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Pointwise compactness and continuity of the integral.

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Homenaje es una palabra larga con la que se acorrala el tiempo...
A Baltasar, mi maestro de aquel tiempo y este tiempo con todo mi afecto.

Abstract

In this paper we brind together the different known ways of establishing the continuity of the integral over a uniformly integrable set of functions endowed with the topology of pointwise convergence. We use these techniques to study Pettis integrability, as well as compactness in $C(K)$ spaces endowed with the topology of pointwise convergence on a dense subset $D \subset K$.

1 Introduction

This is an expository survey concerning the following problem: Given a finite measure space (Ω, Σ, μ) and a family $\mathcal{F}$ of μ-integrable real valued functions, if $f \in \mathcal{F}$ is the pointwise limit of a net $f_\alpha \in \mathcal{F}$, when do any of the following hold?

C1: $\lim_\alpha \int (f_\alpha - f) d\mu = 0$

C2: $\lim_\alpha \int_E (f_\alpha - f) d\mu = 0$ for each $E \in \Sigma$

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C3: $\lim_{\alpha} \int |f_\alpha - f| d\mu = 0$

Note that **C3** $\Rightarrow$ **C2** $\Rightarrow$ **C1**. Most of the results about this problem come from Pettis integrability, since Pettis integrability is really a problem about the continuity of the integral on a pointwise compact set of integrable functions; see [6], [26], [23], [31], [4], [24], [11], [27], [28], [21]. Bearing in mind some results about pointwise compact sets of measurable functions from [12], [9], [1], [7] and [31], we collect basic ideas underlying the work in the above references in order to make a systematic exposition of different means that can be used to state the continuity of the integral. Among others applications we show some results recently obtained in [2].

If $\mathcal{L}^1(\mu)$ is the space of μ-integrable functions $f : \Omega \to \mathbb{R}$ and $L^1(\mu)$ the quotient space obtained by almost everywhere identification, $f \to f^\mu$ will be the quotient mapping. If $A \subset \mathcal{L}^1(\mu)$, then we denote by A^μ its canonical image in $L^1(\mu)$.

If $\mathcal{F}$ is equipped with the pointwise topology, denoted τ_p, condition **C1** means that the integral is continuous on $(\mathcal{F}, \tau_p)$. Condition **C3** is equivalent to the continuity of the canonical mapping $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \ \|_1)$. If $\mathcal{F}$ is $\| \ \|_1$-bounded then condition **C2** is the continuity of $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \text{weak})$.

Recall that $\mathcal{F}$ is said to be uniformly integrable if $\mathcal{F}$ is $\| \ \|_1$-bounded and for each $\epsilon > 0$ there exists $\delta > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta$ then $\sup_{f\in\mathcal{F}} \int_E |f| d\mu < \epsilon$. Dunford's theorem [3, p. 76] states that $\mathcal{F}$ is uniformly integrable if and only if $\mathcal{F}^\mu$ is a weakly relatively compact subset of $L^1(\mu)$. Vitali's theorem tell us that, in this case, the mapping $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \ \|_1)$ is sequentially continuous. Using the fact that every $\| \ \|_1$-convergent sequence has an almost everywhere convergent subsequence and Vitali's convergence theorem, we have

Proposition 1 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly integrable, the following are equivalent*

a) $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \ \|_1)$ *is continuous;*

b) For each $\mathcal{H} \subset \mathcal{F}$, if $f \in \mathcal{F}$ is in the τ_p-closure of $\mathcal{H}$ then f is the almost everywhere limit of a sequence in $\mathcal{H}$.

In a similar way, having in mind that every norm closed convex set is weakly closed, the following can be proved

Proposition 2 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is convex and uniformly integrable, the following are equivalent:*

a) $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), weak)$ is continuous;

b) For each $\mathcal{H} \subset \mathcal{F}$ if $f \in \mathcal{F}$ is in the τ_p-closure of $\mathcal{H}$ then f is the almost everywhere limit of a sequence in $co(\mathcal{H})$ (convex hull of $\mathcal{H}$).

2 Continuity of $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \text{weak})$

Results in this section come from [7], [31], [4] and [27], and they concern a convex family $\mathcal{F}$. We start adapting some of Talagrand's ideas concerning Pettis integrability that can be found in [31, 5-1-2]; (see also [26], [11], and [21], where these ideas have been also used).

If $\epsilon > 0$ and H is a finite set in Ω let $\mathcal{F}(H, \epsilon)$ be the set formed by the functions $f \in \mathcal{F}$ such that $|f(\omega)| \leq \epsilon$ for every $\omega \in H$ and

$$K(\mu, \mathcal{F}) := \bigcap_{(H,\epsilon)} \mathcal{F}(H, \epsilon)^{\mu}.$$

Lemma 1 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly integrable and τ_p-countably compact let $\mathcal{F}_\delta$ be the set $\{f \in \mathcal{F} : \int f \, d\mu \geq \delta\}$, where $\delta > 0$. Then $\mathcal{F}^\mu$ and $\mathcal{F}_\delta^\mu$ are norm closed sets in $L^1(\mu)$. If $\mathcal{F}$ is convex then they are weakly compact.*

Sketch of proof: Use that every $\| \ \|_1$-convergent sequence in $\mathcal{F}$ (resp. $\mathcal{F}_\delta$) has an almost everywhere convergent subsequence which has a τ_p-cluster point in $\mathcal{F}$ (resp. $\mathcal{F}_\delta$). When $\mathcal{F}$ is convex then $\mathcal{F}_\delta$ is convex too, so $\mathcal{F}^\mu$ and $\mathcal{F}_\delta^\mu$ are weakly closed and then they are weakly compact by uniformly integrability. ■

Theorem 1 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is absolutely convex, uniformly integrable and τ_p-countably compact, the following are equivalent:*

a) The integral $f \to \int f \, d\mu$ is continuous on $(\mathcal{F}, \tau_p)$;

b) There exists a countable set $M \subset \Omega$ such that if $f \in \mathcal{F}$ and $f|_M = 0$ then $\int f\,d\mu = 0$;

c) If $f^\mu \in K(\mu, \mathcal{F})$ then $\int f\,d\mu = 0$;

Proof. a) $\Rightarrow$ b) For each $n \in \mathbb{N}$ there exist a finite $M_n \subset \Omega$ and $\delta_n > 0$ such that $|\int f\,d\mu| < 1/n$ whenever $f \in \mathcal{F}$ and $|f(\omega)| < \delta_n$ for all $\omega \in M_n$. If $M := \bigcup_n M_n$ then b) holds.

b) $\Rightarrow$ c) Assume that there exists $h \in \mathcal{L}^1(\mu)$ such that $h^\mu \in K(\mu, \mathcal{F})$ but $\int h\,d\mu \neq 0$. For every (H, ϵ) there exists $f_{(H,\epsilon)} \in \mathcal{F}(H, \epsilon)$ such that $f_{(H,\epsilon)} = h$, μ-a.e. If $M \subset \Omega$ is countable and $H_n \subset M$ is an increasing sequence of finite sets such that $M = \cup_n H_n$, let $f \in \mathcal{F}$ be a τ_p-cluster point of the sequence $f_{(H_n,1/n)} \in \mathcal{F}$. Since $f = h$, μ-a.e we have $\int f\,d\mu \neq 0$ but $f|_M = 0$ and so b) does not hold.

c) $\Rightarrow$ a) If we assume that a) is false then the integral is not continuous at 0 and there exists $\delta > 0$ such that 0 is in the τ_p-closure of $\mathcal{F}_\delta := \{f \in \mathcal{F} : \int f\,d\mu \geq \delta\}$. Then, for each $\epsilon > 0$ and each finite $H \subset \Omega$ the convex set $\mathcal{F}_\delta(H, \epsilon)$ is not empty. Since $\mathcal{F}_\delta(H, \epsilon)^\mu$ is weakly compact in $L^1(\mu)$ by Lemma 1, the intersection of the nested family $\mathcal{F}_\delta(H, \epsilon)^\mu$ is not empty. Hence there exists $h \in \mathcal{L}^1(\mu)$ such that h^μ is in $K(\mu, \mathcal{F}_\delta) \subset K(\mu, \mathcal{F})$ but $\int h\,d\mu \geq \delta > 0$. ■

Theorem 2 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is absolutely convex, uniformly integrable and τ_p-countably compact, the following are equivalent:*

a) $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \text{weak})$ is continuous;

b) $K(\mu, \mathcal{F}) = 0^\mu$.

Proof. a) $\Rightarrow$ b) If b) is false there exists $h^\mu \in K(\mu, \mathcal{F})$ such that $h^\mu \neq 0$ and then we can obtain $E \in \Sigma$ such that $\int_E h\,d\mu \neq 0$. Assume that $\int_E h\,d\mu > 0$. If μ_E denotes the measure $\mu_E(A) = \mu(E \cap A)$ then $h^{\mu_E} \in K(\mu_E, \mathcal{F})$. By Theorem 1 we have that $f \to \int_E f\,d\mu$ is not continuous on $(\mathcal{F}, \tau_p)$.

b) $\Rightarrow$ a) By absolutely convexity it is enough to show the continuity at 0. If $f_\alpha \in \mathcal{F}$ is a net converging to 0, then f_α^μ is eventually in each $\mathcal{F}(H, \epsilon)^\mu$. Since $\mathcal{F}^\mu$ is weakly compact and 0 is the only cluster point (in the weak topology) of this net we have that 0 is the weak limit. ■

The next results try to obtain the continuity of $I : (\mathcal{F}, \tau_p) \to (L^1(\mu),$ weak) by splitting up the mapping I through $(\mathcal{F}|_S, \tau_p(S))$, where S is some subset of Ω.

Definition 1 *We say that $\mathcal{F}$ is determined (resp. separated) by $S \subset \Omega$ if the following holds: If $f, g \in \mathcal{F}$ then $f|_S = g|_S \Rightarrow f = g$, μ-a.e. (resp. $f|_S = g|_S \Leftrightarrow f = g$, μ-a.e.)*

If $\mathcal{F}$ is determined by S we denote by $J : \mathcal{F}|_S \to \mathcal{F}^\mu$ the natural mapping $J(f|_S) := f^\mu$. Note that J is a bijection when $\mathcal{F}$ is separated by S.

Part a) in the next theorem is a reformulation of a result of Stefansson [27], and part b) is an improvement due to Edgar [7] of a previous result of Ionescu-Tulcea [12], where it was assumed that $\Omega = S$. Edgar's improvement consists in a weakening of separation condition and a corresponding weakening of conclusion using weak topology rather than the norm topology.

Theorem 3 *Let $\mathcal{F} \subset \mathcal{L}^1(\mu)$ be convex, uniformly integrable and τ_p-countably compact.*

a) If $\mathcal{F}$ is determined by $S \subset \Omega$ then the natural map $J : (\mathcal{F}|_S, \tau_p(S)) \to (\mathcal{F}^\mu, weak)$ is sequentially continuous;

b) If $\mathcal{F}$ is separated by $S \subset \Omega$ then $J : (\mathcal{F}|_S, \tau_p(S)) \to (\mathcal{F}^\mu, weak)$ is a homeomorphism.

Proof. a) Suppose that $f_n \in \mathcal{F}$ is pointwise convergent to $f \in \mathcal{F}$ on S. Since $\mathcal{F}^\mu$ is weakly compact by Lemma 1, in order to show that f^μ is the weak limit of $f_n^\mu \in \mathcal{F}^\mu$ it is enough to check that for each weak cluster point h^μ of f_n^μ we have $h^\mu = f^\mu$.

For every $k \in \mathbb{N}$, h^μ is in the weak closure of $co\{f_n^\mu : n \geq k\}$ and we can obtain $h_k \in \mathcal{F} \cap \overline{co\{f_n : n \geq k\}}^{\tau_p}$ such that $h_k = h$, μ-a.e. (Let us consider a sequence in the convex hull $co\{f_n : n \geq k\} \subset \mathcal{F}$ converging to h, μ-a.e. and let h_k be a τ_p-cluster point of this sequence). If $g \in \mathcal{F}$ is a τ_p-cluster point of the sequence h_k then $g^\mu = h^\mu$ and $g \in \bigcap_k \overline{co\{f_n : n \geq k\}}^{\tau_p}$. If we can show that $f^\mu = g^\mu$ then $h^\mu = f^\mu$ as required. Suppose that $f^\mu \neq g^\mu$ and take $\omega \in S$ such that $f(\omega) \neq g(\omega)$.

If we assume that $g(\omega) < \delta < f(\omega)$ then for each $k \in \mathbb{N}$ we can find a convex combination ψ_k of $\{f_n : n \geq k\}$ such that $g(\omega) < \psi_k(\omega) < \delta$. For some $n \geq k$ we have $f_n(\omega) < \delta$ and we can obtain a subsequence f_{n_k} such that $f_{n_k}(\omega) < \delta < f(\omega)$ for all $k \in \mathbb{N}$, which contradicts the initial assumption that $f_n(s)$ is converging to $f(s)$ for each $s \in S$.

b) If $\mathcal{F}$ is separated by S the natural mapping J is a bijection and we can consider the inverse bijection $R : \mathcal{F}^\mu \to \mathcal{F}|_S$. The convex sets $\mathcal{F}^\mu$, $\{f \in \mathcal{F} : f(\omega) \geq s\}^\mu$, and $\{f \in \mathcal{F} : f(\omega) \leq t\}^\mu$, $(s, t \in \mathbb{R})$ are weakly compact in $L^1(\mu)$ by Lemma 1. Then $\{f^\mu \in \mathcal{F}^\mu : t < f(\omega) < s\}$ is open in $(\mathcal{F}^\mu, \text{weak})$ for each $\omega \in S$ and each $(s,t) \subset \mathbb{R}$. Thus the mapping $R : (\mathcal{F}^\mu, \text{weak}) \to (\mathcal{F}|_S, \tau_p(S))$ is continuous and J is a homeomorphism because $(\mathcal{F}^\mu, \text{weak})$ is compact and $(\mathcal{F}|_S, \tau_p(S))$ is Hausdorff. ∎

In order to obtain some applications of Theorem 3 we start considering special situations where we can obtain a set S such that $\mathcal{F}$ is separated (determined) by S. If $\mathcal{F}$ is a family of continuous functions on a completely regular topological space Ω and μ is a τ-smooth Baire probability on Ω then μ extends to a Borel probability and the non empty support of this extension is a closed set S such that $\mathcal{F}$ is separated by S. Note that $S = \bigcap\{Z \in \mathcal{Z} : \mu(Z) = 1\}$ where $\mathcal{Z}$ is the family of zero sets. In the general situation, a careful analysis of this case shows that in order to obtain a set S separating a given family $\mathcal{F}$ it is enough to operate on the family $\mathcal{Z}_\mathcal{F}$ formed by the subsets of Ω obtained as finite intersections of the following type of sets $\{\omega \in \Omega : f(\omega) \leq g(\omega) + t\}$, $f, g \in \mathcal{F}$, $t \in \mathbb{R}$.

Definition 2 *Let $\mathcal{F}$ be a family of real valued measurable functions on Ω.*

a) We shall say that μ is $\mathcal{F}$-concentrated on $M \subset \Omega$ if $\mu(Z) = \mu(\Omega)$ for each $Z \in \mathcal{Z}_\mathcal{F}$ such that $M \subset Z$;

b) We shall say that μ is $\mathcal{F}$-smooth if for any net $Z_\alpha \in \mathcal{Z}_\mathcal{F}$ which decreases to the empty set we have $\lim_\alpha \mu(Z_\alpha) = 0$.

Proposition 3 *If μ is $\mathcal{F}$-concentrated on M then $\mathcal{F}$ is determined by M. If μ is $\mathcal{F}$-smooth the set $core_\mathcal{F}(\mu) := \bigcap\{Z \in \mathcal{Z}_\mathcal{F} : \mu(Z) = 1\}$ is not empty and $\mathcal{F}$ is separated by $S = core_\mathcal{F}(\mu)$.*

Proof. The first assertion is immediate because the sets $\{\omega \in \Omega : f(\omega) = g(\omega)\}$, $f, g \in \mathcal{F}$, are in $\mathcal{Z}_{\mathcal{F}}$. For the second one, note that $core_{\mathcal{F}}(\mu) \neq \emptyset$ follows from the definition of a $\mathcal{F}$-smooth measure. It is immediate that given $f, g \in \mathcal{F}$ such that $f = g$, μ-a.e. then $f|_S = g|_S$. On the other hand, if $f, g \in \mathcal{F}$ and $f|_S = g|_S$ the set $E := \{\omega \in \Omega : f(\omega) = g(\omega)\}$, is in $\mathcal{Z}_{\mathcal{F}}$ and it suffices to prove that $\mu(E) = 1$. Suppose $\mu(E) < 1$ and assume that $\mu(\{\omega \in \Omega : f(\omega) > g(\omega)\}) > 0$. Then $\mu(C_n) > 0$ for some $n \in \mathbb{N}$, where $C_n := \{\omega \in \Omega : f(\omega) \geq g(\omega) + 1/n\}$. Now $\{Z \cap C_n : Z \in \mathcal{Z}_{\mathcal{F}}, \mu(Z) = 1\}$ is a net in $\mathcal{Z}_{\mathcal{F}}$ which decreases to the empty set such that every set $Z \cap C_n$ in this net verifies $\mu(Z \cap C_n) = \mu(C_n) > 0$ which contradicts the hypothesis. ■

Corollary 1 *Let $\mathcal{F} \subset \mathcal{L}^1(\mu)$ be convex uniformly integrable and τ_p-countably compact. If μ is $\mathcal{F}$-smooth then the canonical map $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), weak)$ is continuous.*

Corollary 2 *Let $\mathcal{F} \subset \mathcal{L}^1(\mu)$ be convex uniformly integrable and τ_p-countably compact. If $\mathcal{F}$ is determined by $S \subset \Omega$ and the topological space $(\mathcal{F}|_S, \tau_p(S))$ has the property that every real valued sequentially continuous function is continuous then $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), weak)$ is continuous.*

Obviously the condition considered in Corollary 2 holds if the topological space $(\mathcal{F}|_S, \tau_p(S))$ is metrizable but there are weaker conditions to obtain the continuity of sequentially continuous functions. Such a condition is the following one: For every τ_p(S)-cluster point h of a set $A \subset \mathcal{F}|_S$ there exists a sequence in A which converges to h for the topology $\tau_p(S)$. Recall that a topological space T is said to be angelic if and only if every relatively countably compact $C \subset T$ is relatively compact and its closure is formed by the limits of sequences from C. If a topological space S contains a dense $\mathcal{K}$-analytic subset then $(C(S), \tau_p(S))$ is angelic [19] so we obtain

Corollary 3 *Let Ω be a topological space, μ a finite Baire measure on Ω and $\mathcal{F} \subset C(\Omega)$ convex uniformly integrable and τ_p-countably compact. If $\mathcal{F}$ is determined by a set $S \subset \Omega$ that contains a dense $\mathcal{K}$-analytic subset then $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), weak)$ is continuous.*

As an application of Theorem 2 we obtain an abstract reformulation of a result from [4]. Recall that a subset A of a topological space is called countably closed if A contains the closure of each of its countable subsets.

Corollary 4 *Let $\mathcal{F} \subset \mathcal{L}^1(\mu)$ be absolutely convex uniformly integrable and τ_p-compact. Suppose than $\mathcal{F}$ is determined by a set $S \subset \Omega$ such that every convex countably closed subset of $(\mathcal{F}|_S, \tau_p(S))$ is closed. Then $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), weak)$ is continuous.*

Proof. In view of Theorem 2 we only have to prove that $K(\mu, \mathcal{F}) = 0^\mu$. Take $h^\mu \in K(\mu, \mathcal{F})$, so for every ϵ and every finite $H \subset \Omega$ there exists $f_{(H,\epsilon)} \in \mathcal{F}(H, \epsilon)$ such that $f_{(H,\epsilon)} = h$, μ-a.e. Thus 0 is in the τ_p-closure of the convex countably closed set $\mathcal{H} = \{f \in \mathcal{F} : f = h, \mu\text{-a.e.}\}$. By compactness $\mathcal{H}|_S$ is also countably closed in $(\mathcal{F}|_S, \tau_p(S))$ and therefore it is closed and contains the 0 map. Hence there exists $f \in \mathcal{H}$ such that $f|_S = 0$, so $f = 0$, μ-a.e. Then $h(\omega) = 0$ μ-a.e. ∎

3 Continuity of $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \ \|_1)$

Results in this section, with an additional assumption to those considered in above section, but without convexity, give us a stronger conclusion. We start with Theorem 4 and its Corollary 5, essentially due to Edgar [7], in which a separating set S still plays a role, and continue with the deep contributions of Bourgain-Fremlin-Talagrand [1] and Talagrand [31] and with some useful remarks to facilitate applications of these results.

Theorem 4 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly integrable τ_p-countably compact and separated by $S \subset \Omega$ the following are equivalent:*

a) Every sequence $f_n \in \mathcal{F}$ has a μ-almost everywhere convergent subsequence;

b) The natural bijection $J : (\mathcal{F}|_S, \tau_p(S)) \to (\mathcal{F}^\mu, \| \ \|_1)$ is a homeomorphism.

Proof. a) $\Rightarrow$ b) By Lemma 1 $\mathcal{F}^\mu$ is closed in $(L^1(\mu), \| \;\|_1)$ and Vitali's convergence theorem gives us that $\mathcal{F}^\mu$ is compact in $(L^1(\mu), \| \;\|_1)$. In order to obtain b) it is enough to show that the inverse mapping $R := J^{-1}$ is continuous. Take a sequence f_n in $\mathcal{F}$ such that f_n^μ is $\| \;\|_1$-convergent to f^μ, where $f \in \mathcal{F}$, and suppose that $f_n(\omega_0)$ does not converges to $f(\omega_0)$ for some $\omega_0 \in S$. Then for some $\epsilon > 0$ there exists a subsequence f_{n_k} (that can be assumed to be almost everywhere convergent) such that $|f_{n_k}(\omega_0) - f(\omega_0)| > \epsilon$ for every $k \in \mathbb{N}$. If $g \in \mathcal{F}$ is a τ_p-cluster point of this subsequence we have that $f = g$, μ-a.e., but $|f(\omega_0) - g(\omega_0)| > \epsilon$, which contradicts the hypothesis.

b) $\Rightarrow$ a) If b) holds then $\mathcal{F}^\mu$ is compact in $(L^1(\mu), \| \;\|_1)$. Since each $\| \;\|_1$-convergent sequence has a μ-a.e. convergent subsequence we obtain a). ∎

A finite measure space (Ω, Σ, μ) is called *perfect* if for every measurable function $f : \Omega \to \mathbb{R}$ and every $E \subset \mathbb{R}$ such that $f^{-1}(E) \in \Sigma$ there exists a Borel set $B \subset E$ such that $\mu(f^{-1}(E)) = \mu(f^{-1}(B))$. A remarkable result of Fremlin [9] states that a sequence of real valued measurable functions on a perfect measure space either has a μ-a.e. convergent subsequence, or a subsequence all of whose τ_p-cluster points are non-measurable. As a consequence we have [7]

Corollary 5 *Let $\mathcal{F} \subset \mathcal{L}^1(\mu)$ be uniformly integrable, τ_p-countably compact and separated by $S \subset \Omega$. If one of the following conditions holds:*

i) $\mathcal{F}$ is τ_p-sequentially compact;

ii) The measure space (Ω, Σ, μ) is perfect;

then $J : (\mathcal{F}|_S, \tau_p(S)) \to (\mathcal{F}^\mu, \| \;\|_1)$ is a homeomorphism, and so $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \;\|_1)$ is continuous.

Remark In Theorem 4 and Corollary 5 the hypothesis of uniformly integrability can be removed if instead of $L^1(\mu)$ we consider the space $L^0(\mu)$, formed by the (equivalence classes of) real valued measurable functions, equipped with the metrizable topology of convergence in measure.

Subsequently we go on with some results of Talagrand and Bourgain, that can be found in [31] and [24], concerning the continuity of $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \;\|_1)$. In [31] Talagrand introduces the notion of a μ-stable set of functions. These are sets that satisfy an explicit criterion for their relative τ_p-compactness in the space $\mathcal{M}_\mu(\Omega)$ of real valued μ-measurable functions

Definition 3 *A τ_p-relatively compact subset $\mathcal{F}$ of $\mathbb{R}^\Omega$ is called μ-stable if for each $E \in \Sigma$, $\mu(E) > 0$ and for each $s < t$ there exists $k, l \in \mathbb{N}$ such that*

$$\mu^*_{k+l}\Big(\bigcup_{f\in\mathcal{F}}(\{f < s\}^k \times \{f > t\}^l) \cap E^{k+l}\Big) < (\mu(E)^{k+l}).$$

If the measure space (Ω, Σ, μ) is complete every μ-stable set is formed by μ-measurable functions and its τ_p-closure in $\mathbb{R}^\Omega$ is also μ-stable so it is a subset of $\mathcal{M}_\mu(\Omega)$ ([31, Chap. 9]). The notion of μ-stable set is close to relative τ_p-compactness in $\mathcal{M}_\mu(\Omega)$ (it is the same for perfect measure spaces and countable sets of functions) but demanding a natural technical condition avoiding some pathological behaviour of the arbitrary relatively τ_p-compact subset of $\mathcal{M}_\mu(\Omega)$. The most important result on μ-stable sets concerning the problem we are surveying is the following one that can be found in [31, Chap. 9] and [5, Chap. II].

Theorem 5 *If $\mathcal{F} \subset \mathbb{R}^\Omega$ is μ-stable and uniformly integrable then the canonical mapping $I : (\mathcal{F}, \tau_p) \to (\mathcal{F}^\mu, \| \ \|_1)$ is continuous.*

The notion of μ-stable set and the proof of Theorem 5 is rather technical. The following stronger notion due to Bourgain gives us another way to attain the same conclusion with a shorter proof.

Definition 4 *A family $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Bourgain property with respect to μ if for each $A \in \Sigma$, $\mu(A) > 0$, and each $\epsilon > 0$ there is a finite collection of measurable subsets of A, $A_i \in \Sigma$, $\mu(A_i) > 0$, $1 \le i \le n$, such that each $f \in \mathcal{F}$ the oscillation of f in some A_i is less than ϵ.*

Theorem 6 (Bourgain) *If $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Bourgain property (with respect to μ) then the τ_p-closure of $\mathcal{F}$ also has the Bourgain property and it is formed by μ-measurable functions. Each element in the τ_p-closure of $\mathcal{F}$ is the μ-almost everywhere limit of a sequence in $\mathcal{F}$.*

Proof. See [24, Th.11]. ∎

Corollary 6 *If $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly , integrable and has the Bourgain property with respect to μ then the canonical map $I : (\mathcal{F}, \tau_p) \to (L^1(\mu), \| \ \|_1)$ is continuous.*

Proof. It follows from Theorem 6 and Proposition 1. ∎

If μ is a Radon measure on a compact space Ω and $\mathcal{F}$ is formed by continuous functions then there is an useful condition implying Bourgain's property for $\mathcal{F}$. To formulate this condition we recall the notion of independent sequence of functions ([25]).

Definition 5 *A sequence of functions f_n in $\mathbb{R}^\Omega$ is called independent on $A \subset \Omega$ if there exists numbers $s < t$ such that for each pair of finite disjoint subsets $P, Q \subset \mathbb{N}$ we have*

$$[\bigcap_{n \in P} \{\omega \in A : f_n(\omega) < s\}] \cap [\bigcap_{n \in Q} \{\omega \in A : f_n(\omega) > t\}] \neq \emptyset.$$

Theorem 7 *Let μ be a finite Radon measure on a compact space Ω and $\mathcal{F} \subset C(\Omega)$ an uniformly bounded family of continuous functions. If $\mathcal{F}$ does not contain an independent sequence then $\mathcal{F}$ has the Bourgain property with respect to μ.*

Proof. See [16, Prop. 2]. ∎

Remark Every family $\mathcal{F}$ with the Bourgain property is stable but the converse is false [31, 9-5-4, p. 112]. However, for a compact space Ω, a uniformly bounded family $\mathcal{F} \subset C(\Omega)$ has the Bourgain property with respect to each Radon measure if and only if $\mathcal{F}$ is stable with respect to each Radon measure (see Theorem 9).

In order to apply Theorems 6 and 7 it will be interesting to have some useful criterion preventing the existence of independent sequences in the family $\mathcal{F}$. If $\mathcal{F}$ is an uniformly bounded family of continuous functions on a compact space Ω such that every sequence in $\mathcal{F}$ has a pointwise convergent subsequence then it is easy to prove that $\mathcal{F}$ does not contain an independent sequence. This result can be improved and in order to do so we introduce the following definition.

Definition 6 *If f_n is an uniformly bounded sequence in $\mathbb{R}^\Omega$ such that its τ_p-closure in $\mathbb{R}^\Omega$ is not homeomorphic to $\beta\mathbb{N}$ then we shall say that f_n is a narrow sequence.*

It is clear that every uniformly bounded and pointwise convergent sequence is a narrow sequence. Next result, that is based on ideas from [30], is a reformulation of [2, Lemma 1].

Theorem 8 *If Ω is a compact space and $\mathcal{F} \subset C(\Omega)$ is an uniformly bounded family such that every sequence in $\mathcal{F}$ has a narrow subsequence then $\mathcal{F}$ does not contain an independent sequence, so $\mathcal{F}$ has the Bourgain property with respect to each finite Radon measure on Ω.*

Proof. Since subsequences of independent sequences are independent it will be enough to prove that if $f_n \in C(\Omega)$ is an independent sequence such that $\| f_n \| \leq 1$ then the τ_p-compact set $K := \overline{\{f_n : n \in \mathbb{N}\}}^{\tau_p}$ is homeomorphic to $\beta\mathbb{N}$. In order to prove that K is homeomorphic to $\beta\mathbb{N}$ we consider the continuous linear map $T : C(K) \to \ell^\infty$, $\varphi \to (\varphi(f_n))$. Once we prove that T is a surjection then T will be an isometric isomorphism between the spaces $(C(K), \| \ \|)$ and $(C(\beta\mathbb{N}), \| \ \|)$, (supremum norm in both spaces) and the Banach-Stone theorem [14, §25.2.2] applies to conclude that (K, τ_p) is homeomorphic to $\beta\mathbb{N}$. Therefore, we only have to show that for each $x \in \ell^\infty$ there exists $\varphi \in C(K)$ such that $T(\varphi) = x$.

First step: Note that for a compact space Ω and continuous functions f_n Definition 5 of independence is equivalent to the one obtained when the disjoint subsets P, Q of $\mathbb{N}$ are not assumed to be finite. Then there are numbers $s < t$ such that for each subset M of $\mathbb{N}$ there are two points $\omega_1, \omega_2 \in \Omega$ verifying:

$$f_n(\omega_1) \geq t,\ f_n(\omega_2) \leq s \text{ for all } n \in M, \text{ and } f_n(\omega_1) \leq s,\ f_n(\omega_2) \geq t$$

for all $n \notin M$

Second step: Now we shall prove that $\tau = (t - s)/8$ verifies $\tau < 1/4$ and there exists $\varphi \in C(K)$ such that

$$\| \varphi \| \leq \| x \|/4, \text{ and } \| x - T(\varphi) \| \leq (1 - \tau)\| x \|.$$

It is enough to prove this when $\| x \| = 4$. By the first step applied to the set $M := \{n : x_n \geq 2\}$ there exist two points $\omega_1, \omega_2 \in \Omega$ verifying the condition stated above. If $\varphi \in C(K)$ is defined by the formula $\varphi(f) := (f(\omega_1) - f(\omega_2))/2$ then

$$\varphi(f_n) \geq (t - s)/2 = 4\tau \text{ si } n \in M, \text{ y } \varphi(f_n) \leq -(t - s)/2 = -4\tau \text{ si } n \notin M$$

Since $\| f_n \| \leq 1$ we have $\| \varphi \| \leq 1$ so $4\tau \leq 1$ and then

$$n \in M \Rightarrow 0 \leq x_n - \varphi(f_n) \leq 4 - 4\tau = 4(1-\tau), \text{ and}$$

$$n \notin M \Rightarrow -4(1-\tau) = -4 + 4\tau \leq x_n - \varphi(f_n) \leq 2 + 1 < 4(1-\tau).$$

Therefore $\| x - T(\varphi) \| \leq 4(1-\tau)$ and this is the required condition when $\| x \| = 4$.

Third step: Repeating the argument with $(x - T(\varphi)) \in \ell^\infty$ and proceeding by induction we produce a sequence $\varphi_n \in C(K)$ such that

$$\| \varphi_n \| \leq \frac{(1-\tau)^{n-1}}{4} \| x \|, \text{ and } \quad \| x - \textstyle\sum_{i=1}^{n} T(\varphi_i) \| \leq (1-\tau)^n \| x \|.$$

The first inequality implies that $\sum_{n=1}^{\infty} \varphi_n$ defines an element $\varphi \in C(K)$ and the second one tell us that $T(\varphi) = x$ so the surjectivity of T is obtained. ∎

Given a compact space Ω, the uniformly bounded sets $\mathcal{F} \subset C(\Omega)$ which are universally stable (i.e. stable with respect to every Radon measure on Ω) are characterized in [31, 14-1-7] (see also [5, 3.11]). The characterization is formulated by means of the equivalence of several properties concerning either the pointwise relative compactness of $\mathcal{F}$ in nice spaces of measurable functions or the non existence of independent sequences in $\mathcal{F}$. Now we complete the list of properties which are equivalent to the universal stability adding two properties that to the best of our knowledge are new (these are b) and d) in the next theorem).

Theorem 9 *If Ω is a compact space and $\mathcal{F} \subset C(\Omega)$ is uniformly bounded the following are equivalent:*

- *a) $\mathcal{F}$ is μ-stable with respect to each Radon measure μ on Ω;*
- *b) $\mathcal{F}$ has the Bourgain property with respect to each Radon measure μ on Ω;*
- *c) $\mathcal{F}$ is pointwise relatively compact in $M_\mu(\Omega)$ for each Radon measure μ on Ω;*
- *d) Every sequence in $\mathcal{F}$ has a narrow subsequence;*

e) Every sequence in $\mathcal{F}$ has a pointwise convergent subsequence;

f) $\mathcal{F}$ does not contain an independent sequence on Ω;

Proof. f) $\Rightarrow$ b) by Theorem 7; b) $\Rightarrow$ c) follows from Theorem 6; c) $\Rightarrow$ f) corresponds to v) $\Rightarrow$ vi) of Theorem 2F in [1, p. 855] (the proof uses that every free ultrafilter on $\mathbb{N}$ is non-measurable with respect to the canonical probability on $\mathcal{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}}$); e) $\Rightarrow$ d) because $\beta\mathbb{N}$ is not countable; d) $\Rightarrow$ f) is Theorem 8; f) $\Rightarrow$ e) is an old result that comes from [25]; f) $\Rightarrow$ a) $\Rightarrow$ c) can be found in [31, 14-1-7] and [5, 3.11]. ■

4 Applications

Metrizability of pointwise compact sets. The following result of A. Ionescu Tulcea [12] is the first significant result about the following problem: For a given pointwise τ_p-compact set of measurable functions when do the pointwise topology and the topology of convergence in measure agree on $\mathcal{F}$?. (See Chapter 12 in [31] which is dedicated to this problem).

Theorem 10 *If $\mathcal{F}$ is a convex τ_p-countably compact set of μ-measurable functions separated by Ω then $(\mathcal{F}, \tau_p)$ is metrized by the distance of convergence in measure. If $\mathcal{F}$ is not assumed to be convex but either $\mathcal{F}$ is τ_p-sequentially compact or the measure μ is perfect, the same conclusion holds.*

Proof. If $\mathcal{F}$ is uniformly integrable and convex then by Eberlein's theorem and Theorem 3 we have that $\mathcal{F}$ is τ_p-sequentially compact. If $\mathcal{F}$ is not assumed to be uniformly integrable the same conclusion holds because we can consider another measure ν such that $\mathcal{F}$ is uniformly ν-integrable and such that the families of ν-null and μ-null sets agree (since $\{f(\omega) : f \in \mathcal{F}\}$ is bounded for each $\omega \in \Omega$ we can consider a measurable function $h : \Omega \to [0, +\infty)$ which is the essential supremum of $\mathcal{F}$ and we can define $\nu(E) := \int_E \frac{1}{1+h} d\mu$). If we only assume that either $\mathcal{F}$ is τ_p-sequentially compact or μ is perfect the conclusion is obtained as in the proof of Corollary 5 (see the remark following this corollary). ■

Grothendieck measures. Let T be a completely regular topological space, $C_b(T)$ the space of all continuous bounded functions on T and $\mathcal{M}_g(T)$ the space of Grothendieck measures [32]: Baire measures which are τ_p-continuous on absolutely convex and τ_p-compact subset of $C_b(T)$. Note that every τ-smooth Baire measure is in $\mathcal{M}_g$ because every set of continuous functions can be separated by the support of the measure and then we can apply Corollary 1. In [20] was showed that $\mathcal{M}_g(T)$ is sequentially complete for the topology $\sigma(\mathcal{M}_g(T), C_b(T))$. The proof of this result can be obtained as a direct application of Theorem 1, [21]. Indeed, by a standard argument ([20]) it is enough to show that every cluster point of a sequence in $\mathcal{M}_\sigma^+(T)$ (for $\sigma(\mathcal{M}_g(T), C_b(T))$) belongs to $\mathcal{M}_g^+(T)$ and this fact is a trivial consequence of Theorem 1.

A result on norm separable subsets of $C(K)$. As a first application of Theorem 8 we obtain the following result concerning the continuity of a function defined by an integral

Theorem 11 *Let H, K be compact topological spaces and suppose that K does not contain a homeomorphic copy of $\beta\mathbb{N}$. Let $f : H \times K \to \mathbb{R}$ be a bounded function such that*

a) For every $h \in H$ the function $x \to f(h,x)$ is continuous;

b) There exist a dense subset D of K such that for each $x \in D$ the function $h \to f(h,x)$ is continuous.

If μ is a finite Radon measure on H, then all functions $h \to f(h,x)$ are μ-measurable and the integral

$$\varphi(x) = \int_H f(h,x)d\mu(h)$$

defines a continuous function on K.

Proof. Suppose $|f| \leq 1$ and let $F : K \to ([-1,1]^H, \tau_p)$ be the continuous mapping defined by $F(x) = f_x$ where $f_x(h) = f(h,x)$. Then $\mathcal{F} := F(K)$ is τ_p-compact and does not contain a copy of $\beta\mathbb{N}$) (see [8, 6.3.19 c)]). By assumption $F(D)$ is an uniformly bounded set of continuous function on H so it has the Bourgain property with respect to μ by Theorem 8. Since $F(D)$ is τ_p-dense in $\mathcal{F}$ then $\mathcal{F}$ also has Bourgain's property (Theorem 6) so f_x is μ-measurable for each $x \in K$. Now the continuity of φ follows from Corollary 6. ∎

The last theorem, together with some of the main results in this paper, allows us to give a proof of the following result [15]. If K is a compact space and D a dense subset of K then $\tau_p(D)$ (resp. $\tau_p(K)$) denotes the topology on $C(K)$ of pointwise convergence on D (resp. K).

Theorem 12 *Let K be a separable compact space and D a countable dense subset of K. If $(C(K), \tau_p(K))$ is Lindelöf then every bounded $\tau_p(D)$-compact set $H \subset C(K)$ is norm separable.*

Proof. Suppose that $H \subset C(K)$ is a $\tau_p(D)$-compact set such that $\| f \| \leq 1$ for all $f \in H$. By a result of [18] in order to prove that H is norm separable it is enough to show that it is fragmented by the norm. We shall prove this fact showing that for each Radon probability μ on the compact space $(H, \tau_p(D))$ every norm closed subset of H is μ-measurable [13].

Since $(C(K), \tau_p(K))$ is Lindelöf, K has countable tightness [29] so K does not contain a copy of $\beta\mathbb{N}$. Taking $f(h, x) := h(x)$ the hypothesis of Theorem 11 holds and the proof of this theorem shows that $\mathcal{F} := \{f_x : x \in K\}$ has the Bourgain property with respect to μ, so it is formed by μ-measurable functions. Having in mind that functions from $\mathcal{F}$ are continuous on the Lindelöf space $(H, \tau_p(K))$ it is easy to conclude that μ is $\mathcal{F}$-smooth and then, by Proposition 3, there exists a set S separating the family $\mathcal{F}$.

Since every Radon measure is perfect Corollary 5 gives us that the natural mapping $(\mathcal{F}|_S, \tau_p(S)) \to (\mathcal{F}^\mu, \| \ \|_1)$ is a homeomorphism. By standard arguments the metrizability of $(\mathcal{F}|_S, \tau_p(S))$ implies that S is norm separable. Since S is norm closed in $C(K)$ we have that $(S, \| \ \|)$ is a polish space so $(S, \tau_p(D))$ is analytic and a classical result gives us that S is a μ-measurable set. Now it is easy to see that $\mu(S) = 1$ (Assuming that there exists in H a τ_p-open set $U \supset S$ such that $\mu(U) < 1$ then $\bigcap\{Z \setminus U : Z \in \mathcal{Z}_\mathcal{F}, \mu(Z) = 1\} = \emptyset$. By the Lindelöf property of $(H, \tau_p(K))$ we have that $\bigcap_n\{Z_n \setminus U\} = \emptyset$ for some sequence $Z_n \in \mathcal{Z}_\mathcal{F}$, $\mu(Z_n) = 1$. Then $\mu(Z_n \setminus U) < 1 - \mu(U)$ for some $n \in \mathbb{N}$ and we arrive at a contradiction: $\mu(Z_n) = \mu(Z_n \cap U) + \mu(Z_n \setminus U) < 1$).

Now, given a norm closed subset C of H we obtain that C is μ-measurable because $C = (C \cap S) \cup (C \setminus S)$ where $C \cap S$ is μ-measurable (it is analytic) and $C \setminus S$ is μ-null. ∎

Pettis integrability. If X is a Banach space a function $f : \Omega \to X$ is weakly measurable (resp. weakly integrable) if x^*f is measurable (resp. integrable) for each $x^* \in X^*$. If f is weakly integrable then for every $E \in \Sigma$ there exists $x_E^{**} \in X^{**}$ such that $x_E^{**}(x^*) = \int_E x^* f d\mu$ for all $x^* \in X^*$. If $x_E^{**} \in X$ for all $E \in \Sigma$ then f is called Pettis integrable. In this case the indefinite integral $m_f : \Sigma \to X$, $m_f(E) = x_E^{**}$ is countably additive. If $\{x^*f : \| x^* \| \leq 1\}$ is bounded in $\mathcal{L}^\infty(\mu)$ then f is said to be μ-weakly bounded. The Banach space X is said to have the Pettis integral property (shortly PIP) if each μ-weakly bounded function from an arbitrary finite measure space (Ω, Σ, μ) into X is μ-Pettis integrable.

Every weakly measurable function $f : \Omega \to X$ is $Baire(X, \text{weak})$-measurable [6] and the image measure $\lambda(B) := \mu(f^{-1}(B))$ defined on $Baire(X, \text{weak})$ is considered in [6] and [31] in order to characterize some facts concerning weak measurability and Pettis integrability.

In order to show how to obtain results of Pettis integrability from the results in this paper note that the μ-Pettis integrability of a function f is equivalent to the λ-Pettis integrability of the identity $i : (X, Baire(X, \text{weak}), \lambda) \to X$, where $\lambda = \mu f^{-1}$, which is equivalent to the continuity of $I : (B_{X^*}, weak^*) \to (L^1(\mu), \text{weak})$ (see [6],[31, 4-1-7]). Thus we consider a finite measure λ defined on $Baire(X, \text{weak})$ and apply previous results to the measure space $(X, Baire(X, \text{weak}), \lambda)$ and the family $\mathcal{F} := B_{X^*}$. Note that B_{X^*} is uniformly λ-integrable if f is μ-weakly bounded. In this situation the notion of a weakly measurable function f determined by a subspace Y of X, considered in [11], [28] and [21], translates to B_{X^*} is determined by Y (with respect to the image measure). Now $\mathcal{Z}_{\mathcal{F}}$ is formed by the sets which are finite intersections of closed semispaces and if the Baire measure λ $(=\mu f^{-1})$ is B_{X^*}-smooth we shall say that it is c-smooth. In this case the closed convex set S of Proposition 3 will be denoted $core(\lambda)$ and it can be showed that $core(\lambda)$ is the closed convex hull of $AR(f, \mu) = \{\frac{1}{\mu(E)} \int_E f d\mu : E \in \Sigma, \mu(E) > 0\}$.

Note that in the case we are concerned, if f is Pettis integrable then the measure image $\lambda = \mu f^{-1}$ has the following special property: For every $E \in \Sigma$, $\lambda(E) > 0$, there exists $\omega_E \in \Omega$ such that $\int f d\lambda_E = f(\omega_E)$ for each $f \in \mathcal{F}$, $(\lambda_E(B) := \lambda(E \cap B)/\lambda(E))$. This fact allows us to characterize Pettis integrability through the image measure [7], [31].

Theorem 13 *If $f : \Omega \to X$ is μ-weakly bounded and $\lambda := \mu f^{-1}$ is the image measure defined on $Baire(X, weak)$ the following are equivalent*

a) λ is c-smooth;

b) f is μ-integrable Pettis;

Then X has the Pettis integral property if and only if every finite measure on $Baire(X, weak)$ is c-smooth.

Proof. a) $\Rightarrow$ b) by Corollary 1. b) $\Rightarrow$ a) Suppose λ is not c-smooth and take a net $Z_\alpha \in \mathcal{Z}_\mathcal{F}$ decreasing to $\emptyset$ such that $\inf_\alpha \lambda(Z_\alpha) = \epsilon > 0$. If Z_{α_n} is a decreasing sequence such that $\lim_n \lambda(Z_{\alpha_n}) = \epsilon$ then the set $E := \cap_n Z_{\alpha_n}$ has $\lambda(E) = \epsilon$ and by Pettis integrability there exists $x_E \in X$ such that $x^*(x_E) = \lambda(E)^{-1} \int_E x^* \, d\mu$ for each $x^* \in X$. We will find a contradiction showing that $x_E \in Z_\alpha$ for every α. In order to do this we consider a fixed Z_β, which is a finite intersection of closed semispaces $C := \{x \in X : x^*(x) \leq t\}$, and for such a C we prove that $x_E \in C$. Since $Z_\beta \subset C$ we have $\inf_\alpha \lambda(C \cap Z_\alpha) = \epsilon$ and then $\lambda(C \cap E) = \lim_n \lambda(C \cap Z_{\alpha_n}) = \epsilon = \lambda(E)$ so E is essentially contained in C, i.e. $x^*(x) \leq t$ λ-a.e. Then $x^*(x_E) \leq t$ and the proof of b) is finished.

In order to obtain that if X has the PIP then every Baire measure λ is c-smooth it is enough to use the following lemma, applied to the family B_{X^*}, to obtain a sequence of c-smooth measures which is uniformly convergent to λ, so λ is c-smooth. ■

Lemma 2 *If $\mathcal{F}$ is a pointwise bounded family of measurable functions then there exists an increasing sequence of measurable sets $\Omega_n \in \Sigma$ such that $\Omega = \cup_n \Omega_n$ and $\mathcal{F}$ is essentially bounded with respect to each measure $\lambda_n(E) := \lambda(E \cap \Omega_n)$.*

Proof. Let $h : \Omega \to [0, +\infty)$ be a measurable function which is the essential supremum of the family $\{|f| : f \in \mathcal{F}\}$ and take $\Omega_n = \{h < n\}$. ■

A Banach space X has the Mazur property if the sequentially weak* continuous functionals on X^* are in X and has the property C (of Corson) if any collection of closed convex subsets of X with the countable intersection property has non void intersection. Pol's dual characterization of property C [22] states that X has property C if and only if for each $A \subset B_{X^*}$ and each x^* in the weak* closure of A there exists a countable set $M \subset A$ such that x^* is in the weak* closure of $co(M)$.

In the following theorem, parts a) and b) come from [6] and part c) from [26]. Part d) is an improvement of a result in [28] where it is showed that if the dual of X is weakly compactly generated then X has the Pettis integral property

Theorem 14 *The Banach space X has the Pettis integral property in the following cases*

a) X has property C;

b) X has the Mazur's property;

c) If each $x^{**} \in X^{**}$ is in the $\sigma(X^{**}, X^*)$-closure of a countable $M \subset X$;

d) X^* has property C.

Proof. a) follows from Theorem 13 and b) from Corollary 2 with $S = X$. c) is a direct consequence of Theorem 1: Take $x_E^{**} \in X^{**}$ defined by $x_E^{**}(x^*) = \int_E x^* f d\lambda$ which is in the $\sigma(X^{**}, X^*)$-closure of a countable $M \subset X$, so M verifies the hypothesis of this theorem for $\mathcal{F} := B_{X^*}$. d) is a consequence of c) and Pol's dual characterization of property C: For each $x^{**} \in B_{X^{**}}$ there exists a countable set $C \subset B_X$ such that x^{**} is in the weak*-closure of $co(C)$. If M is the countable set formed by the rational convex combinations of elements from C then the condition required in c) holds. ∎

Parts b) and c) of the next theorem are reformulations, in terms of image measure, of Stefansson's characterization of Pettis integrability. Part d), obtained from Drewnowsky's ideas in [4], completes this characterization adding a new case in terms of property C. If λ is a Baire measure on $Baire(X, \text{weak})$ and B_{X^*} is determined by $Y \subset X$ with respect to λ (i.e. if $x^*|_Y = 0$ then $x^* = 0$, λ-a.e.) then we shall say that λ is localized on Y.

Corollary 7 *If X is a Banach space and λ a finite measure on $Baire(X$, weak) the following are equivalent:*

a) λ is c-smooth;

b) λ is localized on a WCG subspace;

c) λ is localized on a subspace having Mazur's property;

d) λ is localized in a subspace having property C.

Proof. a) $\Rightarrow$ b) If λ is c-smooth, Lemma 2 applied to the family B_{X^*} provides us with a sequence of c-smooth measures λ_n such that the identity $i : (X, Baire(X, \text{weak}), \lambda_n) \to X$ is Pettis integrable for each $n \in \mathbb{N}$ (Theorem 13). Every λ_n is localized on the WCG subspace Y_n generated by the range of the corresponding Pettis integral so the WCG subspace Y generated by $\cup_{n\in\mathbb{N}} Y_n$ shows that b) holds.

b) $\Rightarrow$ c) and b) $\Rightarrow$ d) are obvious.

c) $\Rightarrow$ a) If c) holds and λ_n is the sequence given by Lemma 2 then each λ_n verifies c). The Mazur property and part a) of Theorem 3 give us that $i : (X, Baire(X, \text{weak}), \lambda_n) \to X$ is Pettis integrable. Then λ is c-smooth because it is the uniform limit of the sequence λ_n, formed by c-smooth measures (Theorem 13).

d) $\Rightarrow$ a) In a similar way we have that every λ_n verifies d) and Pol's dual characterization of property C ([22]) tell us that every convex countably closed set in (B_{Y^*}, weak^*) is weak* closed. By Corollary 4 the identity $i : (X, Baire(X, \text{weak}), \lambda_n) \to X$ is Pettis integrable and the proof concludes as in the previous case. ∎

Remark *a)* $\Leftrightarrow$ *b)* in Corollary 7 can be used to give a easy proof of a version of Vitali's convergence theorem for the Pettis integral [11],[27, Th.2.10]. In the preliminary version of this result given in [10] the measure was assumed to be perfect. The general result was stated in [17] but it was showen using the difficult theorem of James.

Using Banach spaces techniques such as the Davis-Figiel-Johnson-Pelczynsky theorem, in [27] it was proved that given a bounded weakly measurable function $f : \Omega \to X^*$ into a dual of a weakly compactly generated space, then f is Pettis integrable if and only if f is determined by a separable subspace of X^*. Now we present a measure image version of this result with a short proof based on the main result of this paper.

Theorem 15 *If the Banach space X is isomorphic to a subspace of a dual of a WCG space and λ is a c-smooth measure defined on $Baire(X$, weak$)$ then λ is localized on a separable subspace of X.*

Proof. Let Y be a WCG space such that $X \subset Y^*$. The general case reduces to the case $X = Y^*$ by considering the image measure $\overline{\lambda}$ of λ in Y^* which is c-smooth and verifies $core(\overline{\lambda}) \subset X$. We start assuming that $S := core(\lambda)$ is bounded. It is enough to show that $(B_{X^*}|_S, \tau_p(S))$ is a metrizable compact space. By Theorem 1 we know that it is Eberlein's compact so it is metrizable if and only if it is separable. Since $B_Y|_S$ is $\tau_p(S)$-dense in $(B_{X^*}|_S, \tau_p(S))$ it is enough to prove that $B_Y|_S$ is $\tau_p(S)$-separable.

If λ_0 is the restriction of λ to $Baire(Y^*, \text{weak}^*)$ then λ_0 is B_Y-smooth so there exists a weak* compact set $T \supset S$ such that the family B_Y is separated by T with respect to the measure λ_0. If Y is generated by the weak compact K then Theorem 10 tell us that the compact space $(K|_T, \tau_p(T))$ is metrizable and so it is separable. Using a well known theorem of Troallic we have that $K|_T$ is a norm separable subset of $(C(T), \| \ \|)$. Since $Y|_T$ is a subspace of $(C(T), \| \ \|)$ generated by $K|_T$ it follows that $Y|_T$ is norm separable. Having in mind that $S \subset T$ we obtain that $B_Y|_S$ is $\tau_p(S)$ separable as required.

If we do not assume that $core(\lambda)$ is bounded, the same result can be obtained by considering a sequence of c-smooth measures λ_n such as in Lemma 2 (note that $core(\lambda_n)$ is bounded for each $n \in \mathbb{N}$). ∎

Our last applications concern some results on universal Pettis integrability in [23] and [24]. If K is a compact space let $\mathcal{U}(K)$ be the σ-algebra formed by the universally measurable sets (subsets of K which are measurable with respect each finite Radon measure on K). A function $f : K \to X$ is called weakly universally measurable if it is weakly measurable with respect to $\mathcal{U}(K)$. Moreover, if f is Pettis integrable with respect to each Radon measure μ on K then f is said to be universally Pettis integrable.

If μ is a Radon measure on K, a function $f : K \to X^*$ is called weak* Lusin measurable if for each $\epsilon > 0$ there exists a compact $F \subset K$ such that $\mu(K \setminus F) < \epsilon$ and $f|_F \colon F \to X^*$ is weak*-continuous. If this fact happens for every Radon measure μ on K then f is said to be universally weak* Lusin measurable.

If X is a separable Banach space and K a compact space then every bounded weakly universally measurable function $f : K \to X^*$ is universally Pettis integrable [23]. If X is a separable Banach space then every bounded weakly universally measurable function $f : K \to X^*$ is universally weak* Lusin measurable and the above result can be extended to arbitrary Banach spaces assuming that f is universally weak* Lusin measurable (see [24] and [28]).

The next theorem is a result from [24] where the equivalence with property d) was stated with special martingale techniques assuming that K is a metrizable compact space. Applying Theorem 9 we can provide a short proof without this assumption. Recall that if F is a compact space then a set $Z \subset C(F)$ is said to be weakly precompact if every sequence in Z has a pointwise convergent subsequence.

Theorem 16 *If K is a compact space and $f : K \to X^*$ is a bounded universally $\sigma(X^*, X)$-Lusin measurable function, the following are equivalent:*

a) f is weakly universally measurable;

b) f is universally Pettis integrable;

c) $\{< x, f >: x \in B_X\}$ has the Bourgain property with respect to each Radon measure on K;

d) For each compact $F \subset K$ such that $f|_F : F \to X^*$ is $\sigma(X^*, X)$-continuous the set $\{< x, f > |_F : x \in B_X\}$ is weakly precompact in $C(F)$.

Proof. d) $\Rightarrow$ c) If d) holds then by Lusin measurability and Theorem 9 we have that for each Radon measure μ on K and each $\epsilon > 0$ there exists a compact set $F \subset K$ such that $\mu(K \backslash F) < \epsilon$ and $\{< x, f > |_F : x \in B_X\}$ has the Bourgain property with respect to the Radon measure that μ induces in F. From this fact it is easy to see that $\{< x, f >: x \in B_X\}$ has the Bourgain property with respect to μ.

c) $\Rightarrow$ b) follows from Corollary 6, and b) $\Rightarrow$ a) is immediate.

a) $\Rightarrow$ d) If a) holds and $F \subset K$ is a compact set such that $f|_F : F \to X^*$ is $\sigma(X^*, X)$-continuous, we have that $\{< x, f > |_F : x \in B_X\}$ is τ_p-relatively compact in the space of functions which are universally measurable on F (since its τ_p-closure $\{< x^{**}, f > |_F : x^{**} \in B_{X^{**}}\}$ is a τ_p-compact subset of this space). Other application of Theorem 9 give us that $\{< x, f > |_F : x \in B_X\}$ is weakly precompact in $C(F)$. ■

Corollary 8 *If K is a compact space, X a separable Banach space and $f : K \to X^*$ a bounded function the following are equivalent:*

a) f is universally Pettis integrable;

b) $\{< x, f >: x \in B_X\}$ has the Bourgain property with respect to each Radon measure on K.

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