{t} G(\varphi_{s,\tau}(z),\tau)\,d\tau \Big\| \\
&\quad = \Big\| \int_{T-\varepsilon}^{t} h(\varphi^{\varepsilon}_{s,\tau}(z)) - G(\varphi_{s,\tau}(z),\tau)\,d\tau \Big\| \\
&\quad \le \int_{T-\varepsilon}^{t} \|h(\varphi^{\varepsilon}_{s,\tau}(z)) - G(\varphi^{\varepsilon}_{s,\tau}(z),\tau)\|\,d\tau + \int_{T-\varepsilon}^{t} \|G(\varphi^{\varepsilon}_{s,\tau}(z),\tau) - G(\varphi_{s,\tau}(z),\tau)\|\,d\tau \\
&\quad \le 2L_r\|z\|(t - T + \varepsilon) + L_r \int_{T-\varepsilon}^{t} \|\varphi^{\varepsilon}_{s,\tau}(z) - \varphi_{s,\tau}(z)\|\,d\tau.
\end{aligned}$$

Using the well-known Gronwall lemma (see [13, p. 198]), this implicit estimate for $\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\|$ leads to the explicit estimate

$$\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\| \le 2L_r\varepsilon\|z\|(1 + L_r\varepsilon e^{L_r\varepsilon}) \quad \text{for every } t \in [s,T]. \tag{2.4}$$

In view of the semigroup property $\varphi^{\varepsilon}_{T,t} \circ \varphi^{\varepsilon}_{s,T} = \varphi^{\varepsilon}_{s,t}$, we therefore get that for all $t > T$,

$$\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\| = \|\varphi_{T,t}(\varphi^{\varepsilon}_{s,T}(z)) - \varphi_{T,t}(\varphi_{s,T}(z))\| \le C_r\|\varphi^{\varepsilon}_{s,T}(z) - \varphi_{s,T}(z)\|, \tag{2.5}$$

where $C_r > 0$ is a constant such that $\|\varphi_{s,t}(z) - \varphi_{s,t}(z')\| \le C_r\|z - z'\|$ for all $t \ge s$ and all $z, z' \in \overline{\mathbb{B}}^n_r$. If we combine (2.5) with (2.4), we finally have

$$\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\| \le \gamma_r\varepsilon \quad \text{for all } \|z\| \le r \text{ and all } t \ge s, \tag{2.6}$$

where γ_r depends only on r. This completes the proof of Lemma 2.7. ■

Lemma 2.7 says that the needle variations $(\varphi^{\varepsilon}_{s,t})$ of $(\varphi_{s,t})$ with data (T,h) form a "continuous deformation" of the evolution family $(\varphi_{s,t})$. If $T \in R_G$ is in addition a *regular* point of $G(z,t)$, then this deformation is actually "differentiable" in the following sense.

Theorem 2.8 *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$, let $T \in R_G$ and $h \in \mathcal{M}_n$. For fixed $s \in [0,T]$ denote by $\varphi^{\varepsilon}_{s,t}$ the needle variations of $\varphi_{s,t} := \varphi^{G}_{s,t}$ with data (T,h). Then*

$$\varphi^{\varepsilon}_{s,t} = \varphi_{s,t} + \varepsilon\, d(\varphi_{s,t})_z \cdot \big[d(\varphi_{s,T})_z\big]^{-1} \cdot \big[h(\varphi_{s,T}) - G(\varphi_{s,T},T)\big] + o^{\varepsilon}_{s,t}$$

for any $t \ge T$. Here, $o^{\varepsilon}_{s,t}$ indicates a term, which divided by ε, tends to 0 locally uniformly in $\mathbb{B}^n$ for each fixed $t \ge T$ as $\varepsilon \to 0+$.

Proof Using (2.3), we have

$$\frac{\varphi^{\varepsilon}_{s,T}(z)-\varphi_{s,T}(z)}{\varepsilon}=\frac{\varphi^{\varepsilon}_{s,T}(z)-\varphi_{s,T-\varepsilon}(z)}{\varepsilon}-\frac{\varphi_{s,T}(z)-\varphi_{s,T-\varepsilon}(z)}{\varepsilon}$$
$$=\frac{1}{\varepsilon}\int_{T-\varepsilon}^{T}h(\varphi^{\varepsilon}_{s,\tau}(z))\,d\tau-\frac{1}{\varepsilon}\int_{T-\varepsilon}^{T}G(\varphi_{s,\tau}(z),\tau)\,d\tau.$$

Since $T\in R_G$, we see by using Remark 2.3 and Lemma 2.7 that

$$\frac{\partial^{+}\varphi^{\varepsilon}_{s,T}(z)}{\partial\varepsilon}\Big|_{\varepsilon=0}:=\lim_{\varepsilon\to 0+}\frac{\varphi^{\varepsilon}_{s,T}(z)-\varphi_{s,T}(z)}{\varepsilon}=h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T),$$

where the limit exists locally uniformly in $\mathbb{B}^n$. This proves the claim for $t=T$. We can now handle the general case $t\geq T$. By what we have just proved, we know that $\varphi^{\varepsilon}_{s,t}$ is a solution to

(2.7)
$$\begin{aligned}&\dot{\varphi}^{\varepsilon}_{s,t}(z)=G(\varphi^{\varepsilon}_{s,t}(z),t), \qquad \text{for } t\geq T\\ &\varphi^{\varepsilon}_{s,T}(z)=\varphi_{s,T}(z)+\varepsilon\big[h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T)\big]+r_{\varepsilon}(z),\end{aligned}$$

where $r_\varepsilon\in\operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$ such that $r_\varepsilon/\varepsilon\to 0$ locally uniformly in $\mathbb{B}^n$ as $\varepsilon\to 0+$. We now make use of a standard result from ODE-theory about "differentiability with respect to initial conditions" and differentiate (2.7) with respect to ε; see [25, Theorem 1A, p. 57]. This way, we find that

$$\psi_t(z):=\frac{\partial^{+}\varphi^{\varepsilon}_{s,t}(z)}{\partial\varepsilon}\Big|_{\varepsilon=0}$$

is a solution to the initial value problem

(2.8)
$$\begin{aligned}&\dot{\psi}_t(z)=\frac{\partial G}{\partial z}(\varphi_{s,t}(z),t)\cdot\psi_t(z), \qquad t\geq T,\\ &\psi_T(z)=h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T).\end{aligned}$$

On the other hand, by differentiating the evolution equation (2.1) with respect to z, it is easy to see that

$$t\mapsto d(\varphi_{s,t})_z\cdot\big[d(\varphi_{s,T})_z\big]^{-1}\cdot\big[h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T)\big]$$

is also a solution to (2.8). By uniqueness, we deduce that for every $t\geq T$,

$$\frac{\partial^{+}\varphi^{\varepsilon}_{s,t}(z)}{\partial\varepsilon}\Big|_{\varepsilon=0}=d(\varphi_{s,t})_z\cdot\big[d(\varphi_{s,T})_z\big]^{-1}\cdot\big[h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T)\big].$$

We have hence shown that for fixed $t\geq T$,

$$\varphi^{\varepsilon}_{s,t}(z)=\varphi_{s,t}(z)+\varepsilon\,d(\varphi_{s,t})_z\cdot\big[d(\varphi_{s,T})_z\big]^{-1}\cdot\big[h(\varphi_{s,T}(z))-G(\varphi_{s,T}(z),T)\big]+o^{\varepsilon}_{s,t}(z),$$

where

$$\lim_{\varepsilon\to 0+}\frac{o^{\varepsilon}_{s,t}(z)}{\varepsilon}=0 \qquad \text{for every } z\in\mathbb{B}^n.$$

It is not difficult to prove that this limit actually exists locally uniformly with respect to $z\in\mathbb{B}^n$. In fact, note that for fixed $0<r<1$, $\varphi^{\varepsilon}_{s,t}(z)\in\overline{\mathbb{B}}^n_r$ for every $z\in\overline{\mathbb{B}}^n_r$ and all

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$0 \le s \le t$. Again using the compactness of $\mathcal{M}_n$, we see that there is a constant $L_r > 0$ such that $\|g(z) - g(z')\| \le L_r \|z - z'\|$ for all $g \in \mathcal{M}_n$ and all $z, z' \in \overline{\mathbb{B}}^n_r$. Therefore,

$$\|\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)\| = \left\| \varphi^{\varepsilon}_{s,T}(z) - \varphi_{s,T}(z) + \int_T^t \left(G(\varphi^{\varepsilon}_{s,\tau}(z), \tau) - G(\varphi_{s,\tau}(z), \tau) \right) d\tau \right\|$$
$$\le \|\varphi^{\varepsilon}_{s,T}(z) - \varphi_{s,T}(z)\| + L_r \int_T^t \|\varphi^{\varepsilon}_{s,\tau}(z) - \varphi_{s,\tau}(z)\| \, d\tau.$$

Now, Gronwall's lemma implies that

$$\left\| \frac{\varphi^{\varepsilon}_{s,t}(z) - \varphi_{s,t}(z)}{\varepsilon} \right\| \le \left\| \frac{\varphi^{\varepsilon}_{s,T}(z) - \varphi_{s,T}(z)}{\varepsilon} \right\| e^{L_r(t-T)}$$
$$= \left\| h(\varphi_{s,T}(z)) - G(\varphi_{s,T}(z), T) + \frac{r_\varepsilon(z)}{\varepsilon} \right\| e^{L_r(t-T)}.$$

Hence, $o^{\varepsilon}_{s,t}(z)/\varepsilon$ is uniformly bounded on $\overline{\mathbb{B}}^n_r$ as $\varepsilon \to 0+$. Since we have already proved that $o^{\varepsilon}_{s,t}(z)/\varepsilon \to 0$ pointwise in $\mathbb{B}^n$, Vitali's theorem shows that actually $o^{\varepsilon}_{s,t}/\varepsilon \to 0$ locally uniformly in $\mathbb{B}^n$. ∎

3 Variations in $\mathcal{S}^0_n$ and Variations of Normal Loewner Chains

By definition, every $f \in \mathcal{S}^0_n$ has the form

$$f = \lim_{t \to \infty} e^t \varphi^G_{0,t}$$

for some Herglotz vector field $G(z,t)$ in the class $\mathcal{M}_n$. Therefore the following result for $s = 0$ and $t = \infty$ is exactly the statement of Theorem 1.3 and provides us with a variational formula in the class $\mathcal{S}^0_n$.

Theorem 3.1 *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$, let $T \in R_G$, and let $h \in \mathcal{M}_n$. For fixed $s \in [0,T]$ consider the needle variations $(\varphi^{\varepsilon}_{s,t})$ of $(\varphi_{s,t}) := (\varphi^G_{s,t})$ with data (T, h). Then*

$$e^t \varphi^{\varepsilon}_{s,t} = e^t \varphi_{s,t} + \varepsilon \, d(e^t \varphi_{s,t})_z \cdot \left[d(e^T \varphi_{s,T})_z \right]^{-1} \cdot e^T \left[h(\varphi_{s,T}) - G(\varphi_{s,T}, T) \right] + r^{\varepsilon}_{s,t} \tag{3.1}$$

for any $t \in [T, \infty]$. Here, the error term $r^{\varepsilon}_{s,t} \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ has the property that $r^{\varepsilon}_{s,t}/\varepsilon \to 0$ locally uniformly in $\mathbb{B}^n$ for every fixed $t \in [T, \infty]$ as $\varepsilon \to 0+$.

Remark 3.2 Note that Theorem 3.1 holds in particular for $t = \infty$, where we have used the convenient notation

$$e^t \varphi^{\varepsilon}_{s,t} := \lim_{\tau \to \infty} e^{\tau} \varphi^{\varepsilon}_{s,\tau} \quad \text{for } t = \infty\,.$$

In this case, we define the error term $r^{\varepsilon}_{s,\infty} \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ as

$$r^{\varepsilon}_{s,\infty} := \lim_{t \to \infty} r^{\varepsilon}_{s,t}\,.$$

This limit clearly exists locally uniformly in $\mathbb{B}^n$ in view of (3.1).

Proof of Theorem 3.1 Let $r^{\varepsilon}_{s,t}$ be defined by (3.1). We need to show that $r^{\varepsilon}_{s,t}/\varepsilon \to 0$ locally uniformly in $\mathbb{B}^n$ for every fixed $t \in [T, \infty]$ as $\varepsilon \to 0+$. The cases $t < \infty$ follow directly from Theorem 2.8, so we only need to deal with the case $t = \infty$.

(i) In order to handle the error term $r^{\varepsilon}_{s,\infty}$ we first derive a convenient expression for the error term $r^{\varepsilon}_{s,t}$ for all $0 \le s \le t < \infty$. Let $v^{\varepsilon}_{s,t}(z) := e^t \varphi^{\varepsilon}_{s,t}(z)$ and $v^0_{s,t}(z) := e^t \varphi_{s,t}(z)$. If we set $\widetilde{G}(z,t) := z + e^t G(e^{-t}z, t)$, then

$$\begin{aligned} &\dot{v}^{\varepsilon}_{s,t}(z) = \widetilde{G}(v^{\varepsilon}_{s,t}(z), t) \quad \text{for } t \ge T, \\ &v^{\varepsilon}_{s,T}(z) = v^0_{s,T}(z) + \varepsilon e^T \big[h(\varphi_{s,T}(z)) - G(\varphi_{s,T}(z), T) \big] + r^{\varepsilon}_{s,T}(z), \end{aligned} \tag{3.2}$$

where $r^{\varepsilon}_{s,T}/\varepsilon \to 0$ locally uniformly in $\mathbb{B}^n$ as $\varepsilon \to 0+$ by applying Theorem 2.8 for $t = T$. For $t \ge T$ let

$$E^{\varepsilon}_{s,t}(z) := \int_0^1 \frac{\partial \widetilde{G}}{\partial z}\big(v^0_{s,t}(z) + \alpha\big(v^{\varepsilon}_{s,t}(z) - v^0_{s,t}(z)\big), t\big)\, d\alpha, \quad E^0_{s,t}(z) = \frac{\partial \widetilde{G}}{\partial z}\big(v^0_{s,t}(z), t\big),$$

so that the difference $\Psi^{\varepsilon}_{s,t}(z) := v^{\varepsilon}_{s,t}(z) - v^0_{s,t}(z)$ has the property

$$\begin{aligned} &\dot{\Psi}^{\varepsilon}_{s,t}(z) = E^{\varepsilon}_{s,t}(z) \cdot \Psi^{\varepsilon}_{s,t}(z) \quad \text{for } t \ge T, \\ &\Psi^{\varepsilon}_{s,T}(z) = \varepsilon e^T \big[h(\varphi_{s,T}(z)) - G(\varphi_{s,T}(z), T) \big] + r^{\varepsilon}_{s,T}(z). \end{aligned} \tag{3.3}$$

In order to analyze the behaviour of $\Psi^{\varepsilon}_{s,t}$ as $t \to \infty$, we consider the linear matrix-ODE

$$\begin{aligned} &\dot{Y}^{\varepsilon}_{s,t}(z) = E^{\varepsilon}_{s,t}(z) \cdot Y^{\varepsilon}_{s,t}(z) \quad \text{for } t \ge T, \\ &Y^{\varepsilon}_{s,T}(z) = I. \end{aligned} \tag{3.4}$$

The motivation for doing so comes from the observation that in view of (3.3) we can write

$$\begin{aligned} \Psi^{\varepsilon}_{s,t}(z) &= Y^{\varepsilon}_{s,t}(z) \cdot \Psi^{\varepsilon}_{s,T}(z) \\ &= Y^{\varepsilon}_{s,t}(z) \cdot \Big\{ \varepsilon e^T \big[h(\varphi_{s,T}(z)) - G(\varphi_{s,T}(z), T) \big] + r^{\varepsilon}_{s,T}(z) \Big\}. \end{aligned} \tag{3.5}$$

In a similar way, since differentiating (3.2) for $\varepsilon = 0$ with respect to z shows that

$$\frac{d}{dt}\big[d(v^0_{s,t})_z \big] = \frac{\partial \widetilde{G}}{\partial z}\big(v^0_{s,t}(z), t\big) \cdot \big[d(v^0_{s,t})_z \big] = E^0_{s,t}(z) \cdot \big[d(v^0_{s,t})_z \big], \qquad t \ge T,$$

we get

$$d(v^0_{s,t})_z = Y^0_{s,t}(z) \cdot d(v^0_{s,T})_z, \qquad \text{for } t \ge T. \tag{3.6}$$

Now, formulas (3.5) and (3.6) and the definition of the error term $r^{\varepsilon}_{s,t}$ show that

$$r^{\varepsilon}_{s,t}(z) = \varepsilon e^T \big[Y^{\varepsilon}_{s,t}(z) - Y^0_{s,t}(z) \big] \cdot \big[h(\varphi_{s,T}(z)) - G(\varphi_{s,T}(z), T) \big] + Y^{\varepsilon}_{s,t}(z) r^{\varepsilon}_{s,T}(z). \tag{3.7}$$

(ii) We now examine $Y^{\varepsilon}_{s,t}$ with the help of the linear matrix-ODE (3.4) and show that for any $r \in (0,1)$ there exists a constant $M_r > 0$ such that

$$\| Y^{\varepsilon}_{s,t}(z) - Y^0_{s,t}(z) \| \le M_r \varepsilon \text{ and } \| Y^{\varepsilon}_{s,t}(z) \| \le M_r \text{ for all } \|z\| \le r \text{ and all } t \ge T. \tag{3.8}$$

In view of (3.7) this then implies that $r^{\varepsilon}_{s,\infty}(z)/\varepsilon \to 0$ uniformly in $\|z\| \le r$ as $\varepsilon \to 0+$. It therefore remains to prove (3.8). We fix $r \in (0,1)$. In the following, C_r always denotes a constant, which depends only on r, but the value of C_r may be different at each occurence. We first note $\| \operatorname{id} + d(h)_z \| \le C_r \cdot \|z\|$ for all $\|z\| \le r$ and all $h \in \mathcal{M}_n$. This follows from the compactness of $\mathcal{M}_n$ and the normalization $d(h)_0 = -\operatorname{id}$.

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Moreover, $\|\varphi_{s,\tau}(z)\| \le C_r e^{-\tau}$ for all $\tau \ge T$ and all $\|z\| \le r$; see [20, Lemma 8.1.4]. Hence, from the definition of $E^{\varepsilon}_{s,\tau}(z)$ we infer that

$$
\begin{aligned}
(3.9)\qquad \|E^{\varepsilon}_{s,\tau}(z)\| &\le \int_0^1 \Big\| \mathrm{id} + \frac{\partial G}{\partial z}\big(\varphi_{s,\tau}(z) + \alpha\big(\varphi^{\varepsilon}_{s,\tau}(z) - \varphi_{s,\tau}(z)\big), \tau\big) \Big\| \, d\alpha \\
&\le C_r \int_0^1 \big\| \varphi_{s,\tau}(z) + \alpha\big(\varphi^{\varepsilon}_{s,\tau}(z) - \varphi_{s,\tau}(z)\big) \big\| \, d\alpha \\
&\le C_r\, e^{-\tau} \text{ for all } \|z\| \le r \text{ and all } \tau \ge T.
\end{aligned}
$$

In a similar way, we can deduce

$$
\begin{aligned}
(3.10)\qquad \|E^{\varepsilon}_{s,\tau}(z) - E^{0}_{s,\tau}(z)\| &\le C_r \|\varphi^{\varepsilon}_{s,\tau}(z) - \varphi_{s,\tau}(z)\| \\
&= C_r \|\varphi_{T,\tau}(\varphi^{\varepsilon}_{s,T}(z)) - \varphi_{T,\tau}(\varphi_{s,T}(z))\| \\
&\le C_r e^{-\tau} \|\varphi^{\varepsilon}_{s,T}(z) - \varphi_{s,T}(z)\| \\
&\le C_r e^{-\tau} \varepsilon \text{ for all } \|z\| \le r \text{ and all } \tau \ge T.
\end{aligned}
$$

The last estimate comes from (2.6). We are now prepared to prove (3.8). Since $t \mapsto Y^{\varepsilon}_{s,t}(z)$ is a solution to (3.4), we get

$$
\begin{aligned}
\|Y^{\varepsilon}_{s,t}(z) - Y^{0}_{s,t}(z)\| &= \Big\| \int_T^t E^{\varepsilon}_{s,\tau}(z) Y^{\varepsilon}_{s,\tau}(z)\, d\tau - \int_T^{\tau} E^{0}_{s,\tau}(z) V^{0}_{s,\tau}(z)\, d\tau \Big\| \\
&\le \int_T^t \|E^{\varepsilon}_{s,\tau}(z)\| \cdot \|Y^{\varepsilon}_{s,\tau}(z) - Y^{0}_{s,\tau}(z)\| d\tau \\
&\quad + \int_T^t \|E^{\varepsilon}_{s,\tau}(z) - E^{0}_{s,\tau}(z)\| \|Y^{0}_{s,\tau}(z)\|\, d\tau.
\end{aligned}
$$

Now (3.6) shows

$$
Y^{0}_{s,\tau}(z) = e^{\tau} d\left(\varphi_{s,\tau}\right)_z \cdot \left[d(e^T \varphi_{s,T})_z\right]^{-1},
$$

so the inequality $\|\varphi_{s,\tau}(z)\| \le C_r e^{-\tau}$ holds for all $\|z\| \le r$ and all $\tau \ge T$, which leads to $\|d(\varphi_{s,\tau})_z\| \le C_r e^{-\tau}$, therefore implies that $\|Y^{0}_{s,\tau}(z)\| \le C_r$ for all $\tau \ge T$. Hence, in combination with (3.9) and (3.10), we get the implicit estimate

$$
\|Y^{\varepsilon}_{s,t}(z) - Y^{0}_{s,t}(z)\| \le C_r \int_T^t e^{-\tau} \|Y^{\varepsilon}_{s,\tau}(z) - Y^{0}_{s,\tau}(z)\| d\tau + C_r \varepsilon \int_T^t e^{-\tau}\, d\tau,
$$

which is valid for all $\|z\| \le r$ and all $t \ge T$. Again using Gronwall's lemma, we obtain $\|Y^{\varepsilon}_{s,t}(z) - Y^{0}_{s,t}(z)\| \le C_r \varepsilon$ and then also $\|Y^{\varepsilon}_{s,t}(z)\| \le \|Y^{\varepsilon}_{s,t}(z) - Y^{0}_{s,t}(z)\| + \|Y^{0}_{s,t}(z)\| \le C_r$ for all $\|z\| \le r$ and all $t \ge T$. This proves (3.8) and finishes the proof of Theorem 3.1 for the case $t = \infty$. ■

Theorem 3.1 enables us to construct variations for a certain class of Loewner chains. We first recall the basic concepts.

Definition 3.3 A *normalized Loewner chain* $(f_t)_{t \ge 0}$ is a family of univalent mappings $f_t \colon \mathbb{B}^n \to \mathbb{C}^n$ such that $f_t(0) = 0$, $d(f_t)_0 = e^t\, \mathrm{id}$ for all $t \ge 0$ and such that for every $0 \le s \le t$ there exists a holomorphic map $\varphi_{s,t} \colon \mathbb{B}^n \to \mathbb{B}^n$ with $f_s = f_t \circ \varphi_{s,t}$. A normalized Loewner chain $(f_t)_{t \ge 0}$ is called a *normal Loewner chain* if the family $\{e^{-t} f_t\}$ is normal.

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Remark 3.4 We note the following well-known facts; see [20].

(i) If (f_t) is a normalized Loewner chain, then there is a unique Herglotz vector field $G(z,t)$ in the class $\mathcal{M}_n$ such that the Loewner–Kufarev PDE

$$\frac{\partial f_t}{\partial t}(z) = -d(f_t)_z \cdot G(z,t) \tag{3.11}$$

holds.

(ii) If $G(z,t)$ is a Herglotz vector field in the class $\mathcal{M}_n$, we define

$$f_s^G := \lim_{t\to\infty} e^t \varphi_{s,t}^G.$$

Then $(f_t^G)_{t\geq 0}$ is a normal Loewner chain. In fact, $(f_t^G)_{t\geq 0}$ is the unique normal Loewner chain such that f_t^G is a solution to (3.11) for the Herglotz vector field $G(z,t)$. The Loewner chain (f_t^G) is called the *canonical solution of the Loewner PDE* (3.11).

(iii) It follows from part (ii) that the class $\mathcal{S}_n^0$ consists precisely of all normalized univalent mappings $f \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ for which there is a normal Loewner chain $(f_t)_{t\geq 0}$ with $f_0 = f$.

We now construct for a given normal Loewner chain $(f_t)_{t\geq 0}$ a differentiable family of deformations $(f^\varepsilon)_{t\geq 0}$ that coincide with $(f_t)_{t\geq 0}$ from a certain time on.

Theorem 3.5 (Variations of normal Loewner chains) *Let $(f_t)_{t\geq 0}$ be a normal Loewner chain with associated Herglotz vector field $G(z,t)$ in the class $\mathcal{M}_n$. Let $(\varphi_{s,t})_{0\leq s\leq t}$ denote the evolution family generated by $G(z,t)$. Then for any $T \in R_G$, any $h \in \mathcal{M}_n$ and any $\varepsilon \in (0,T)$ there exists a normal Loewner chain $(f_t^\varepsilon)_{t\geq 0}$ such that*

$$f_t^\varepsilon = \begin{cases} f_t & \text{if } t \geq T, \\ f_t + \varepsilon\, d(f_t)_z \cdot [d(e^T\varphi_{t,T})_z]^{-1} \cdot e^T[h(\varphi_{t,T}) - G(\varphi_{t,T},T)] + o_t^\varepsilon & \text{if } t < T. \end{cases}$$

Here, o_t^ε indicates a term, which divided by ε, tends to 0 locally uniformly in $\mathbb{B}^n$ as $\varepsilon \to 0+$.

Proof Let $G(z,t)$ denote the Herglotz vector field in the class $\mathcal{M}_n$ such that the Loewner PDE (3.11) holds, so $f_t = \lim_{\tau\to\infty} e^\tau \varphi_{t,\tau}$ for any $t \geq 0$ in view of Remark 3.4(ii). Denote by $\varphi_{s,t}^\varepsilon$ the needle variations of $\varphi_{s,t}$ with data (T,h). Define $f_t^\varepsilon = \lim_{\tau\to\infty} e^\tau \varphi_{t,\tau}^\varepsilon$ for any $t \geq 0$. Since $\varphi_{t,\tau}^\varepsilon = \varphi_{t,\tau}$ for $T \leq t \leq \tau$, we have $f_t^\varepsilon = f_t$ for any $t \geq T$. Now let $t < T$ and choose $\tau \geq T$. Then Theorem 3.1 shows that

$$e^\tau \varphi_{t,\tau}^\varepsilon = e^\tau \varphi_{t,\tau} + \varepsilon\, d(e^\tau \varphi_{t,\tau})_z \cdot \left[d(e^T\varphi_{t,T})_z\right]^{-1} \cdot e^T\left[h(\varphi_{t,T}) - G(\varphi_{t,T},T)\right] + o_{t,\tau}^\varepsilon.$$

Here, $o_{t,\tau}^\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0+$ locally uniformly in $\mathbb{B}^n$. The proof is finished by letting $\tau = \infty$. ∎

As we have already pointed out in the introduction, Theorem 3.5 is related to the recent work [8], where variations of a specific class of Loewner chains (so-called geräumig Loewner chains) have been introduced; see, in particular, [8, Theorem 3.1].

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4 Extremal Problems on $\mathcal{S}_n^0$

In order to apply our variational formulas, we need to consider a suitable class of "differentiable nonlinear functionals" on the Fréchet space $\mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$. We use the Fréchet space calculus as developed by R. Hamilton [23]. This approach is more general than the one used in the standard monographs [26] or [11].

Definition 4.1 (Complex derivative, [23, p. 73]) Let $U \subseteq \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ be an open set and let $\Phi\colon U \to \mathbb{C}$ be continuous. We call $\Phi\colon U \to \mathbb{C}$ *differentiable at* $f \in U$ *along* $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$, if the limit

$$\Lambda(f;h) := \lim_{\mathbb{C}\ni\delta\to 0} \frac{\Phi(f+\delta h)-\Phi(f)}{\delta}$$

exists. In this case, $\Lambda(f;h)$ is called the *directional derivative of* Φ *at* f *along* h. We say that $\Phi\colon U \to \mathbb{C}$ is *complex differentiable at* $F \in U$ if there is an open neighborhood $V \subseteq U$ of F such that Φ is differentiable at any $f \in V$ along any $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ and if the map $\Lambda\colon V \times \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$ is continuous. In this case, $L := \Lambda(F, \cdot)$ is called the *complex derivative of* Φ *at* F.

Lemma 4.2 *Let* $U \subseteq \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ *be an open set and let* $\Phi\colon U \to \mathbb{C}$ *be complex differentiable at* $F \in U$ *with complex derivative* $L = \Lambda(F; \cdot)$.

(i) *The continuous functional* $L\colon \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$ *is linear.*

(ii) *If* $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ *and* $f^\varepsilon = F + \varepsilon h + r_\varepsilon$, *where* $r_\varepsilon/\varepsilon \to 0$ *locally uniformly in* $\mathbb{B}^n$ *as* $\varepsilon \to 0+$, *then*

$$\lim_{\varepsilon\to 0+} \frac{\Phi(f^\varepsilon)-\Phi(F)}{\varepsilon} = L(h).$$

Proof (i) See [23, pp. 76–77].

(ii) Let $U \subseteq \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ be an open neighborhood of F such that

$$\Lambda\colon U \times \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$$

is continuous. We may assume that U is convex. Lemma 3.3.1 in [23] shows that there is a continuous mapping $\widehat{L}\colon U \times U \times \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$ so that $h \mapsto \widehat{L}(f_1, f_2, h)$ is linear and such that $\Phi(f_2) - \Phi(f_1) = \widehat{L}(f_1, f_2, f_2 - f_1)$ for all $f_1, f_2 \in U$. In addition, $\widehat{L}(f_1, f_1, h) = \Lambda(f_1, h)$ for every $f_1 \in U$ and every $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$. Therefore,

$$\frac{\Phi(f^\varepsilon)-\Phi(F)}{\varepsilon} = \frac{\widehat{L}(F, f^\varepsilon, f^\varepsilon - F)}{\varepsilon} = \frac{\widehat{L}(f^\varepsilon, F, \varepsilon h + r_\varepsilon)}{\varepsilon} \to \widehat{L}(F, F, h) = L(h)$$

as $\varepsilon \to 0+$. ∎

Corollary 4.3 *Let* $G(z,t)$ *be a Herglotz vector field in the class* $\mathcal{M}_n$, *let* $T \in R_G$ *and* $h \in \mathcal{M}_n$. *Consider the needle variations* (φ_t^ε) *of* $(\varphi_t) := (\varphi_t^G)$ *with data* (T, h). *Suppose that* Φ *is a complex functional with complex derivative* L *at* $e^\tau\varphi_\tau$ *for some fixed* $\tau \in (T, \infty]$. *Then*

$$\lim_{\varepsilon\to 0+} \frac{\Phi(e^\tau\varphi_\tau^\varepsilon)-\Phi(e^\tau\varphi_\tau)}{\varepsilon} = L\Big(d(e^\tau\varphi_\tau)_z \cdot \big[d(e^T\varphi_T)_z\big]^{-1} \cdot e^T\big[h(\varphi_T) - G(\varphi_T, T)\big]\Big).$$

Proof This follows from Theorem 3.1 for $s = 0$ and Lemma 4.2(ii). ∎

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Theorem 4.4 *Let Φ be a complex functional with complex derivative L at $F \in \mathcal{S}_n^0$ and suppose that F maximizes* $\operatorname{Re}\Phi$ *over $\mathcal{S}_n^0$. Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$ with $(\varphi_t) := (\varphi_t^G)$ such that $F = e^\tau \varphi_\tau$ for some $\tau \in (0,\infty]$. Then the following hold.*

(i) *For every $t \in (0,\tau) \cap R_G$, the mapping $G(\,\cdot\,,t) \in \mathcal{M}_n$ maximizes the real part of the continuous linear functional*

$$L_t(h) := L\big(d(F)_z \cdot [d(\varphi_t)_z]^{-1} \cdot h(\varphi_t)\big)$$

over $\mathcal{M}_n$, that is,

$$\max_{h \in \mathcal{M}_n} \operatorname{Re} L_t(h) = \operatorname{Re} L_t(G(\,\cdot\,,t)).$$

(ii) *The function $t \mapsto \max_{h\in\mathcal{M}_n} \operatorname{Re} L_t(h)$ is constant on $[0,\tau)$.*

Theorem 1.4 is the special case $\tau = \infty$ of Theorem 4.4(i). Theorem 4.4(i) for $\tau < \infty$ and $n = 1$ is exactly [31, Theorem 4.1].

Proof of Theorem 4.4 (i) Let $h \in \mathcal{M}_n$, $T \in (0,\tau) \cap R_G$ and let (φ_t^ε) denote the needle variations of (φ_t) with data (T,h). Since F maximizes $\operatorname{Re}\Phi$ on $\mathcal{S}_n^0$, we have $\operatorname{Re}\Phi(F) \geq \operatorname{Re}\Phi(e^\tau \varphi_\tau^\varepsilon)$ for every $\varepsilon > 0$. Corollary 4.3 therefore implies

$$\operatorname{Re} L\left(d(F)_z \cdot \left[d(e^T\varphi_T)_z\right]^{-1} \cdot \left[h(\varphi_T) - G(\varphi_T,T)\right]\right) \leq 0\,.$$

(ii) Let $H(t,h) := L_t(h) = L\left(d(F)_z \cdot [d(\varphi_t)_z]^{-1} \cdot h(\varphi_t)\right)$ and

$$m(t) := \max_{h\in\mathcal{M}_n} \operatorname{Re} H(t,h).$$

In order to show that m is constant on $[0,\tau)$ we proceed in several steps.

Step 1 We first show that $m\colon [0,\tau) \to \mathbb{R}$ is locally Lipschitz continuous. Since L is a continuous linear functional on the Fréchet space $\operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$, there are finite complex Borel measures $\mu_1,\dots,\mu_n$ that are supported on compact subsets $E_1,\dots,E_n$ of $\mathbb{B}^n$ such that

$$L(h) = \sum_{k=1}^{n} \iint_{E_k} h_k(z)\,d\mu_k(z), \qquad h = (h_1,\dots,h_n) \in \operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n); \tag{4.1}$$

see *e.g.*, [22, p. 65]. Let E be a closed ball in $\mathbb{B}^n$ centered at the origin such that $E_k \subset E$ for $k = 1,\dots,n$. Since $\|\varphi_t(z)\| \leq \|z\|$ for every $z \in \mathbb{B}^n$ and every $t \geq 0$, it follows that

$$\varphi_t(z) \in E \text{ for all } z \in E \text{ and all } t \geq 0. \tag{4.2}$$

As $\mathcal{M}_n$ is a compact subset of $\operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$, we see as before that there exists a constant $\gamma > 0$ such that

$$\|h(z)\| \leq \gamma\|z\| \text{ and } \|h(z) - h(z')\| \leq \gamma\|z - z'\| \qquad \text{for all } z, z' \in E,\ h \in \mathcal{M}_n; \tag{4.3}$$

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see *e.g.*, [20, formula (8.1.2)]. Since $\varphi_t(z)$ is a solution to (1.1) and $G(\cdot,t) \in \mathcal{M}_n$ for a.e. $t \geq 0$, the estimate (4.3) combined with (4.2) implies

$$\|\varphi_\beta(z) - \varphi_\alpha(z)\| = \Big\| \int_\alpha^\beta G(\varphi_t(z),t)\,dt \Big\| \\ \leq \gamma \cdot |\beta - \alpha| \quad \text{for all } \alpha, \beta \geq 0, z \in E. \tag{4.4}$$

In a similar way, since $t \mapsto [d(\varphi_t)_z]^{-1}$ has the property that

$$\frac{d}{dt}\big([d(\varphi_t)_z]^{-1}\big) = -[d(\varphi_t)_z]^{-1} \cdot \frac{\partial G}{\partial z}(\varphi_t(z),t) \quad \text{for a.e. } t \geq 0,$$

an application of Gronwall's lemma leads to

$$\big\|[d(\varphi_\beta)_z]^{-1}\big\| \leq e^{\gamma\beta} \quad \text{for all } \beta \geq 0, z \in E, \tag{4.5}$$

and

$$\big\|[d(\varphi_\beta)_z]^{-1} - [d(\varphi_\alpha)_z]^{-1}\big\| \leq |e^{\gamma\beta} - e^{\gamma\alpha}| \quad \text{for all } \alpha, \beta \geq 0, z \in E. \tag{4.6}$$

We can now prove that $m\colon [0,\tau) \to \mathbb{R}$ is locally Lipschitz continuous. Let $\alpha, \beta \geq 0$ be given. Since $\mathcal{M}_n$ is a compact subset of $\operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$, there is a mapping $h_\beta \in \mathcal{M}_n$ such that $m(\beta) = \operatorname{Re} H(\beta, h_\beta)$. In view of $m(\alpha) \geq \operatorname{Re} H(\alpha, h_\beta)$ it follows that

$$\begin{aligned} &m(\beta) - m(\alpha) \\ &\quad \leq \operatorname{Re} H(\beta, h_\beta) - \operatorname{Re} H(\alpha, h_\beta) \\ &\quad = \operatorname{Re} L\big(d(F)_z \cdot [d(\varphi_\beta)_z]^{-1} \cdot h_\beta(\varphi_\beta)\big) - \operatorname{Re} L\big(d(F)_z \cdot [d(\varphi_\alpha)_z]^{-1} \cdot h_\beta(\varphi_\alpha)\big) \\ &\quad = \operatorname{Re} L\big(d(F)_z \cdot [d(\varphi_\beta)_z]^{-1} \cdot (h_\beta(\varphi_\beta) - h_\beta(\varphi_\alpha))\big) \\ &\qquad + \operatorname{Re} L\big(d(F)_z \cdot ([d(\varphi_\beta)_z]^{-1} - [d(\varphi_\alpha)_z]^{-1}) \cdot h_\beta(\varphi_\alpha)\big) \\ &\quad = \operatorname{Re} \sum_{k=1}^n \iint_{E_k} \big(d(F)_z \cdot [d(\varphi_\beta)_z]^{-1} \cdot (h_\beta(\varphi_\beta(z)) - h_\beta(\varphi_\alpha(z)))\big)_k \, d\mu_k(z) \\ &\qquad + \operatorname{Re} \sum_{k=1}^n \iint_E \big(d(F)_z \cdot \big([d(\varphi_\beta)_z]^{-1} - [d(\varphi_\alpha)_z]^{-1}\big) \cdot h_\beta(\varphi_\alpha(z))\big)_k \, d\mu_k(z), \end{aligned}$$

where we have used the representation formula (4.1). In view of the estimates (4.2)–(4.6), we now see that for every compact subintervall I of $[0,\tau)$ there is a constant $C = C_I$ such that $m(\beta) - m(\alpha) \leq C|\beta - \alpha|$ for all $\alpha, \beta \in I$. This shows that $m\colon [0,\tau) \to \mathbb{R}$ is locally Lipschitz.

Step 2 We next show that

$$\frac{\partial H}{\partial t}(t, G_t) = 0 \quad \text{for a.e.} \quad t \geq 0.$$

As above, using the fact that φ_t is a solution of the Loewner equation (1.1), we first see that there is a set $E \subseteq \mathbb{R}^+$ of measure 0 such that for any $t \in \mathbb{R}^+ \backslash E$,

$$\frac{d}{dt}\big([d(\varphi_t)_z]^{-1}\big) = -[d(\varphi_t)_z]^{-1} \cdot \frac{\partial G}{\partial z}(\varphi_t(z),t) \text{ locally uniformly with respect to } z \in \mathbb{B}^n.$$

Now, for any $t, t^* \in \mathbb{R}^+$, we have

$$\frac{[d(\varphi_t)_z]^{-1}G(\varphi_t(z),t^*) - [d(\varphi_{t^*})_z]^{-1}G(\varphi_{t^*}(z),t^*)}{t-t^*} =$$
$$\frac{[d(\varphi_t)_z]^{-1} - [d(\varphi_{t^*})_z]^{-1}}{t-t^*}G(\varphi_t(z),t^*) + [d(\varphi_{t^*})_z]^{-1}\frac{G(\varphi_t(z),t^*) - G(\varphi_{t^*}(z),t^*)}{t-t^*},$$

and this expression converges for $t \to t^* \in \mathbb{R}^+ \setminus E$ to

$$-[d(\varphi_{t^*})_z]^{-1} \cdot \frac{\partial G}{\partial z}(\varphi_{t^*}(z),t^*)G(\varphi_{t^*}(z),t^*)$$
$$+ [d(\varphi_{t^*})_z]^{-1}\frac{\partial G}{\partial z}(\varphi_{t^*}(z),t^*)G(\varphi_{t^*}(z),t^*) = 0$$

locally uniformly with respect to $z \in \mathbb{B}^n$. Hence, by definition of H,

$$\lim_{t\to t^*}\frac{H(t,G(\,\cdot\,,t^*)) - H(t^*,G(\,\cdot\,,t^*))}{t-t^*} = 0.$$

for every $t^* \in \mathbb{R}^+ \setminus E$.

Step 3 Next note that $m(t) = \operatorname{Re} H(t, G_t)$ for every $t \in (0,\tau) \cap R_G$ by part (i). Therefore, we obtain for all $t, t^* \in (0,\tau) \cap R_G$ such that $t^* < t$,

$$\frac{\operatorname{Re} H(t,G_{t^*}) - \operatorname{Re} H(t^*,G_{t^*})}{t-t^*} \le \frac{m(t)-m(t^*)}{t-t^*} \le \frac{\operatorname{Re} H(t,G_t) - \operatorname{Re} H(t^*,G_t)}{t-t^*}.$$

Since $m\colon [0,\tau) \to \mathbb{R}$ is locally Lipschitz continuous, it is differentiable for a.e. $t \ge 0$, so

$$\frac{d}{dt}m(t) = \frac{\partial \operatorname{Re} H}{\partial t}(t,G_t) \qquad \text{for a.e. } t \in (0,\tau).$$

By what we have proved in (b2), we see that

$$\frac{d}{dt}m(t) = 0 \qquad \text{for a.e. } t \in (0,\tau).$$

Therefore, the locally Lipschitz continuous function $m\colon [0,\tau) \to \mathbb{R}$ is constant on $[0,\tau)$. ∎

Proof of Theorem 1.6 Since $\mathcal{M}_n$ is compact and $h(0) = 0$, $dh_0 = -\operatorname{id}$ for every $h \in \mathcal{M}_n$, there exists for every $0 < r < 1$ a constant $M_r > 0$ such that

$$\|h(z) + z\| \le M_r\|z\|^2 \qquad \text{for all } \|z\| \le r \text{ and every } h \in \mathcal{M}_n.$$

By [20, formula (8.1.11)] this implies

$$\|h(\varphi_t(z)) + \varphi_t(z)\| \le M_r\|\varphi_t(z)\|^2 \le M_r e^{-2t}\frac{\|z\|^2}{(1-\|z\|)^4}$$

for all $\|z\| \le r$ and every $h \in \mathcal{M}_n$. Therefore,

$$e^t h(\varphi_t(z)) = -e^t\varphi_t(z) + e^t\big(h(\varphi_t(z)) + \varphi_t(z)\big) \to -F(z) \qquad (t \to \infty)$$

locally uniformly for $z \in \mathbb{B}^n$ and uniformly for $h \in \mathcal{M}_n$. Since we also have $[d(e^t\varphi_t)]^{-1} \to [d(F)_z]^{-1}$ locally uniformly in $\mathbb{B}^n$ as $t \to \infty$, we get

$$L_t(h) = L\big(d(F)_z \cdot [d(\varphi_t)_z]^{-1}h(\varphi_t)\big) = L\big(d(F)_z \cdot [d(e^t\varphi_t)_z]^{-1}e^t h(\varphi_t)\big) \to -L(F)$$

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uniformly for $h \in \mathcal{M}_n$, so

$$m(t) = \max_{h\in\mathcal{M}_n} \operatorname{Re} L_t(h) \to -\operatorname{Re} L(F) \qquad (t \to \infty).$$

On the other hand,

$$m(0) = \max_{h\in\mathcal{M}_n} \operatorname{Re} L_0(h) = \max_{h\in\mathcal{M}_n} \operatorname{Re} L(d(F)_z \cdot h).$$

Therefore, Theorem 4.4(ii) completes the proof of Theorem 1.6. ∎

We next show that under the condition that L is not constant on $\mathcal{S}_n^0$, the continuous linear functionals L_t in Theorem 4.4 are support points of $\mathcal{M}_n$. First we recall the following definition.

Definition 4.5 Let $\mathcal{A} \subseteq \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$. A mapping $G \in \mathcal{A}$ is called a *support point* of $\mathcal{A}$, if there exists a continuous linear functional $L\colon \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$ such that $\operatorname{Re} L(h) \le \operatorname{Re} L(G)$ for every $h \in \mathcal{A}$ and L is not constant on $\mathcal{A}$. We denote by $\operatorname{supp} \mathcal{A}$ the set of all support points of $\mathcal{A}$.

Proposition 4.6 *Let Φ be a complex functional with complex derivative L at $F \in \mathcal{S}_n^0$ and suppose that F maximizes $\operatorname{Re}\Phi$ over $\mathcal{S}_n^0$. Let G be a Herglotz vector field in the class $\mathcal{M}_n$ with $(\varphi_t) := (\varphi_t^G)$ such that $F = e^\tau \varphi_\tau$ for some $\tau \in (0, \infty]$. Suppose that L is not constant on $\mathcal{S}_n^0$. Then for any $t \in [0, \tau]$, the continuous linear functional*

$$h \mapsto L_t(h) := L\big(d(F)_z \cdot [d(e^t\varphi_t)_z]^{-1} \cdot h(\varphi_t)\big)$$

is not constant on $\mathcal{M}_n$.

Proof We show that if L_t is constant on $\mathcal{M}_n$ for some $t \in [0, \tau]$, then L is constant on $\mathcal{S}_n^0$. Hence, let $t \in [0, \tau]$ such that $L_t(h)$ is constant on $\mathcal{M}_n$. Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with $P(0) = 0$ and $d(P)_0 = 0$. Then there is a number $\delta > 0$ such that $-z + \varepsilon P(z) \in \mathcal{M}_n$ for every $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, so $L_t(-z + \varepsilon P) = -L_t(z) + \varepsilon L_t(P)$ is constant in ε. This implies that $L_t(P) = 0$. Now let $g \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ with $g(0) = 0$ and $d(g)_0 = 0$. Since $\varphi_t(\mathbb{B})$ is Runge (see [5]), g is the locally uniform limit of $(P_k \circ \varphi_t)_k$ for a sequence of polynomials P_k with $P_k(0) = 0$ and $d(P_k)_0 = 0$. Hence

$$0 = \lim_{k\to\infty} L_t(P_k) = L\big(d(F)_z \cdot [d(e^t\varphi_t)_z]^{-1} \cdot g\big)$$

for all $g \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ with $g(0) = 0$ and $d(g)_0 = 0$. This clearly implies $L(g) = 0$ for all such g. Since we can take $g = f - \mathrm{id}$ for any $f \in \mathcal{S}_n^0$, we get $L(f) = L(\mathrm{id})$, $f \in \mathcal{S}_n^0$, so L is constant on $\mathcal{S}_n^0$. ∎

Definition 4.7 A mapping $F \in \mathcal{S}_n^0$ is called *extremal* if there exists a complex functional Φ with complex derivative L at F such that

(i) $\operatorname{Re}\Phi(f) \le \operatorname{Re}\Phi(F)$ for all $f \in \mathcal{S}_n^0$, and
(ii) L is not constant on $\mathcal{S}_n^0$.

Theorem 4.8 *Let $G(z, t)$ be a Herglotz vector field in the class $\mathcal{M}_n$. Let $e^\tau \varphi_\tau^G$ be extremal for some $\tau \in (0, \infty]$. Then $G(\cdot\,, t) \in \operatorname{supp} \mathcal{M}_n$ for every $t \in R_G \cap (0, \tau]$.*

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Proof This follows immediately from Theorem 4.4(i) and Proposition 4.6. ∎

Theorem 1.5 now follows directly from Corollary 4.9 for $\tau = \infty$ and Lemma 2.2, which shows that the set R_G of regular points of any Herglotz vector field $G(z,t)$ in the class $\mathcal{M}_n$ has full measure.

Corollary 4.9 *Let $G(z,t)$ be a Herglotz vector field in the class $\mathcal{M}_n$ and assume that*

$$\sup_{z\in\mathbb{B}^n\setminus\{0\}} \operatorname{Re}\langle G(z,T), z/\|z\|^2\rangle < 0$$

for some $T \in R_G$. Then $e^t\varphi_t \in \mathcal{S}_n^0$ is not extremal for any $t \in (T,\infty]$.

Proof In view of Theorem 4.8, it suffices to show that $h := G(\cdot\,,T)$ cannot be a support point of the class $\mathcal{M}_n$ on $\mathbb{B}^n$. By assumption, there is a constant $a > 0$ such that $\operatorname{Re}\langle G(z,T), z\rangle \le -a\|z\|^2$ for all $z \in \mathbb{B}^n$. This implies that for any polynomial mapping $P\colon \mathbb{C}^n \to \mathbb{C}^n$ with $P(0) = 0$ and $d(P)_0 = 0$, there is a number $\delta > 0$ such that $h + \varepsilon P \in \mathcal{M}_n$ for every $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$. If $h \in \operatorname{supp}\mathcal{M}_n$, there is a continuous linear functional L on $\operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$ such that $\max_{g\in\mathcal{M}_n}\operatorname{Re}L(g) = \operatorname{Re}L(h)$ and L is not constant on $\mathcal{M}_n$. We can now argue as in the proof of Proposition 4.6. In particular, $\operatorname{Re}L(h+\varepsilon P) \le \operatorname{Re}L(h)$ for any $|\varepsilon| < \delta$, so $L(P) = 0$ for every polynomial mapping $P\colon \mathbb{C}^n \to \mathbb{C}^n$ with $P(0) = 0$ and $d(P)_0 = 0$. Hence $L = 0$ on the set of mappings $g \in \operatorname{Hol}(\mathbb{B}^n,\mathbb{C}^n)$ with $g(0) = 0$ and $d(g)_0 = 0$, so L is constant on $\mathcal{M}_n$, a contradiction. ∎

We end with a remark that extends [8, Lemma 4.3].

Remark 4.10 Corollary 4.9 shows that if $(f_t)_{t\ge 0}$ is a normal Loewner chain such that

$$\inf_{z\in\mathbb{B}^n\setminus\{0\}} \operatorname{Re}\Big\langle [d(f_t)_z]^{-1}\frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2}\Big\rangle > 0$$

for all $t \in E$, where $E \subseteq \mathbb{R}^+$ is a set of positive measure, then f_0 is not extremal (in particular, $f_0 \notin \operatorname{supp}\mathcal{S}_n^0$).

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Department of Mathematics, University of Würzburg, Emil Fischer Straße 40, 97074 Würzburg, Germany
e-mail: roth@mathematik.uni-wuerzburg.de