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Pooling Modalities and Pointwise Intersection: Axiomatization and Decidability

Abstract. We establish completeness and the finite model property for logics featuring the *pooling modalities* that were introduced in Van De Putte and Klein (Pooling modalities and pointwise intersection: semantics, expressivity, and applications). The definition of our canonical models combines standard techniques with a so-called "puzzle piece construction", which we first illustrate informally. After that, we apply it to the weakest classical logics with pooling modalities and investigate the technique's potential for the axiomatization of stronger logics, obtained by imposing well-known frame conditions on the models.

Keywords: Pointwise intersection, Pooling modalities, Classical modal logics, Completeness, Finite model property, Puzzle piece construction.

1. Introduction

Pooling modalities are a specific type of modal operators, interpreted in terms of neighbourhood models and the operation of pointwise intersection. Logics featuring such modalities were introduced in [8], where their relation to normal modal logics, their expressive power, and some potential applications of these logics are studied. The main aim of this more technical paper is to give sound and strongly complete axiomatizations for various logics with pooling modalities and to show that they satisfy the finite model property (and hence are decidable). We do so using an altogether new technique for building canonical models in combination with other, familiar techniques from the general theory of modal logics.

In the remainder of this introduction, we briefly recapitulate the central notions from [8] in an informal, loose way. Exact definitions are recalled in Section 2. We refer the reader to [8] for an elaborate discussion, motivation, and investigation of the language and semantics of pooling modalities.

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Somewhat loosely speaking, *pooling modalities* are classical modal operators in the sense of [3] that allow us to express that a given proposition φ coincides with the intersection of various propositions that are each associated with a given index i. According to one specific interpretation, such modalities allow us to formalize reasoning about the aggregation of evidence of a given agent i or of several agents $i_1, \ldots, i_n$. In this context, $\Box_{\{1,1,2\}}p$ may be read as: "aggregating two pieces of evidence of agent 1 with one piece of evidence of agent 2, we can obtain the evidence that p".

This way, pooling modalities allow to express information that is implicit among agents having various pieces of evidence, but that can be obtained by pooling their evidence. The multiset $\{1, 1, 2\}$ that is attached to the modal operator indicates (a) the agents whose evidence we are combining, and (b) the number of pieces of evidence that we may take from each agent.

It should however be noted that this is only one among many potential applications of pooling modalities. Alternatively, one may use them to interpret notions of group agency and coalitional abilities, distributed (non-normal) belief or knowledge, implicit obligations, or collective norms [8, Section 6]. In the present paper we abstract from those various applications of the logics and focus on their axiomatization and related issues.

Semantically, pooling modalities are interpreted over multi-index neighbourhood models, i.e. possible worlds models with a domain W that have a distinctive neighbourhood function $\mathcal{N}_i : W \to \wp(\wp(W))$ for each index i in a given index set I. Here, each neighbourhood function $\mathcal{N}_i$ records the set of propositions that are associated with the index i at each world w in the model. Pooling modalities are interpreted over such models in terms of the operation of *pointwise intersection* of neighbourhood sets. Loosely speaking, pointwise intersection is the intersections of neighbourhoods $X \in \mathcal{N}_i(w)$ (rather than the entire neighbourhood sets or collections $\mathcal{N}_i(w)$), where the number of neighbourhoods that we may take from each neighbourhood set is specified by the index of the pooling modality, cf. (a) and (b) above. So for instance $\Box_{\{1,1,2\}}(p \wedge q)$ will be true at the world w if and only if the proposition denoted by $p \wedge q$ in the model coincides with the intersection of three sets X, Y, Z, where $X, Y \in \mathcal{N}_1(w)$ and $Z \in \mathcal{N}_2(w)$.

The plan for this paper is as follows. In Section 2 we recapitulate the notation and definitions from [8] that are required for present purposes. In the three subsequent sections, we focus on the weakest modal logics that can be interpreted in terms of pointwise intersection. While arguably, many applications require stronger logics, focusing on the weakest modal logics first allows us to present our results in their most general form. We present the axiomatization of these logics in Section 3 and show our axioms to be logically independent. Section 4 provides an informal explanation of the new technique for constructing canonical models that is implemented in Section 5. In Section 6, we turn to a range of stronger logics that are suited for various applications. These logics are obtained by imposing various well-known frame conditions on our models. An overview of our main results is given in Section 7, after which we conclude with a number of pointers to open issues and future work (Section 8).

2. Preliminaries

In this section, we recall the notion of a pooling profile, and subsequently introduce the formal languages and semantics of classical logics with pooling modalities from [8]. We will introduce some new notation and observations along the way, but clearly indicate this when doing so.

2.1. Pooling Profiles

Let $\mathbb{N}$ denote the set of natural numbers, and let $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$. Let $I = \{1, 2, \ldots\}$ be a countable set of *indices*. We start with some crucial concepts and notation:

Definition 1. ***Pooling profiles*** are functions of the type $M : I \to \mathbb{N} \cup \{\infty\}$, where (a) for only finitely many $i \in I$, $M(i) \neq 0$ and (b) for at least one $i \in I$, $M(i) > 0$.

$\mathbb{M}_\infty$ denotes the set of all pooling profiles, while $\mathbb{M}_f \subset \mathbb{M}_\infty$ is the set of all ***finitary*** pooling profiles, i.e. pooling profiles of the type $M : I \to \mathbb{N}$. Where $M \in \mathbb{M}_\infty$, $I(M) =_{\mathsf{df}} \{i \in I \mid M(i) \neq 0\}$.

Note that pooling profiles can be seen as finite, non-empty multisets. We will often switch from functional to a simplified relational notation, writing every pooling profile as a finite set of pairs (i, k) for $k \in \mathbb{N}^+ \cup \{\infty\}$, thus omitting all pairs $(j, 0)$.

The sets $\mathbb{M}_\infty$, resp. $\mathbb{M}_f$ can also be interpreted in algebraic terms. In order to explain this, we first introduce a natural notion of adding up two pooling profiles:

Definition 2. Let $M, M' \in \mathbb{M}_\infty$. The ***union of*** M ***and*** M' is defined as $M \sqcup M' := \{(i, k + k') \mid (i, k) \in M, (i, k') \in M'\}$, where $\infty + k = k + \infty = \infty$ for all $k \in \mathbb{N} \cup \{\infty\}$.

It can then be easily verified that $\mathbb{M}_\infty$ is the closure of the set

$$\{\{(i, 1)\}, \{(i, \infty)\} \mid i \in I\}$$

under $\sqcup$. Analogously, $\mathbb{M}_f$ is the closure of $\{\{(i, 1)\} \mid i \in I\}$ under $\sqcup$.

Table 1. Languages with pooling modalities

$\mathfrak{L}_\infty$	$\varphi{:=}p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_\infty$
$\mathfrak{L}_\infty^{[\forall]}$	$\varphi{:=}p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi \mid [\forall]\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_\infty$
$\mathfrak{L}_f$	$\varphi{:=}p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$
$\mathfrak{L}_f^{[\forall]}$	$\varphi{:=}p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi \mid [\forall]\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$

Intuitively, a pooling profile M indicates, for each $i \in I$, the number of sets $X \in \mathcal{N}_i(w)$ that we can use in order to obtain a member Y of the neighbourhood set $\mathcal{N}_M(w)$. The symbol ∞ can be read as "arbitrarily many". Thus, $X \in \mathcal{N}_{\{(1,\infty)\}}(w)$ means that X is the result of intersecting an arbitrary number of members of $\mathcal{N}_1(w)$, whereas $Y \in \mathcal{N}_{\{(1,2),(2,\infty)\}}(w)$ expresses that Y can be obtained by intersecting two members of $\mathcal{N}_1(w)$ with an arbitrary number of members of $\mathcal{N}_2(w)$.[1]

2.2. Formal Languages

Table 1 gives the Backus Naur Forms of the four formal languages that will be studied in this paper. All of them extend the standard classical propositional language (based on a countable set of propositional variables $\mathfrak{P}$) with unary modal operators of the type $\Box_M$. The richest of these, $\mathfrak{L}_\infty^{[\forall]}$, contains pooling modalities $\Box_M$ for any $M \in \mathbb{M}_\infty$ and a universal modal operator $[\forall]$. The other three languages are obtained by skipping $[\forall]$ and/or restricting to ***finitary pooling modalities***, i.e. operators $\Box_M$ for $M \in \mathbb{M}_f$.

In examples and informal digressions, we will sometimes use more sloppy notation, writing e.g. $\Box_{1,1,2}p$ instead of $\Box_{\{(1,2),(2,1)\}}p$. We will also abbreviate a sequence of index i repeated k times by i^k, and use i^∞ to refer to "any arbitrary number of i's". Finally we require some additional notation that has not yet been introduced in [8]:

Definition 3. (a) For $M \in \mathbb{M}_\infty$, let $\delta(\mathbf{M}){:=}\{(i,k) \in I \times \mathbb{N}^+ \mid k \leq M(i)\}$, where we stipulate that $k < \infty$ for all $k \in \mathbb{N}$. We will refer to $(i,k) \in \delta(M)$ as ***the*** k***th occurrence of*** i ***in*** M. (b) Where $M, N \in \mathbb{M}_\infty$, the ***weak partial order*** $\sqsubseteq$ is defined by $M \sqsubseteq N$ iff for all i, $M(i) \leq N(i)$.

2.3. Semantics

The formal language $\mathfrak{L}_\infty^{[\forall]}$ is interpreted in terms of neighbourhood models, where each index $i \in I$ receives a distinct neighbourhood function $\mathcal{N}_i$:

[1] "Arbitrary" should be interpreted here in the strongest possible sense, corresponding to the third item of Definition 5.

Definition 4. A ***model*** $\mathfrak{M}$ is a triple $\langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$, where (a) $W \neq \emptyset$ is the ***domain*** of $\mathfrak{M}$, (b) for every $i \in I$, $\mathcal{N}_i : W \to \wp(\wp(W))$ is a ***neighbourhood function for*** i, and (c) $V : \mathfrak{P} \to \wp(W)$ is a ***valuation function***.

Where $w \in W$ and $i \in I$, we refer to the members of a given $\mathcal{N}_i(w)$ as the *neighbourhoods for* i *at* w, where we call $\mathcal{N}_i(w)$ the *neighbourhood set for* i *at* w. In order to spell out the semantics for the pooling modalities in exact terms, we first define the notion of pointwise intersection, after which we apply it to the models from Definition 4.

Definition 5. Let D be a set, let $\mathcal{X}, \mathcal{Y} \subseteq \wp(\wp(D))$, and let $k \in \mathbb{N}^+$.

1. $\mathcal{X} \Cap \mathcal{Y} {:=} \{X \cap Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ is the ***pointwise intersection of*** $\mathcal{X}$ ***and*** $\mathcal{Y}$.
2. $\Cap^k \mathcal{X} {:=} \{X_1 \cap \ldots \cap X_k \mid X_1, \ldots, X_k \in \mathcal{X}\}$ is the ***pointwise*** k***-intersection of*** $\mathcal{X}$ ***with itself.***
3. $\Cap^\infty \mathcal{X} {:=} \{\bigcap \mathcal{Y} \mid \mathcal{Y} \subseteq \mathcal{X}\}$ is the ***pointwise arbitrary intersection of*** $\mathcal{X}$ ***with itself.***

Definition 6. Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a neighbourhood model and let $M \in \mathbb{M}_\infty$, with $I(M) = \{i_1, \ldots, i_n\}$. The ***neighbourhood function*** $\mathcal{N}_M$ ***in*** $\mathfrak{M}$ is defined as follows: for every $w \in W$,

$$\mathcal{N}_M(w) = (\Cap^{M(i_1)} \mathcal{N}_{i_1}(w)) \Cap \ldots \Cap (\Cap^{M(i_n)} \mathcal{N}_{i_n}(w)).$$

With these preliminary definitions, we can now give the semantic clauses for the formulas in $\mathfrak{L}_\infty^{[\forall]}$, and hence, by restriction, also for $\mathfrak{L}_\infty$, $\mathfrak{L}_f^{[\forall]}$ and $\mathfrak{L}_f$

Definition 7. Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is a model and $w \in W$:

0. $\mathfrak{M}, w \not\models \bot$
1. $\mathfrak{M}, w \models \varphi$ iff $w \in V(\varphi)$ for all $\varphi \in \mathfrak{P}$
2. $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
3. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
4. $\mathfrak{M}, w \models \Box_M \varphi$ iff $\|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_M(w)$
5. $\mathfrak{M}, w \models [\forall]\varphi$ iff for all $w' \in W$, $\mathfrak{M}, w' \models \varphi$

where $\|\varphi\|^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models \varphi\}$.

Validity ($\Vdash \varphi$) and semantic consequence ($\Gamma \Vdash \varphi$) are defined in the standard way, viz. as truth, resp. truth-preservation at all worlds in all models. Since we consider four distinct formal languages (cf. Table 1), these definitions give us four distinct logics. We will henceforth refer to the latter as the *base logics*, and denote them by $\mathbf{BL}_\infty$, $\mathbf{BL}_\infty^{[\forall]}$, $\mathbf{BL}_f$, and $\mathbf{BL}_f^{[\forall]}$ respectively. Note that up to this point, we did not impose any specific frame conditions on the models. Such frame conditions and the associated logics are discussed in Section 6.

New facts Before continuing, we note some properties of pointwise intersections that we will rely upon in the remainder of this paper. The verification of these facts is safely left to the reader.

FACT 1. *For all sets $\mathcal{X}$ of sets, each of the following hold:*

1. $\Cap^1 \mathcal{X} = \mathcal{X}$
2. *for all* $k, l \in \mathbb{N}^+$ *with* $k < l$, $\Cap^k \mathcal{X} \subseteq \Cap^l \mathcal{X}$
3. *for all* $k \in \mathbb{N}^+$, $\Cap^k \mathcal{X} \subseteq \Cap^\infty \mathcal{X}$.

FACT 2. *For all sets $\mathcal{X}, \mathcal{Y}, \ldots$ of sets, each of the following hold:*

1. *for all* $k, l \in \mathbb{N}^+$, $(\Cap^k \mathcal{X}) \Cap (\Cap^l \mathcal{X}) = \Cap^{k+l} \mathcal{X}$
2. *for all* $k \in \mathbb{N}^+$, $(\Cap^k \mathcal{X}) \Cap (\Cap^\infty \mathcal{X}) = \Cap^\infty \mathcal{X}$
3. $\mathcal{X} \Cap \mathcal{Y} = \mathcal{Y} \Cap \mathcal{X}$
4. $(\mathcal{X} \Cap \mathcal{Y}) \Cap \mathcal{Z} = \mathcal{X} \Cap (\mathcal{Y} \Cap \mathcal{Z})$

3. Axiomatization of the Four Base Logics

In this section, we start by noting a number of key validities of the four base logics (Section 3.1) and showing the independence of these validities (Section 3.2). After that, we present the axiomatization of the four base logics (Section 3.3) and clarify an issue related to their compactness (Section 3.4).

3.1. Validities

First, as usual, $[\forall]$ behaves like a normal operator of type **S5** in $\mathbf{BL}_\infty^{[\forall]}$ and $\mathbf{BL}_f^{[\forall]}$. Second, since the models we work with are a specific type of neighbourhood models, it follows immediately that if two propositions φ, ψ are equivalent in any of the logics, then $\Box_M \varphi$ entails $\Box_M \psi$ and vice versa. In other words, pooling modalities are *classical* modal operators in the sense of [3]:

$$\text{if } \Vdash \varphi \leftrightarrow \psi, \text{ then } \Vdash \Box_M \varphi \rightarrow \Box_M \psi \tag{RE}$$

In the languages with the global modality $[\forall]$, we can moreover express that in a given model, φ and ψ are co-extensive (i.e. they are true at the same set of worlds). This gives us the stronger property of replacement under global equivalents:

$$\Vdash [\forall](\varphi \leftrightarrow \psi) \to (\Box_M \varphi \to \Box_M \psi) \tag{RGE}$$

Third, the base logics satisfy four central *bridging principles*, i.e. validities that link distinct operators $\Box_M, \Box_{M'}, \ldots$ to one another. For the fourth of these, we use the convention that $M^\infty :=_{\mathsf{df}} \{(i, \infty) \mid i \in I(M)\} \cup \{(i, 0) \mid i \notin I(M)\}$. The four key bridging principles to be discussed can now be written as follows:

$$(\Box_M \varphi \wedge \Box_N \psi) \to \Box_{M \sqcup N}(\varphi \wedge \psi) \tag{B1}$$

$$\Box_{M \sqcup N} \top \to \Box_M \top \tag{B2}$$

$$(\Box_M \varphi \wedge \Box_{M \sqcup N \sqcup N'} \varphi) \to \Box_{M \sqcup N} \varphi \tag{B3}$$

$$\Box_M \varphi \to \Box_{M^\infty} \varphi \tag{B4}$$

THEOREM 1. *Each of (B1)–(B4) are valid in every model.*

PROOF. (B1) follows from a simple observation, that is itself an immediate consequence of Fact 2:

FACT 3. *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a model. For all $M, M' \in \mathbb{M}_\infty$ and all $w \in W$, $\mathcal{N}_{M \sqcup M'}(w) = \mathcal{N}_M(w) \Cap \mathcal{N}_{M'}(w)$.*

For (B2), suppose that $\mathfrak{M}, w \models \Box_{M \sqcup N} \top$. Hence, for every $i \in I(M \sqcup N)$, there is an $X_i \in \mathcal{N}_i(w)$ such that $X_i = W$. Note that $I(M) \subseteq I(M \sqcup N)$, and $\bigcap_{i \in I(M)} X_i = W \in \mathcal{N}_M(w)$. Hence, $W \in \mathcal{N}_M(w)$.

For (B3), suppose that $\mathfrak{M}, w \models \Box_M \varphi$ and $\mathfrak{M}, w \models \Box_{M \sqcup N \sqcup N'} \varphi$. Applying Fact 3 twice, we obtain:

$$\mathcal{N}_{M \sqcup N \sqcup N'}(w) = \mathcal{N}_M(w) \Cap \mathcal{N}_N(w) \Cap \mathcal{N}_{N'}(w)$$

and, hence, (i) there is an $X \in \mathcal{N}_M(w)$ such that $X = \|\varphi\|^{\mathfrak{M}}$ and (ii) there are $X' \in \mathcal{N}_M(w)$, $Y' \in \mathcal{N}_N(w)$, and $Z' \in \mathcal{N}_{N'}(w)$ such that $X' \cap Y' \cap Z' = \|\varphi\|^{\mathfrak{M}}$. It follows that $Y' \supseteq \|\varphi\|^{\mathfrak{M}}$, and hence $X \cap Y' = X = \|\varphi\|^{\mathfrak{M}}$. Again by Fact 3, $X \cap Y' \in \mathcal{N}_{M \sqcup N}(w)$, and hence $\mathfrak{M}, w \models \Box_{M \sqcup N} \varphi$.

Finally, for (B4), it suffices to observe that, by Fact 1, for all $M \in \mathbb{M}_\infty$, $\mathcal{N}_M(w) \subseteq \mathcal{N}_{M^\infty}(w)$. ∎

Let us briefly comment on each of the bridging principles. First, (B1) is probably the most expected among the four; it simply expresses that whenever $X \in \mathcal{N}_M(w)$ and $Y \in \mathcal{N}_N(w)$, then $X \cap Y \in \mathcal{N}_{M \sqcup N}(w)$. When applied to $M = N = \{i\}$, this principle gives us a more fine-grained variant of the aggregation axiom (C), keeping track of the number of pieces of information one needs to combine in order to arrive at a given proposition: $(\Box_i\varphi \wedge \Box_i\psi) \rightarrow \Box_{i,i}(\varphi \wedge \psi)$.

Principle (B2) concerns the border cases where $W \in \mathcal{N}_{M \sqcup N}(w)$; in that case, each $i \in I(M \sqcup N)$ must contain the unit W. Note that the consequent of this principle is vacuously true for models in which every neighbourhood contains the unit.

(B3) is perhaps the least intuitive property of the four. Applied to the case with $M = \{1\}$, $N = \{2\}$, and $N' = \{3\}$, it expresses the following: if X is a member of $\mathcal{N}_1(w)$ and X is a member of the pointwise intersection of $\mathcal{N}_1(w)$, $\mathcal{N}_2(w)$, and $\mathcal{N}_3(w)$, then X is also a member of the pointwise intersection of $\mathcal{N}_1(w)$ and $\mathcal{N}_2(w)$.

Finally, (B4) deals explicitly with the pooling profiles $M \in \mathbb{M}_\infty \backslash \mathbb{M}_f$. As we show below, one can use the other axioms to derive a kind of iteration principle (see Theorem 3.3). (B4) essentially generalizes this principle to the case where we have infinitely many indices in the consequent.

3.2. Proof of Independence

Importantly, none of the four principles (B1), (B2), (B3), and (B4) can be derived from any combination of the others. In order to show this, we need to define a more general semantics that allows us to invalidate some of the principles while validating the others.

Definition 8. A g-model is a triple $\mathfrak{G} = \langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V \rangle$, where $W \neq \emptyset$ is ***the domain of*** $\mathfrak{G}$, for every $M \in \mathbb{M}_\infty$, $\mathcal{N}_M : W \rightarrow \wp(\wp(W))$ is ***a neighbourhoud function***, and $V : \mathfrak{P} \rightarrow \wp(W)$ is a valuation function.

Definition 9. Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V \rangle$ is a g-model and $w \in W$:

0. $\mathfrak{M}, w \not\models \bot$
1. $\mathfrak{M}, w \models \varphi$ iff $w \in V(\varphi)$ for $\varphi \in \mathfrak{P}$
2. $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
3. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
4. $\mathfrak{M}, w \models \Box_M\varphi$ iff $\|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_M(w)$
5. $\mathfrak{M}, w \models [\forall]\varphi$ iff for all $w' \in W$, $\mathfrak{M}, w' \models \varphi$

where $\|\varphi\|^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models \varphi\}$.

The basic idea behind a g-model is that it associates a distinct neighbourhood function with each pooling profile, rather than taking $\mathcal{N}_M$ to be defined in terms of the primitive neighbourhood functions $\mathcal{N}_i$. In other words, pointwise intersection corresponds to the additional assumption on g-models that they be *pooled*, i.e., that for every $M \in \mathbb{M}_\infty$, $\mathcal{N}_M(w)$ satisfies the equation in Definition 6. With this generalized semantics at hand we can prove the following theorem:

THEOREM 2. *Each of (B1), (B2), (B3), and (B4) are logically independent.*

PROOF. *Ad (B1).* Let $I = \{1, 2\}$. Define the g-model $\mathfrak{G}=\langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V\rangle$ as follows. First, $W = \{w\}$. Second, whenever $I(M) = \{1\}$ or $I(M) = \{2\}$, $\mathcal{N}_M = \{W\}$. For all other M, $\mathcal{N}_M(w) = \emptyset$. Finally, for all $\varphi \in \mathfrak{P}$, $V(\varphi) = W$. Note that, since $\mathfrak{G}, w \models \Box_1 \top \wedge \Box_2 \top \wedge \neg \Box_{\{1,2\}} \top$, (B1) is not valid in $\mathfrak{G}$. In general, $\mathfrak{G}, w \models \Box_M \varphi$ for some φ iff either $I(M) = \{1\}$ or $I(M) = \{2\}$. Using this observation, it is a matter of routine to verify that (B2), (B3), and (B4) are all valid in $\mathfrak{G}$.

Ad (B2). Let $I = \{1\}$. Define the g-model $\mathfrak{G} = \langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V\rangle$ as follows. Let $W = \{w\}$, let $\mathcal{N}_1(w) = \emptyset$, and for all proper pooling profiles M of the type $\{(1, k)\}$ $(k \geq 2)$, $\mathcal{N}_M(w) = \{W\}$. Finally, for all $\varphi \in \mathfrak{P}$, $V(\varphi) = W$. Note that $\Box_{1,1} \top$ is true at w in $\mathfrak{G}$ and $\Box_1 \top$ is false, contradicting (B2). It is safely left to the reader to check that all instances of (B1), (B3), and (B4) are valid in this model.

Ad (B3). Let $I = \{1, 2, 3\}$ and let $\mathfrak{G} = \langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V\rangle$ be defined as follows. First, $W = \{w_1, w_2\}$. For all $\varphi \in \mathfrak{P}, V(\varphi) = \{w_1\}$. The neighbourhood functions are defined by cases:

Case 1: $I(M) = \{1\}$ or $I(M) = \{1, 2, 3\}$. Then $\mathcal{N}_M(w) = \{\{w\}\}$ for all $w \in W$

Case 2: $I(M) \neq \{1\}$ and $I(M) \neq \{1, 2, 3\}$. Then $\mathcal{N}_M(w) = \emptyset$ for all $w \in W$

Note that at w_1, $\Box_1 p \wedge \Box_{1,2,3} p$ holds, but $\Box_{1,2} p$ fails. So it remains to check the validity of the other axioms. For (B1), suppose that $\mathfrak{G}, w \models \Box_M \varphi \wedge \Box_N \psi$ for a $w \in W$. This implies that, for both M and N, Case 1 of the above definition applies. From this we can infer that Case 1 also applies to $M \sqcup N$, and that $\|\varphi\|^{\mathfrak{G}} = \|\psi\|^{\mathfrak{G}} = \|\varphi \wedge \psi\|^{\mathfrak{G}} = \{w\}$, whence also $\mathfrak{G}, w \models \Box_{M \sqcup N}(\varphi \wedge \psi)$.

(B2) holds trivially, since, for any M, $\Box_M \top$ is false at both w_1 and w_2. Finally, for (B4), suppose that $\mathfrak{G}, w \models \Box_M \varphi$. This again means that Case 1 of our definition applies, and hence also Case 1 applies to M^∞. Consequently, $\mathfrak{G}, w \models \Box_{M^\infty} \varphi$.

Ad (B4). Let $I = \{1\}$. Let $\mathfrak{G} = \langle W, \langle \mathcal{N}_M \rangle_{M \in \mathbb{M}_\infty}, V\rangle$, where $W = \{w\}$, $V(\varphi) = W$ for all $\varphi \in \mathfrak{P}$, for all $k \in \mathbb{N}$, $\mathcal{N}_{\{(1,k)\}} = W$, and $\mathcal{N}_{\{(1,\infty)\}} = \emptyset$. We safely leave it to the reader to check that each of (B1), (B2), and (B3) are satisfied in $\mathfrak{G}$, whereas (B4) is violated for $M = \{1\}$ and $\varphi = \top$. ∎

Table 2. Axioms and rules and the four base logics. "✓" indicates that the axiom schema is included in the logic; "D" indicates that the axiom schema or rule is derivable

	Axiom/rule	$\mathbf{BL}_\infty^{[\forall]}$	$\mathbf{BL}_f^{[\forall]}$	$\mathbf{BL}_\infty$	$\mathbf{BL}_f$
(B1)	$(\Box_M\varphi \wedge \Box_N\psi) \rightarrow \Box_{M\sqcup N}(\varphi \wedge \psi)$	✓	✓	✓	✓
(B2)	$\Box_{M\sqcup N}\top \rightarrow \Box_M\top$	✓	✓	✓	✓
(B3)	$(\Box_M\varphi \wedge \Box_{M\sqcup N\sqcup N'}\varphi) \rightarrow \Box_{M\sqcup N}\varphi$	✓	✓	✓	✓
(B4)	$\Box_M\varphi \rightarrow \Box_{M^\infty}\varphi$	✓		✓	
(RE)	If $\vdash \varphi \leftrightarrow \psi$, then $\Box_M\varphi \rightarrow \Box_M\psi$	D	D	✓	✓
(RGE)	$[\forall](\varphi \leftrightarrow \psi) \rightarrow (\Box_M\varphi \rightarrow \Box_M\psi)$	✓	✓		
(S5)	All **S5**-axioms and rules for $[\forall]$	✓	✓		
(CL)	Classical propositional logic	✓	✓	✓	✓

3.3. Axiomatization

Throughout this paper, we work with Hilbert-style axiomatizations in terms of axiom schemata. As usual, φ is a ***theorem*** of a given logic ($\vdash \varphi$) if and only if φ can be derived from the axioms of the logic by application of the rules that are valid in the logic. Syntactic consequence is defined as follows: $\Gamma \vdash \varphi$ iff there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$. Note that by this very definition, the syntactic consequence relations we will be working with are compact.

In Section 5 we show that, using (B1)–(B3) in combination with either (RE) or (RGE), we obtain sound and strongly complete axiomatizations for the languages $\mathfrak{L}_f$, resp. $\mathfrak{L}_f^{[\forall]}$. If we moreover add (B4), then we obtain sound and strongly complete axiomatizations for the languages $\mathfrak{L}_\infty$ and $\mathfrak{L}_\infty^{[\forall]}$. See Table 2 for an overview of the resulting logics.

In order to illustrate their strength, we list a number of schemata that can be derived from the above principles in combination with (RE). Theorem 3.1 gives a (weak) sufficient condition for deriving $\Box_{M\sqcup N}\varphi$ from $\Box_M\varphi$: there has to be some ψ such that $\Box_N(\varphi\vee\psi)$ holds. Theorem 3.2 illustrates the strength of (B3), when combined with (B1). Theorem 3.3 gives a kind of iteration property: e.g. it allows us to derive $\Box_{1,1}p$ from $\Box_1 p$, and $\Box_{1,1,2,2,3}q$ from $\Box_{1,2,3}q$. Finally, Theorem 3.4 shows that, in the presence of necessitation, bigger pooling profiles inherit the neighbourhoods of smaller ones.

Theorem 3. *Each of the following are derivable from (B1)–(B4):*

1. $(\Box_M\varphi \wedge \Box_N(\varphi \vee \psi)) \rightarrow \Box_{M\sqcup N}\varphi$
2. $((\Box_M\varphi \wedge \Box_{N\sqcup N'}(\varphi \vee \psi)) \rightarrow \Box_{M\sqcup N}\varphi$

3. $\Box_M \varphi \to \Box_N \varphi$ *whenever* $I(M) = I(N)$ *and* $M(i) \le N(i)$ *for all* $i \in I$
4. $(\Box_M \varphi \wedge \Box_N \top) \to \Box_{M \sqcup N} \varphi$

Moreover, when the mentioned pooling profiles are finitary, we can derive them from (B1)–(B3).

Proof. *Ad 1.* By (B1), (RE), and since $\vdash \varphi \leftrightarrow (\varphi \wedge (\varphi \vee \psi))$.
Ad 2. Suppose the antecedent holds. By item 1, $\Box_{M \sqcup N \sqcup N'} \varphi$. By (B3), $\Box_{M \sqcup N} \varphi$.
Ad 3. Suppose the antecedent holds. Case 1: there is a $k \in \mathbb{N}$ such that $N(i) \le k$ for all $i \in I$. Then by k applications of (B1), we can derive $\Box_\epsilon \varphi$, where the ϵ stands for M followed by a sequence of $k-1$ times "$\sqcup M$". Then by (B3) and the supposition, we obtain $\Box_N \varphi$. Case 2: there is no such k. Hence, $N \in \mathbb{M}_\infty \backslash \mathbb{M}_f$. Then we use (B4) to derive $\Box_{M^\infty} \varphi$, and next apply (B3) to derive $\Box_N \varphi$.
Ad 4. By (B1), (RE), and since $\vdash \varphi \leftrightarrow (\varphi \wedge \top)$. ■

3.4. Compactness

As we will show in the next sections, the above axiomatizations are strongly complete with respect to the respective base logics. As an immediate corollary, the semantic consequence relations of these logics are compact. This may come as a surprise, in view of the seemingly infinitary character of pointwise intersection for pooling profiles $M \in \mathbb{M}_\infty \backslash \mathbb{M}_f$.

In order to explain this feature, we note two crucial design choices of the logics. First, consider the case of a simple pooling profile $M = \{(1, \infty)\}$. Note that $\|p\|_{\mathfrak{M}} \in \mathcal{N}_M(w)$ means that there is *some* subset $\mathcal{X} \subseteq \mathcal{N}_M(w)$ such that $\bigcap \mathcal{X} = \|p\|_{\mathfrak{M}}$. In particular, and as pointed out in Section 2.1, the size of $\mathcal{X}$ is arbitrary. As a result, the following infinite set is satisfiable:

$$\Delta = \{\neg \Box_1 p, \neg \Box_{1,1} p, \neg \Box_{1,1,1} p, \ldots\} \cup \{\Box_M p\}$$

All it takes to satisfy Δ is a countably infinite model, in which only the intersection of a (countably) infinite number of neighbourhoods $X_i \in \mathcal{N}_1(w)$ gives us the truth set of p. An example in case is one where $W = \mathbb{N}$, $\|p\|_{\mathfrak{M}} = \emptyset$ and $\mathcal{N}_1(w) = \{W \backslash \{0, \ldots, k\} \mid k \in \mathbb{N}\}$ for some $w \in W$.

Those familiar with Propositional Dynamic Logic (**PDL**) may compare this to the fact that the set $\{\neg\langle\pi\rangle p, \neg\langle\pi;\pi\rangle p, \neg\langle\pi;\pi;\pi\rangle p, \ldots\} \cup \{\langle\pi^*\rangle p\}$ is *not* satisfiable in **PDL**, even though all its finite subsets are. This is so because in **PDL**, $\langle\pi^*\rangle p$ expresses that a p-world can be reached by some *finite* sequence of π-executions. In "Appendix", we discuss a notion of "finitary pooling" and the associated modal operators. There, we show that the resulting logic is not compact and that, in view of our main results, one can easily give a sound and weakly complete axiomatization for it.

Second, note that we excluded pooling profiles whose range is infinite, cf. our clause (a) in Definition 1. As a result, the formal languages cannot express that some $X \subseteq W$ is obtained by intersecting a countably infinite set of neighbourhoods $X_1 \in \mathcal{N}_1(w), X_2 \in \mathcal{N}_2(w), \ldots$. Suppose, indeed, that we would allow for an "infinitary pooling profile" such as $M^\omega = \{(1,1),(2,1),(3,1),\ldots\}$, which we interpret as follows:

$$\mathfrak{M}, w \models \Box_{M^\omega}\varphi \text{ iff } \|\varphi\|_{\mathfrak{M}} \in \mathcal{N}_1(w) \Cap \mathcal{N}_2(w) \Cap \ldots$$

With such an enrichment, compactness fails, as witnessed by the following example:

$$\{\Box_1 p, \Box_2 p, \ldots\} \Vdash \Box_{M^\omega} p \tag{1}$$

$$\text{for all } n \in \{1,2,\ldots\} : \{\Box_1 p, \ldots, \Box_n p\} \nVdash \Box_{M^\omega} p \tag{2}$$

To see why (1) holds, it suffices to check that if we take infinitely many intersections of $\|p\|^{\mathfrak{M}}$, we end up with $\|p\|^{\mathfrak{M}}$ again. It is also fairly easy to construct a model that serves as witness for the failure of implication in (2), for any given $n \in \mathbb{N}$. Let $\mathfrak{M} = \langle \{w\}, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$, where for all $j \leq n$, $\mathcal{N}_j(w) = \{\{w\}\}$, and for all $j > n$, $\mathcal{N}_j(w) = \emptyset$. The valuation function is such that p is true at w. Then $\mathcal{N}_{M^\omega}(w) = \emptyset$, and hence $\mathfrak{M}, w \models \neg\Box_{M^\omega} p$.[2]

In the example we just used, the frame has the particular property that for all $j \in I$ with $j > n$, $\mathcal{N}_j(w)$ is empty. Note however that one can also construct different examples where each of the $\mathcal{N}_j(w)$ for $j > n$ are non-empty, but there are no $X \in \mathcal{N}_j(w)$ with $X \subseteq \|p\|$. One condition that does exclude such examples, and more generally trumps cases like (2), is the requirement that $W \in \mathcal{N}_i(w)$ for all $i \in I$, $w \in W$, corresponding to necessitation (NEC) for the operators $\Box_i$. Indeed, if that requirement is satisfied, then we already have $\Box_1 p \to \Box_{M^\omega} p$.

One other obvious difficulty with logics that allow for "infinitary pooling profiles" is that the language may not be countable, hence making it impossible to set up the standard Lindenbaum construction in the completeness proof. This is not a definitive argument against studying such languages, but rather an invitation to consider additional conditions or new techniques for their axiomatization. Having said this much, we leave a full study of such

[2]Note that there is an important difference with allowing for infinitely many occurrences of one or several (finitely many) indices in some M. For instance, the set $\{\Box_1 p, \Box_{\{1^2\}} p, \Box_{\{1^3\}} p, \ldots\} \cup \{\neg\Box_{\{1^\infty\}} p\}$ is not satisfiable, but neither is $\{\Box_1 p, \neg\Box_{\{1^\infty\}} p\}$.

languages for future work and focus again on the four languages introduced in Table 1.

4. Canonical Model Construction: Puzzle Pieces

Before diving into technicalities, let us sketch the main novelty of our construction, compared to existing completeness proofs for normal and classical modal logics. This should give the reader a feel of how this construction works, and serve as a guideline in reading the definitions and lemmata of the actual proofs that are given in Section 5. We assume familiarity with the technique of constructing canonical models as it is applied in the context of normal modal logics, cf. [2].

General Outline Focusing on the richest of the four base logics, the general outline of our proof goes as follows. Suppose we have a set $\Theta \subseteq \mathfrak{L}_{\infty}^{[\forall]}$ that is consistent with respect to the base logic $\mathbf{BL}_{\infty}^{[\forall]}$. To prove strong completeness, we should show that this set is satisfiable in some model $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ of that logic, at some state $w \in W$. Moreover—in order to obtain the finite model property in one fell swoop—we should show that W is finite whenever Θ is finite. We will focus first on completeness; we return to the finite model property at the end of this section.

Following common practice in modal logic, one can start by taking $W = \mathsf{MCS}$, where MCS is the set of all maximal consistent subsets of $\mathfrak{L}_{\infty}^{[\forall]}$. This way, we are guaranteed that some of the worlds in W will contain Θ as a subset. If we can then prove the *Truth Lemma* (TL):

$$\text{for all } \Lambda \in \mathsf{MCS}, \text{ for all } \varphi \in \mathfrak{L}_{\infty}^{[\forall]} : \mathfrak{M}, \Lambda \models \varphi \text{ iff } \varphi \in \Lambda \tag{3}$$

we are done. For non-modal formulas, it is easy to obtain (TL). To do so just put $V(\Lambda) = \{p \in \mathfrak{P} \mid p \in \Lambda\}$ for all $\Lambda \in \mathsf{MCS}$ and apply the well-known structural induction for the classical connectives. For formulas of the type $\Box_i \varphi$, hence, "degenerate" pooling modalities that only speak about a single index i, the standard way to ensure the truth lemma is by defining the neighbourhoods for the indexes $i \in I$ as follows:

$$\mathcal{N}_i(\Lambda) = \{\{\Delta \in \mathsf{MCS} \mid \varphi \in \Delta\} \mid \Box_i \varphi \in \Lambda\} \tag{4}$$

Relying on (RGE), one can then show in relatively few steps that (TL) holds for formulas of the form $\Box_i \varphi$ as well.[3]

[3] For languages without the universal modality, one relies on (RE) here.

Problem: Pooling Modalities Trouble arises however if we want to preserve (TL) for proper pooling modalities. In order to sketch the problem and our solution to it, we will use the following set of formulas as our running example:

$$\Theta = \{\Box_{1,1,2}p, \neg\Box_1 p, \neg\Box_2 p, \neg\Box_{1,2}p, \neg\Box_{1,1}p, \Box_1 q\} \tag{5}$$

The set Θ is consistent, and hence in line with the preceding, we need to construct a model $\mathfrak{M}$ such that, for some w in the domain of this model, $\mathfrak{M}, w \models \psi$ for all $\psi \in \Theta$. As noted before, we do this by associating worlds with maximal consistent sets, one of those sets including all members of Θ, and then proving a truth lemma about the model: each formula ψ is true at a world w in $\mathfrak{M}$ iff ψ is a member of the maximal consistent set associated with w.

In particular, where Θ is associated with w, we need to ensure that a given formula $\Box_M\varphi$ is true at w in our constructed model, iff $\Box_M\varphi \in \Theta$. So, moving to the object level: where $M = \{1, 1, 2\}$ and $\varphi = p$, we should "add" neighbourhoods X and Y to $\mathcal{N}_1(w)$, and a neighbourhood Z to $\mathcal{N}_2(w)$, in such a way that $X \cap Y \cap Z$ is (exactly) the set of all w' that are associated with an MCS Δ for which $p \in \Delta$. Let us henceforth denote the latter set of worlds by $|p|$. More generally, we let

$$|\psi| =_{\mathsf{df}} \{w \in W \mid \psi \in \Delta \text{ for the } \Delta \text{ that is associated to } w\}.^4$$

Note however that, by thus making $\Box_{1,1,2}p$ true, we should make sure we are not making other formulas like $\Box_1 p$, $\Box_2 p$, $\Box_{1,2}p$, or $\Box_{1,1}p$ true—otherwise we cannot end up with a model of Θ. Consequently, neither X, Y, Z, nor $X \cap Y$, $X \cap Z$, $Y \cap Z$ should themselves already correspond to $|p|$ in this model. To put it differently, *only* by intersecting all three sets, we should arrive at $|p|$.

Solution: Copying and Puzzle Pieces To solve the problem just sketched, we take two steps. First, rather than identifying worlds with maximal consistent sets, we make copies of those maximal consistent sets and distinguish them by labels. For the time being, let us distinguish the copies of a given $\Delta \in \mathsf{MCS}$ by one of the following three labels: $(1, 1)$, $(1, 2)$, and $(2, 1)$. Note that

[4] We distinguish between w and Δ here, since later on we will associate various worlds of the canonical model with the same MCS Δ. More technically, in constructing the canonical model W, we introduce several copies of each $\Lambda \in \mathsf{MCS}$, each creating a distinct member of W.

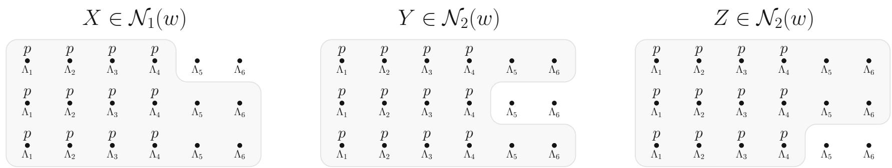

Figure 1. Illustration of the puzzle pieces for the formula $\Box_{1,1,2}p$. Indexes 1 and 2 are equipped with puzzle pieces that, when intersected, give $|p|$, but any combination of just two puzzle pieces contains $\neg p$-worlds. Λ_1, $\Lambda_2, \ldots$ denote distinct maximal consistent subsets of $\mathfrak{L}_\infty^{[\forall]}$

these are the three members of $\delta(M)$ for $M = \{(1,2),(2,1)\}$. We use $f, f', \ldots$ as a metavariables for these three labels.

Second, we refine our definition of the neighbourhoods. For all formulas of the type $\Box_i\varphi$, we do just as before, following the above definition. However, this time we ensure that we take *every copy* (Δ, f), for every Δ that contains φ. We describe the neighborhoods in greater detail. First, note that, since $\Box_1 q \in \Theta$ in our example, we also need to add the set

$$\{(\Delta, f) \mid f \in \{(1,1),(1,2),(2,1)\}, q \in \Delta\} \tag{6}$$

to $\mathcal{N}_1(\Theta, f')$ for each $f' \in \{(1,1),(1,2),(2,1)\}$.

For formulas $\Box_M\varphi \in \Lambda$ where M is a non-degenerate pooling profile, we do the following. Let $i \in I(M)$. For every $(i,k) \in \delta(M)$, we add a distinct "puzzle piece" $X_{i,k}^{M,\varphi}$ to the neighbourhood of i, in such a way that two key desiderata are fulfilled:

(a) if one intersects all the puzzle pieces, then one obtains exactly $|\varphi|$.

(b) none of these puzzle pieces (for all $i \in I(M)$ and all $(i,k) \in \delta(M)$), nor any intersection of such puzzle pieces–with the exception of the intersection of *all* of them—corresponds to any set $|\psi|$ in the newly constructed model.

If we are able to add such puzzle pieces to the neighbourhood sets $\mathcal{N}_i(\Lambda, f)$ whenever $\Box_M\varphi \in \Lambda$, then we can again obtain the truth lemma for the entire language. Sticking to our earlier metaphor, these puzzle pieces hence serve as unique witnesses for the formula in question.

It is in view of desideratum (b) that we need copies of the maximal consistent sets. The underlying idea is as follows. Let labels f and f_0 be distinct. Then, whenever φ is not a theorem (hence not a member of every maximal consistent set Δ), the set of worlds $X = \{(\Delta, f) \mid \Delta \in \mathsf{MCS}, \varphi \in \Delta\} \cup \{(\Delta, f_0) \mid \Delta \in \mathsf{MCS}\}$ is not the truth set of any formula, i.e. there can be no ψ such that $X = |\psi|$. In other words, by making copies of the maximal consistent sets, we can construct neighbourhoods that cannot be "seen" by the formal language $\mathfrak{L}_{\infty}^{[\forall]}$. This is illustrated by Figure 1, for the simple case of $\varphi = p$.

In order to obtain (a), we need to coordinate the construction of these neighbourhoods in such a way that (only) intersecting all of them yields $|\varphi|$. For the set Θ we have been working with so far, this is relatively easy. That is, we may define:

$$X = X_{(1,1)}^{\{1,1,2\},p} = \{(\Delta, f) \mid p \in \Delta \text{ or } f \neq (1,1)\}$$

$$Y = X_{(1,2)}^{\{1,1,2\},p} = \{(\Delta, f) \mid p \in \Delta \text{ or } f \neq (1,2)\}$$

$$Z = X_{(2,1)}^{\{1,1,2\},p} = \{(\Delta, f) \mid p \in \Delta \text{ or } f \neq (2,1)\}$$

and put both X and Y in $\mathcal{N}_1(\Lambda)$, and Z in $\mathcal{N}_2(\Lambda)$.

The reader can easily verify that $X \cap Y \cap Z = \{(\Delta, f) \mid p \in \Delta\}$. Hence, $\Box_{1,1,2}p$ will be true at every world (Λ, f), , while each of $\Box_{1,1}p, \Box_{1,2}p, \Box_1 p$, $\Box_2 p$ will be false.

So far, things were relatively easy since we focused on a single formula of the type $\Box_M \varphi$. In the general case, we need to use as labels functions f that map *every* pair (M, φ) to some $(i, k) \in I \times \mathbb{N}^+ \cup \{\infty\}$, where $(i, k) \in \delta(M)$. Importantly, $f(M, \varphi)$ can be distinct from $f(N, \psi)$, both when $M = N$ (but $\varphi \neq \psi$) and when $\varphi = \psi$ (but $M \neq N$). This will ensure that, by intersecting puzzle pieces that have been added as witnesses for distinct formulas, we can again not obtain any truth set $|\psi|$—see Figure 2 for an illustration. Of course, in the limiting case where we do use all the witnessing puzzle pieces for $\Box_M \varphi$ and all the witnessing puzzle pieces for $\Box_N \psi$, we will obtain $|\varphi \wedge \psi|$. This is however as it should be, since the axiom (B1) ensures that whenever $\Box_M \varphi \in \Lambda$ and $\Box_N \psi \in \Lambda$, then $\Box_{M \sqcup N}(\varphi \wedge \psi) \in \Lambda$.

Finite Models If we apply the above strategy in its full generality, we make uncountably many copies of each maximal consistent set. As far as completeness is concerned, this is all fine. However, to prove the finite model property, two additional tweaks are called for:

(i) restricting the language; and

(ii) restricting the number of copies.

We discuss each of these adaptations in turn.

Technique (i) is well-known from the study of classical modal logics, dating back at least to [4], cf. [5, 2.4.1.1]. The basic idea is that we only look at the single $([\forall], \neg)$-closure of the set of all subformulas of members of Θ. This way we get a finite set of maximal consistent sets to start with. Moreover, we only need to make puzzle pieces for a finite number of formulas $\Box_M \varphi$, since we need only worry about the members of the finite language when proving the truth lemma. We refer to Section 5.1 for the exact way this technique is implemented here.

Adaptation (ii) is more specific to the context of pooling modalities. Here, it is instructive to consider the following set:

$$\Theta' = \{\Box_{1,2^\infty} p, \neg\Box_1 p, \neg\Box_{1,2} p, \neg\Box_{1,2,2} p, \neg\Box_{1,2,2,2} p\} \tag{7}$$

The set Θ' expresses that, by intersecting an arbitrary number of neighbourhoods associated with index 2 with a single neighbourhood associated with index 1, we obtain the proposition $\|p\|$. In contrast, $\|p\|$ is not a member of the neighbourhood of 1, and it can also not be obtained by combining just one, two, or three neighbourhoods of 2 with a neighbourhood of 1.

In order to arrive at the truth lemma, we need to construct the neighbourhoods for our canonical model in such a way that we make $\Box_{1,2^\infty} p$ true, but without thereby making $\Box_1 p$, $\Box_{1,2} p$, $\Box_{1,2,2} p$, or $\Box_{1,2,2,2} p$ true. Naively, one may proceed by making copies (Λ, f) of every maximal consistent set (in the finite language), for every label $(i, k) \in \delta(\{1, 2^\infty\})$. Unfortunately, the latter set is infinite, which would mean our canonical model is also infinite.

Note however that Θ' does not say anything about what would happen if we intersect one neighbourhood of 1 with, say, *four* or more neighbourhoods of 2. That is, Θ' is compatible with $\Box_{1,2,2,2,2} p$. So we needn't make infinitely many copies; it suffices to make sure that we make more copies than can be "seen" from the viewpoint of the finitary pooling profiles that occur in Θ'. In other words, it suffices to take as labels of our worlds all and only those functions f that map each of the pooling profiles that occur in Θ' to some $(i, k) \in \delta(M)$ where $k \in \{1, 2, 3, \infty\}$.

Again, in our above example we were focusing on only a few pooling profiles of increasing magnitude. Things get slightly more complicated if the set Θ under consideration also refers to other, unrelated pooling profiles. Still, as long as Θ is finite, one can always specify a natural number k such that, beyond k, arbitrary intersection (∞) and finite intersections can be safely identified from the viewpoint of Θ. We refer to Section 5.2 for the exact definitions that make this idea exact.

Figure 2. Puzzle pieces for distinct formulas, viz. $\Box_{1,1,2}p$ (top) and $\Box_{3,3}q$ (bottom left). Unless one combines all puzzle pieces for both formulas, there's always a function f_0 such that, for all $\Lambda \in \mathsf{MCS}$, (Λ, f_0) is in the intersection of the puzzle pieces (bottom right). As a result, this intersection is not modally expressible in the model

5. The Base Logics

In this section, we prove completeness and the finite model property for the four base logics. In Sections 5.1–5.5 we focus on $\mathbf{BL}_{\infty}^{[\forall]}$, as it requires various complications. In Section 5.6, we show how (slightly simpler) variants of our proof work for the other three logics with pooling modalities.

5.1. Sublanguage and Maximal Consistent Sets

Throughout Sections 5.1–5.4, we hold fixed a consistent set $\Theta \subseteq \mathfrak{L}_\infty^{[\forall]}$. We start by defining a sublanguage $\mathfrak{L}_\Theta$ that will be used to construct maximal consistent subsets, following the general methodology mentioned under (i) at the end of Section 4.

Let SF_Θ be the closure of Θ under subformulas. Let

$$\mathfrak{L}_\Theta^{[\forall]} =_{\mathsf{df}} \{\varphi, \neg\varphi, [\forall]\varphi, [\forall]\neg\varphi, \neg[\forall]\varphi, \neg[\forall]\neg\varphi, \top, \neg\top \mid \varphi \in \mathsf{SF}_\Theta\} \tag{8}$$

Note that $\mathfrak{L}_\Theta^{[\forall]}$ is closed under subformulas and

Fact 4. *If Θ is finite, then $\mathfrak{L}_\Theta^{[\forall]}$ is finite.*

For the border case where Θ is itself a maximal consistent subset of $\mathfrak{L}_\infty^{[\forall]}$, $\mathfrak{L}_\Theta^{[\forall]} = \mathfrak{L}_\infty^{[\forall]}$. Where $\varphi \in \mathfrak{L}_\Theta^{[\forall]}$ is of the form $\neg\psi$, let $\overline{\varphi} = \psi$; otherwise let $\overline{\varphi} = \neg\psi$. Note that for all $\varphi \in \mathfrak{L}_\Theta^{[\forall]}$, $\overline{\varphi} \in \mathfrak{L}_\Theta^{[\forall]}$; i.e., $\mathfrak{L}_\Theta^{[\forall]}$ is closed under single negation.

Fix a $\subseteq$-maximal consistent set $\Gamma \subseteq \mathfrak{L}_\Theta^{[\forall]}$ such that $\Theta \subseteq \Gamma$. Define $\mathsf{MCS}_\Gamma^\Theta$ as the set of all $\subseteq$-maximal consistent subsets $\Delta \subseteq \mathfrak{L}_\Theta^{[\forall]}$ with the property that $\{[\forall]\varphi \mid [\forall]\varphi \in \Delta\} = \{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\}$. Note that for all $\varphi \in \mathfrak{L}_\Theta^{[\forall]}$ and $\Delta \in \mathsf{MCS}_\Gamma^\Theta$, either $\varphi \in \Delta$ or $\overline{\varphi} \in \Delta$. Where $\varphi \in \mathfrak{L}_\Theta^{[\forall]}$, let $|\varphi|_\Gamma^\Theta = \{\Delta \in \mathsf{MCS}_\Gamma^\Theta \mid \varphi \in \Delta\}$. Where $\Phi \subseteq \mathfrak{L}_\Theta^{[\forall]}$, let $|\Phi|_\Gamma^\Theta = \{\Delta \in \mathsf{MCS}_\Gamma^\Theta \mid \psi \in \Delta \text{ for every } \psi \in \Phi\}$.

Lemma 5. *Where $\varphi \in \mathfrak{L}_\Theta^{[\forall]}$ and $\Phi \subseteq \mathfrak{L}_\Theta^{[\forall]}$: if $|\Phi|_\Gamma^\Theta \subseteq |\varphi|_\Gamma^\Theta$, then there is a finite $\Phi' \subseteq \Phi$ such that, for all $\Delta \in \mathsf{MCS}_\Gamma^\Theta$, $\Delta \vdash [\forall](\bigwedge \Phi' \to \varphi)$.*

Proof. Suppose the antecedent holds. Hence, there is no maximal consistent $\Delta \subseteq \mathfrak{L}_\Theta^{[\forall]}$ such that $\{[\forall]\varphi \mid [\forall]\varphi \in \Delta\} = \{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\}$, $\Phi \subseteq \Delta$, and $\overline{\varphi} \in \Delta$. Hence, since $\overline{\varphi} \in \mathfrak{L}_\Theta^{[\forall]}$, for every maximal consistent $\Delta \subseteq \mathfrak{L}_\Theta^{[\forall]}$ such that $\{[\forall]\varphi \mid [\forall]\varphi \in \Delta\} = \{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\}$ and $\Phi \subseteq \Delta$, $\varphi \in \Delta$. Hence,

$$\{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\} \cup \{\neg[\forall]\varphi \mid \neg[\forall]\varphi \in \Gamma\} \cup \Phi \vdash \varphi \tag{9}$$

By compactness and the deduction theorem, there is a finite $\Phi' \subseteq \Phi$ such that

$$\{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\} \cup \{\neg[\forall]\varphi \mid \neg[\forall]\varphi \in \Gamma\} \vdash \bigwedge \Phi' \to \varphi \tag{10}$$

By well-known **S5**-properties,

$$\{[\forall]\varphi \mid [\forall]\varphi \in \Gamma\} \cup \{\neg[\forall]\varphi \mid \neg[\forall]\varphi \in \Gamma\} \vdash [\forall](\bigwedge \Phi' \to \varphi) \tag{11}$$

But that means that, for every $\Delta \in \mathsf{MCS}^{\Theta}_{\Gamma}$, $\Delta \vdash [\forall](\bigwedge \Phi' \to \varphi)$. ■

Corollary 1. *For every* $\varphi, \psi \in \mathfrak{L}^{[\forall]}_{\Theta}$*:* $|\varphi|^{\Theta}_{\Gamma} \subseteq |\psi|^{\Theta}_{\Gamma}$ *iff for all* $\Delta \in \mathsf{MCS}^{\Theta}_{\Gamma}$*,* $\Delta \vdash [\forall](\varphi \to \psi)$*.*

Corollary 2. *For every* $\varphi, \psi \in \mathfrak{L}^{[\forall]}_{\Theta}$*,* $|\varphi|^{\Theta}_{\Gamma} = |\psi|^{\Theta}_{\Gamma}$ *iff for all* $\Delta \in \mathsf{MCS}^{\Theta}_{\Gamma}$*, then* $\Delta \vdash [\forall](\varphi \leftrightarrow \psi)$*.*

Corollary 3. *For every* $\psi \in \mathfrak{L}^{[\forall]}_{\Theta}$*,* $|\psi|^{\Theta}_{\Gamma} = |\top|^{\Theta}_{\Gamma} = \mathsf{MCS}^{\Theta}_{\Gamma}$ *iff for all* $\Delta \in \mathsf{MCS}^{\Theta}_{\Gamma}$*,* $\Delta \vdash [\forall]\psi$*.*

5.2. Some More Preparatory Definitions

Just like in standard completeness proofs for classical modal logics, $\mathsf{MCS}^{\Theta}_{\Gamma}$ will be used to construct a model of Γ, and hence also of Θ. However, as explained in Section 4, we will need to make a number of copies of each such Δ in order to handle the pooling modalities. We should moreover do so with care, making sure that we only introduce finitely many copies whenever Θ is finite. This requires several preparatory steps.

Let $\mathbb{M}_{\Theta} =_{\mathsf{df}} \{M {\in} \mathbb{M}_{\infty} \mid \Box_M \varphi {\in} \mathfrak{L}^{[\forall]}_{\Theta}$ for some $\varphi\}$. Let $I_{\Theta} =_{\mathsf{df}} \bigcup_{M \in \mathbb{M}_{\Theta}} I(M)$ and, $K^0_{\Theta} =_{\mathsf{df}} \{j \in \mathbb{N}^+ \mid$ for some $M \in \mathbb{M}_{\Theta} : M(i) \neq \infty$ and $j \leq M(i)\}$. We define the set $K_{\Theta} \subseteq \mathbb{N}^+ \cup \{\infty\}$ by cases:

(C1) If for all $M \in \mathbb{M}_{\Theta}$ and all $i \in I(M)$, $M(i) = \infty$, then $K_{\Theta} =_{\mathsf{df}} \{1, 2\}$

(C2) If $\mathbb{M}_{\Theta} \subseteq \mathbb{M}_f$, then $K_{\Theta} =_{\mathsf{df}} K^0_{\Theta}$.

(C3) Otherwise, $K_{\Theta} =_{\mathsf{df}} K^0_{\Theta} \cup \{\infty\}$

Intuitively, our definition of K_{Θ} guarantees that this is a subset of $\mathbb{N} \cup \{\infty\}$ that is "large enough" that we can make all the required distinctions in view of Θ, but also small enough to ensure finiteness whenver Θ is finite. In our example from Section 4, $K_{\Theta} = \{1, 2, 3, \infty\}$.

Fact 6. *If there is some* $M \in \mathbb{M}_{\Theta}$ *and some* $i \in I(M)$ *such that* $M(i) \neq 1$*, then* $|K_{\Theta}| \geq 2$*.*

Let $\mathcal{D}_{\Theta} =_{\mathsf{df}} \{(M, \varphi) \mid \Box_M \varphi \in \mathfrak{L}^{[\forall]}_{\Theta}\}$ and for all $M \in \mathbb{M}_{\Theta}$, define $\delta^{\Theta}(M) =_{\mathsf{df}} \{(i, k) \in \delta(M) \mid k \in K_{\Theta}\}$. Finally, let

$$\mathbb{F}_{\Theta} =_{\mathsf{df}} \{f : \mathcal{D}_{\Theta} \to I_{\Theta} \times K_{\Theta}\} \mid \text{ for all } (M, \varphi) \in \mathcal{D}_{\Theta}, f(M, \varphi) \in \delta^{\Theta}(M)\}.$$

Note that, if Θ is finite, then so is $\mathcal{D}_{\Theta}$. Also, if that is the case, then I_{Θ} and K_{Θ} are both finite. As a result,

Fact 7. *If* Θ *is finite, then* $\mathbb{F}_{\Theta}$ *is finite.*

On the other hand, if Θ is a maximal consistent subset of $\mathfrak{L}^{[\forall]}$, then it holds that $\mathcal{D}_\Theta = \mathbb{M} \times \mathfrak{L}$, $I_\Theta = I$, $K_\Theta = \mathbb{N}^+ \cup \{\infty\}$, and $\delta^\Theta(M) = \delta(M)$ for all $M \in \mathbb{M}_\Theta$.

5.3. The Relativized Canonical Model

With $\mathsf{MCS}_\Gamma^\Theta$ and $\mathbb{F}_\Theta$ in hand, we are finally ready to define the construction that plays central stage in our completeness proofs of this paper.

Definition 10. The ***relativized canonical model for*** $\langle \Theta, \Gamma \rangle$ is defined as $\mathfrak{M}_\Gamma^\Theta = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$, where

1. $W = \{(\Lambda, f) \mid \Lambda \in \mathsf{MCS}_\Gamma^\Theta \text{ and } f \in \mathbb{F}_\Theta\}$;
2. $V(p) = \{(\Lambda, f) \in W \mid p \in \Lambda\}$ for all $p \in \mathfrak{P}$; and
3. for every $i \in I$, $\mathcal{N}_i(\Lambda, f) = \{X_{i,k}^{M,\varphi} \mid \Box_M \varphi \in \Lambda, (i, k) \in \delta^\Theta(M)\}$ where,
4. for all $(M, \varphi) \in \mathcal{D}_\Theta$ and $(i, k) \in \delta^\Theta(M)$,
$$X_{i,k}^{M,\varphi} = \{(\Lambda, f) \in W \mid \varphi \in \Lambda \text{ or } f(M, \varphi) \neq (i, k)\}$$

In this and the next two subsections, we hold both Θ and Γ fixed and continue to use W, V, $\mathcal{N}_i$ to refer to the sets defined from them.

Note that, by the second clause of Definition 10, whenever some p is not in $\mathfrak{L}_\Theta^{[\forall]}$, then $V(p) = \emptyset$. It is not hard to check that $\mathfrak{M}_\Gamma^\Theta$ is well-defined; it suffices to show that W is non-empty, which follows immediately from the fact that $\Gamma \in \mathsf{MCS}_\Gamma^\Theta$. Moreover, in view of Fact 4, the definition of $\mathsf{MCS}_\Gamma^\Theta$, and Fact 7, we have:

Fact 8. *If Θ is finite, then $\mathfrak{M}_\Gamma^\Theta$ is finite.*

So we have constructed a model that is finite whenever Θ is finite. It remains to prove that this model verifies all the members of Θ at the state (Γ, f). As usual, this is done by proving the truth lemma for $\mathfrak{M}_\Gamma^\Theta$ (cf. Lemma 12 in Section 5.4). In preparation for this, we state three auxiliary lemmata.

Intuitively, Lemma 9 expresses that all puzzle pieces X that are used in the construction are distinct, and hence one can associate a unique pair (M, φ) with each of them—with the exception of the neighbourhoods that were added for "degenerate" pooling modalities of the type $\Box_i$ and neighbourhoods that coincide with the unit W. Lemma 10 implies that, unless we intersect *all* puzzle pieces that are associated with a given pair (M, φ), we will always end up with a subset of W that contains all worlds (Λ, f_0) for some $f_0 \in \mathbb{F}_\Theta$. Lemma 11 says that if we do intersect all those puzzle pieces for (M, φ), we get exactly the set of all worlds (Λ, f) for which $\varphi \in \Lambda$.

Both Lemmas 10 and 11 were illustrated in Section 4, cf. Figures 1 and 2.

Lemma 9. *For all $(N,\psi),(N',\psi')\in\mathcal{D}_\Theta$, $(j,l)\in\delta^\Theta(N)$ and $(j',l')\in\delta^\Theta(N')$ such that $\delta^\Theta(N)\neq\{(j,1)\}$ and $|\psi|^\Theta_\Gamma\neq W$: if $X^{N,\psi}_{j,l}=X^{N',\psi'}_{j',l'}$, then $(N,\psi,j,l)=(N',\psi',j',l')$.*

Proof. Suppose that $\delta^\Theta(N)\neq\{(j,1)\}$, that $|\psi|^\Theta_\Gamma\neq W$, and that $(N,\psi,j,l)\neq(N',\psi',j',l')$. By the former two suppositions, there is no $\tau\in\mathfrak{L}^{[\forall]}_\infty$ such that $X^{N,\psi}_{j,l}=\{(\Lambda,f)\in W\mid\tau\in\Lambda\}$. Hence if $|\psi'|^\Theta_\Gamma=W$ or $\delta^\Theta(N')=\{(j',1)\}$, then it follows immediately that $X^{N,\psi}_{j,l}\neq X^{N',\psi'}_{j',l'}$. So suppose that $\delta^\Theta(N')\neq\{(j',1)\}$ and $|\psi'|^\Theta_\Gamma\neq W$. Pick an $f\in\mathbb{F}_\Theta$ such that $f(N,\psi)=(j,l)$ and $f(N',\psi')\neq(j',l')$. (If $(N,\psi)=(N',\psi')$, then we immediately have that $(j,l)\neq(j',l')$; if $(N,\psi)\neq(N',\psi')$, then we can let $f(N',\psi')$ be an arbitrary $(i,k)\in\delta^\Theta(N')\backslash\{(j',l')\}$.) Let $\Lambda\in\mathsf{MCS}^\Theta_\Gamma$ be such that $\psi\notin\Lambda$—such Λ exists since $|\psi|^\Theta_\Gamma\neq W$ and $\psi\in\mathfrak{L}^{[\forall]}_\Theta$. By construction, $(\Lambda,f)\in X^{N',\psi'}_{j',l'}\backslash X^{N,\psi}_{j,l}$ and hence $X^{N,\psi}_{j,l}\neq X^{N',\psi'}_{j',l'}$. ■

Lemma 10. *Let $\mathcal{Y}$ be a set of sets $X^{N,\psi}_{j,l}$ with $(j,l)\in\delta^\Theta(N)$ and $(N,\psi)\in\mathcal{D}_\Theta$, such that for no $(M,\varphi)\in\mathcal{D}_\Theta$, $\{X^{M,\varphi}_{i,k}\mid(i,k)\in\delta^\Theta(M)\}\subseteq\mathcal{Y}$. Then there is an $f_0\in\mathbb{F}_\Theta$ such that $\{(\Lambda,f_0)\in W\}\subseteq\bigcap\mathcal{Y}$.*

Proof. Suppose the antecedent holds. Let $f_0\in\mathbb{F}_\Theta$ be such that, for every $X^{N,\psi}_{j,l}\in\mathcal{Y}$, $f_0(N,\psi)=(i,k)$ for some $(i,k)\in\delta^\Theta(N)$ such that $X^{N,\psi}_{i,k}\notin\mathcal{Y}$. In view of the supposition, there is at least one such f_0. Note that, for all $X^{N,\psi}_{j,l}\in\mathcal{Y}$, $f_0(N,\psi)\neq(j,l)$. By Definition 10.4, for all $X^{N,\psi}_{j,l}\in\mathcal{Y}$ and all $\Lambda\in\mathsf{MCS}^\Theta_\Gamma$, $(\Lambda,f_0)\in X^{N,\psi}_{j,l}$. Consequently, for all $\Lambda\in\mathsf{MCS}^\Theta_\Gamma$, $(\Lambda,f_0)\in\bigcap\mathcal{Y}$. ■

Lemma 11. *Let $\mathcal{Y}=\{X^{M,\varphi}_{i,k}\mid(i,k)\in\delta^\Theta(M)\}$ for some $(M,\varphi)\in\mathcal{D}_\Theta$. Then $\bigcap\mathcal{Y}=\{(\Lambda,f)\in W\mid\varphi\in\Lambda\}$.*

Proof. By Definition 10.4,

$$\bigcap_{(i,k)\in\delta^\Theta(M)}X^{M,\varphi}_{i,k}=\bigcap_{(i,k)\in\delta^\Theta(M)}\{(\Lambda,f)\in W\mid\varphi\in\Lambda\text{ or }f(M,\varphi)\neq(i,k)\}\tag{12}$$

In view of the definition of $\mathbb{F}_\Theta$, for every $f'\in\mathbb{F}_\Theta$ there is some $(i,k)\in\delta^\Theta(M)$ with $f'(M,\varphi)=(i,k)$. So if $\varphi\notin\Lambda$, for every $f'\in\mathbb{F}_\Theta$ there is some $X^{M,\varphi}_{i,k}$ with $(i,k)\in\delta^\Theta(M)$ such that $(\Lambda,f')\notin X^{M,\varphi}_{i,k}$. Hence,

$$\bigcap_{(i,k)\in\delta^{\Theta}(M)} \{(\Lambda, f) \in W \mid \varphi \in \Lambda \text{ or } f(M,\varphi) \neq (i,k)\} = \{(\Lambda, f) \in W \mid \varphi \in \Lambda\}. \tag{13}$$

■

5.4. Nothing but the Truth Lemma

With the above lemmas in hand, we can finally state and prove the truth lemma for $\mathbf{BL}_{\infty}^{[\forall]}$:

LEMMA 12. (Truth Lemma) *For all* $(\Lambda, f) \in W$ *and all* $\varphi \in \mathfrak{L}_{\Theta}^{[\forall]}$ *holds:*

$$\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \varphi \text{ iff } \varphi \in \Lambda.$$

PROOF. By an induction on the complexity of φ. The base case and the induction step for the classical connectives are safely left to the reader. For $\varphi = [\forall]\psi$, we have that $\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \varphi$ iff [by the semantic clause for $[\forall]$ and the induction hypothesis] for all $\Delta \in \mathsf{MCS}_{\Gamma}^{\Theta}$, $\psi \in \Delta$ iff [by Corollary 3] for all $\Delta \in \mathsf{MCS}_{\Gamma}^{\Theta}$, $\Delta \vdash [\forall]\psi$ iff [by the definition of $\mathsf{MCS}_{\Gamma}^{\Theta}$ and since $\varphi \in \mathfrak{L}_{\Theta}^{[\forall]}$] $[\forall]\psi \in \Lambda$.

So it remains to prove that, for all $\Box_M\varphi \in \mathfrak{L}_{\Theta}^{[\forall]}$,

$$\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \Box_M\varphi \text{ iff } \Box_M\varphi \in \Lambda \tag{TL$\Box$}$$

Right to left direction of (TL$\Box$). Suppose that $\Box_M\varphi \in \Lambda$. By Lemma 11,

$$\bigcap_{(i,k)\in\delta^{\Theta}(M)} X_{i,k}^{M,\varphi} = \{(\Lambda', f') \in W \mid \varphi \in \Lambda'\} \tag{14}$$

By the induction hypothesis and since $\varphi \in \mathfrak{L}_{\Theta}^{[\forall]}$,

$$\{(\Lambda', f') \in W \mid \varphi \in \Lambda'\} = \{(\Lambda', f') \in W \mid \mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda', f') \models \varphi\} \tag{15}$$

Taking everything together, we obtain:

$$\bigcap_{(i,k)\in\delta^{\Theta}(M)} X_{i,k}^{M,\varphi} = \|\varphi\|^{\mathfrak{M}_{\Gamma}^{\Theta}} \tag{16}$$

Moreover, by Definition 10.3, for every $(i,k) \in \delta^{\Theta}(M)$, $X_{i,k}^{M,\varphi} \in \mathcal{N}_i(\Lambda, f)$. Note that $\bigcap_{(i,k)\in\delta^{\Theta}(M)} X_{i,k}^{M,\varphi} \in \mathcal{N}_M(\Lambda, f)$. By Definition 7, $\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \Box_M\varphi$.

Left to right direction of (TL$\Box$). Suppose that $\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \Box_M\varphi$. For every $(i,k) \in \delta(M)$, fix an N_k^i, ψ_k^i, l_k^i such that $(N_k^i, \psi_k^i) \in \mathcal{D}_{\Theta}$, $(i, l_k^i) \in \delta^{\Theta}(N_k^i)$, $X_{i,l_k^i}^{N_k^i,\psi_k^i} \in \mathcal{N}_i(\Lambda, f)$ and

$$\bigcap\{X_{i,l_k^i}^{N_k^i,\psi_k^i} \mid (i,k) \in \delta(M)\} = \|\varphi\|^{\mathfrak{M}_\Gamma^\Theta} \tag{17}$$

Let $\mathcal{X} = \{X_{i,l_k^i}^{N_k^i,\psi_k^i} \mid (i,k) \in \delta(M)\}$. We distinguish two cases:

Case 1: $\Gamma \vdash [\forall]\varphi$ and hence also $\Gamma \vdash [\forall](\top \leftrightarrow \varphi)$. Note that $[\forall]\varphi \in \mathfrak{L}_\Theta^{[\forall]}$ and hence, for all $\Delta \in \mathsf{MCS}_\Gamma^\Theta$, $[\forall]\varphi \in \Delta$ and $\varphi \in \Delta$. By the induction hypothesis, $\|\varphi\|^{\mathfrak{M}_\Gamma^\Theta} = W$. Hence for all $X \in \mathcal{X}$, $X = W$. In view of Definition 10.4, for all $(i,k) \in \delta(M)$, $|\psi_k^i|_\Gamma^\Theta = W$ and hence, by Corollary 3, $\Lambda \vdash [\forall]\psi_k^i$. So for all $(i,k) \in \delta(M)$, $\Lambda \vdash [\forall](\psi_k^i \leftrightarrow \top)$. By (RGE), for all $(i,k) \in \delta(M)$, $\Lambda \vdash \Box_{N_k^i}\top$. By (B2), for all $j \in I(N_k^i)$, $\Lambda \vdash \Box_j\top$. It follows that $\Lambda \vdash \Box_i\top$ for all $i \in I(M)$. If M is finite, then using (B1) finitely many times, we can derive that $\Lambda \vdash \Box_M\top$ and hence by (RGE), $\Lambda \vdash \Box_M\varphi$. If M is infinite, then we use (B4) to derive that $\Lambda \vdash \Box_{M^\infty}\top$. By (RGE), $\Lambda \vdash \Box_{M^\infty}\varphi$. By (B3), $\Lambda \vdash \Box_M\varphi$. Since $\Box_M\varphi \in \mathfrak{L}_\Theta^{[\forall]}$, it follows in both cases that $\Box_M\varphi \in \Lambda$.

Case 2: $\Gamma \not\vdash [\forall]\varphi$. We first prove that $\Lambda \vdash \Box_K\varphi$ holds for some $K \sqsubseteq M$. Fix an $\epsilon \subseteq \delta(M)$ such that $\{X_{i,l_k^i}^{N_k^i,\psi_k^i} \mid (i,k) \in \epsilon\} = \mathcal{X}$ but, whenever $(i,k) \neq (j,n)$ for $(i,k),(j,n) \in \epsilon$, then $X_{i,l_k^i}^{N_k^i,\psi_k^i} \neq X_{j,l_n^j}^{N_n^j,\psi_n^j}$.

Let $\mathcal{A}_\dagger = \{(N_k^i,\psi_k^i,i,l_k^i) \mid (i,k) \in \epsilon\}$ and $\mathcal{A} = \{(N_k^i,\psi_k^i) \mid (i,k) \in \epsilon\}$. Let $\mathcal{B}_\dagger = \{(N,\psi,j,l) \in \mathcal{A}_\dagger \mid \ |\psi|_\Gamma^\Theta \neq W$ and for all $(j',l') \in \delta^\Theta(N), X_{j',l'}^{N,\psi} \in \mathcal{X}\}$ and let

$$\mathcal{B} = \{(N,\psi) \mid (N,\psi,j,l) \in \mathcal{B}_\dagger \text{ for some } j,l \in \delta^\Theta(N)\}.$$

We define a one-to-one map t from $\mathcal{E} = \{(N,\psi,j,l) \mid (N,\psi) \in \mathcal{B}, (j,l) \in \delta(N)\}$ into $\delta(M)$, such that $t(N,\psi,j,l) = (j,k)$ for some $k \in \mathbb{N}^+ \cup \{\infty\}$, for every $(N,\psi,j,l) \in \mathcal{E}$.

Let $(N,\psi,j,l) \in \mathcal{E}$ and let $(i,k) \in \delta(M)$ be such that $X_{j,l}^{N,\psi} = X_{i,l_k^i}^{N_k^i,\psi_k^i}$. We define $t(N,\psi,j,l)$ by cases:

Case (i): $\delta(N) = \{(j,1)\}$. Hence, $(N,\psi,j,l) = (N,\psi,j,1) \in \mathcal{A}_\dagger$. Then we put $t(N,\psi,j,l) = (i,k)$.

Case (ii): $\delta(N) \neq \{(j,1)\}$ and $N(j) \neq \infty$. By Fact 6, $|K_\Theta| \geq 2$ and hence $\delta^\Theta(N) \neq \{(j,1)\}$. By Lemma 9, $(N_k^i,\psi_k^i,i,l_k^i) = (N,\psi,j,l)$. By the construction, again $(N,\psi,j,l) \in \mathcal{A}_\dagger$, so that we can put $t(N,\psi,j,l) = (i,k)$.

Case (iii): $N(j) = \infty$. In this case, for all $k \in K_\Theta$, $(j,k) \in \delta^\Theta(N)$. By Lemma 9, there are $|K_\Theta|$ distinct sets of the type $X_{j,m}^{N,\psi}$ in $\mathcal{X}$. Since each of those sets correspond to a unique $(j,n) \in \epsilon$, it follows that for all $k \in K_\Theta$, $(j,k) \in \epsilon$ and hence also $(j,k) \in \delta(M)$. But given our construction of K_Θ, that implies that $M(j) = \infty$. Note that $\mathcal{E}_j = \{(N', \psi', j, l') \mid (N', \psi') \in \mathcal{B}, (j, l') \in \delta(N')\}$ is countable. So with every $\kappa' \in \mathcal{E}_j$, we can associate a unique $(j, k_{\kappa'}) \in \delta(M)$.

We rewrite the intersection of the members of $\mathcal{X}$ as follows:

$$\bigcap \mathcal{X} = \bigcap_{(N^i_k, \psi^i_k) \in \mathcal{A} \setminus \mathcal{B}} X^{N^i_k, \psi^i_k}_{i, l^i_k} \cap \bigcap_{(N,\psi) \in \mathcal{B}, (j,l) \in \delta^{\Theta}(N)} X^{N,\psi}_{j,l} \tag{18}$$

By Lemma 10, there is an $f_0 \in \mathbb{F}_\Theta$ such that

$$\{(\Lambda', f_0) \in W\} \subseteq \bigcap_{(N^i_k, \psi^i_k) \in \mathcal{A} \setminus \mathcal{B}} X^{N^i_k, \psi^i_k}_{i, l^i_k}. \tag{19}$$

In view of Definition 10, Lemma 11, and the induction hypothesis,

$$\bigcap_{(N,\psi) \in \mathcal{B}} \{(\Lambda', f_0) \in W \mid \psi \in \Lambda'\} = \{(\Lambda', f_0) \in W \mid \varphi \in \Lambda'\}. \tag{20}$$

So, putting $\Phi = \{\psi \mid (N, \psi) \in \mathcal{B}\}$, we have:

$$|\Phi|^{\Theta}_{\Gamma} = |\varphi|^{\Theta}_{\Gamma} \tag{21}$$

and hence, for all $\psi \in \Phi$, $|\varphi|^{\Theta}_{\Gamma} \subseteq |\psi|^{\Theta}_{\Gamma}$. Applying Lemma 5 and Corollary 1, we can derive that there is a finite $\mathcal{C} \subseteq \mathcal{B}$ such that, for all $\Delta \in \mathsf{MCS}^{\Theta}_{\Gamma}$, holds that $\Delta \vdash [\forall](\bigwedge_{(N,\psi) \in \mathcal{C}} \psi \to \varphi)$ and $\Delta \vdash [\forall](\varphi \to \bigwedge_{(N,\psi) \in \mathcal{C}} \psi)$. Consequently,

$$\Lambda \vdash [\forall] \left(\bigwedge_{(N,\psi) \in \mathcal{C}} \psi \leftrightarrow \varphi \right). \tag{22}$$

By Definition 10.3, $\Box_N \psi \in \Lambda$ for all $(N, \psi) \in \mathcal{C}$. Also, since $\Gamma \nvdash [\forall]\varphi$, $\mathcal{B}$ and $\mathcal{C}$ must be non-empty. Let $K = \bigsqcup_{(N,\psi) \in \mathcal{C}} N$. Note that, in view of the mapping t and since $\mathcal{C} \subseteq \mathcal{B}$, $K \sqsubseteq M$. Applying (B1) a suitable number of times, we can derive that $\Lambda \vdash \Box_K \bigwedge_{(N,\psi) \in \mathcal{C}} \psi$. By (RGE) and (22),

$$\Lambda \vdash \Box_K \varphi. \tag{23}$$

Now let $i \in I(M)$. In view of the construction, there is an $X^{N_i, \psi_i}_{i, l_i} \in \mathcal{X}$ such that

$$\Box_{N_i} \psi_i \in \Lambda. \tag{24}$$

Since $\bigcap \mathcal{X} = \|\varphi\|^{\mathfrak{M}^{\Theta}_{\Gamma}}$, it follows that $X^{N_i, \psi_i}_{i, l_i} \supseteq \|\varphi\|^{\mathfrak{M}^{\Theta}_{\Gamma}}$. Let $f_i \in \mathbb{F}_\Theta$ be such that $f_i(N_i) = (i, l_i)$. Hence, $X^{N_i, \psi_i}_{i, l_i} \cap \{(\Lambda, f_i) \in W\} = \{(\Lambda, f_i) \in W \mid \psi_i \in \Lambda\}$. This implies that $\{(\Lambda, f_i) \in W \mid \psi_i \in \Lambda\} \supseteq \{(\Lambda, f_i) \in W \mid \varphi \in \Lambda\}$ and hence $|\varphi|^{\Theta}_{\Gamma} \subseteq |\psi_i|^{\Theta}_{\Gamma}$. By Corollary 1, for all

$i \in I(M)$

$$\Lambda \vdash [\forall](\varphi \to \psi_i). \tag{25}$$

This implies that

$$\Lambda \vdash [\forall]\left(\varphi \leftrightarrow \left(\varphi \wedge \bigwedge_{i \in I(M)} \psi_i\right)\right). \tag{26}$$

Let $L = K \sqcup \bigsqcup_{i \in I(M)} N_i$. By (B1), (24) and (23),

$$\Lambda \vdash \Box_L \left(\varphi \wedge \bigwedge_{i \in I(M)} \psi_i\right) \tag{27}$$

By (26) and (RGE),

$$\Lambda \vdash \Box_L \varphi.$$

We distinguish two cases. If M is finite, then there must be some $k \in \{1, 2, \ldots\}$ such that $K \sqsubseteq M \sqsubseteq L^k$. By the derived rule of iteration (see Theorem 3.3), we can derive that $\Lambda \vdash \Box_{L^k}\varphi$. Then, using (B3) and (23), we derive that $\Lambda \vdash \Box_M \varphi$ and hence $\Box_M \varphi \in \Lambda$.

If M is infinite, then we first derive that $\Lambda \vdash \Box_{L^\infty}\varphi \in \Lambda$, using (B4). Finally, by (B3) and (23), we derive that $\Lambda \vdash \Box_M \varphi$ and hence $\Box_M \varphi \in \Lambda$. ■

5.5. Completeness and Finite Model Property

From here, it is a matter of routine to derive the following two key results:

THEOREM 4. (Strong Soundness and Completeness) $\Gamma \vdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$.

PROOF. Soundness is safely left to the reader, relying on Theorem 1. For completeness, suppose that $\Delta \nvdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$. Let $\Theta = \Delta \cup \{\neg\varphi\}$. Let Γ be a maximal consistent subset of $\mathfrak{L}_\Theta^{[\forall]}$ with $\Theta \subseteq \Gamma$. Let $\mathfrak{M}_\Gamma^\Theta$ be the relativized canonical model for $\langle \Theta, \Gamma \rangle$ (cf. Definition 10). Let $f \in \mathbb{F}_\Theta$. By Lemma 12, for all $\psi \in \Gamma$, $\mathfrak{M}_\Gamma^\Theta, (\Gamma, f) \models \psi$. Hence, $\Delta \nVdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$. ■

THEOREM 5. (Finite Model Property for $\mathbf{BL}_\infty^{[\forall]}$) *If* $\nvdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$*, then there is a finite model* $\mathfrak{M}$ *and a world* w *in the domain of* $\mathfrak{M}$ *such that* $\mathfrak{M}, w \not\models \varphi$.

PROOF. Suppose $\nvdash_{\mathbf{BL}_\infty^{[\forall]}} \varphi$. Let $\Theta = \{\neg\varphi\}$ and let Γ be a maximal consistent subset of $\mathfrak{L}_\Theta^{[\forall]}$ with $\Theta \subseteq \Gamma$. Note that $\mathfrak{M}_\Gamma^\Theta$ is finite by Fact 8. Let $f \in \mathbb{F}_\Theta$. By the same reasoning as for Theorem 4, we can show that $\mathfrak{M}_\Gamma^\Theta, (\Gamma, f) \not\models \varphi$. ■

As shown in [6], the finite model property does not always guarantee decidability, if one is working with a recursively enumerable, rather than a finite, axiomatization. Roughly, the question whether some structure is indeed a model of the logic (hence validates all its axioms) may itself not be decidable. One may be checking this ad infinitum for a single model, before ever being able to turn to the next model that could falsify a given formula.

This problem does not arise for our models. More generally, both the set of $\mathbf{BL}_\infty^{[\forall]}$-proofs and the set of finite $\mathbf{BL}_\infty^{[\forall]}$-models are easily seen to be recursively enumerable, and hence we have:

Corollary 4. $\mathbf{BL}_\infty^{[\forall]}$ *is decidable.*

5.6. Adaptations for $\mathfrak{L}_f^{[\forall]}$, $\mathfrak{L}_f$, and $\mathfrak{L}_\infty$

The above proofs can be adapted in a relatively straightforward way to obtain similar completeness and decidability results for each of $\mathbf{BL}_f^{[\forall]}$, $\mathbf{BL}_\infty$, and $\mathbf{BL}_f$. We will not state the theorems as such, but refer to Section 7 for the overview of all our results.

Let us start with the simplest adaptation. In our completeness proof, we only relied on the axiom schema (B4) when dealing with infinite pooling profiles. In fact, we only applied those axiom schemas at the very end in the two cases of Lemma 12, in order to obtain the conclusion that $\varphi \in \Lambda$. Hence, leaving out those applications immediately gives us completeness and the finite model property for $\mathbf{BL}_f^{[\forall]}$.

For the languages without the universal modality, the proof also consists in a simplification of the above proof, with some minor adjustments. We sketch them here.

First, we define $\mathfrak{L}_\Theta$ as the single negation-closure of $\mathsf{SF}_\Theta \cup \{\top\}$. Rather than $\mathsf{MCS}_\Gamma^\Theta$ for some maxiconsistent $\Gamma \supseteq \Theta$, we just use the set MCS^Θ of all maximal consistent subsets of $\mathfrak{L}_\Theta$ in our construction of the relativized canonical model. We simplify other notation accordingly: $|\varphi|^\Theta = \{\Delta \in \mathsf{MCS}^\Theta \mid \varphi \in \Delta\}$ and $|\Phi|^\Theta = \{\Delta \in \mathsf{MCS}^\Theta \mid \Phi \subseteq \Delta\}$. Lemma 5 and its corollaries are rephrased as follows:

Lemma 5 (rewritten). *Where* $\varphi \in \mathfrak{L}_\Theta$ *and* $\Phi \subseteq \mathfrak{L}_\Theta$: *if* $|\Phi|^\Theta \subseteq |\varphi|^\Theta$, *then there is a finite* $\Phi' \subseteq \Phi$ *such that* $\vdash \bigwedge \Phi' \to \varphi$.

This lemma is proven by standard means, relying on the compactness of $\vdash$ and the deduction theorem.

Corollary 1 (rewritten). *For every* $\varphi, \psi \in \mathfrak{L}_\Theta$: $|\varphi|^\Theta \subseteq |\psi|^\Theta$ *iff* $\vdash \varphi \to \psi$.

Corollary 2 (rewritten). *For every* $\varphi, \psi \in \mathfrak{L}_\Theta$, $|\varphi|^\Theta = |\psi|^\Theta$ *iff* $\vdash \varphi \leftrightarrow \psi$.

Corollary 3 (rewritten). *For every* $\psi \in \mathfrak{L}_\Theta$, $|\psi|^\Theta = |\top|^\Theta = \mathsf{MCS}^\Theta$ *iff* $\vdash \psi$.

The definitions in Section 5.2 remain unchanged. We define the canonical model $\mathfrak{M}^{\Theta}$ just like $\mathfrak{M}^{\Theta}_{\Gamma}$, but replacing $\mathsf{MCS}^{\Theta}_{\Gamma}$ with MCS^{Θ}. The facts and lemmas from Sections 5.2 and 5.3 remain unchanged.

In the proof of the Truth Lemma (Lemma 12), we obviously skip the case $\varphi = [\forall]\psi$. Further down in that proof, we replace every expression of the type "$\Gamma \vdash [\forall]\tau$" with "$\vdash \tau$", and use (RE) instead of (RGE). So for instance, in the proof of the left-to-right direction of (TL$\Box$), Case 1 becomes: $\vdash \varphi$. In Case 2, equation (22) is rewritten to

$$\vdash \bigwedge_{(N,\psi)\in\mathcal{C}} \psi \leftrightarrow \varphi. \tag{28}$$

Then, using (28), we can derive $\Box_K\varphi$ from $\Box \bigwedge_{(N,\psi)\in\mathcal{C}} \psi$, by means of (RE) instead of (RGE). Similar rewriting of the proof's final steps results in the conclusion that $\Lambda \vdash \Box_M\varphi$ and hence $\Box_M\varphi \in \Lambda$.

Finally, a completeness proof for $\mathbf{BL}_f$ is immediately obtained from the completeness proof for $\mathbf{BL}_\infty$, just by skipping the parts that concern pooling profiles in $\mathbb{M}_\infty \backslash \mathbb{M}_f$. By the same token, we obtain the finite model property and hence decidability for each of the three logics $\mathbf{BL}_f^{[\forall]}$, $\mathbf{BL}_\infty$, and $\mathbf{BL}_f$.

Let us insert a brief aside. Relying on Lewis' finitary construction developed in [4], one can also establish the decidability of these logics, as they are all axiomatized by non-iterative modal axioms.[5] However, this method cannot be used to prove completeness with respect to our intended semantics, let alone strong completeness and the semantic compactness that follows from it.[6]

6. Extensions

In this section we prove completeness and the finite model property for a range of stronger logics that are obtained by imposing one or several standard frame conditions on the neighbourhood functions. In all cases, the proofs are variations on the one for the base logic (Section 5), making essential use of the puzzle construction that we illustrated in Section 4. We will also indicate simplified axiomatizations for the studied extensions whenever possible. An overview of all completeness results is given in Section 7.

[5]An axiom (schema) is non-iterative if it contains no nested modal operators. An example of an iterative axiom schema is (4): $\Box\varphi \rightarrow \Box\Box\varphi$.

[6]Lewis' method would allow us to prove *weak* completeness with respect to the quasi-models that we refer to in Section 8. Still, one would require something along the lines of our puzzle piece construction to turn such quasi-models into our intended models.

Some notation will prove useful in the statement of theorems and properties. In the remainder, let **BL** be a metavariable for any of the four base logics. Where (X_1), (X_2), ... are axioms (axiom schemata), we write $\mathbf{BL} + \mathbf{X_1} + \mathbf{X_2} + \cdots$ to denote the logic obtained by adding (all instances of) (X_1), (X_2), ... to the axioms of **BL**, and closing the resulting set under (RE) and (MP). When we state soundness and completeness results of a given logic with respect to a certain class of frames, we always assume fixed the underlying language of that logic, and we only require the addition of axioms in that language.

6.1. Monotonic Models

Recall that a neighourhood model $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is *monotonic* iff for all $i \in I$ and all $w \in W$, $\mathcal{N}_i(w)$ is closed under supersets. In the remainder, we will use $\mathcal{X}^{\uparrow}$ to refer to the closure of $\mathcal{X}$ under supsersets. Monotony thus means that $\mathcal{N}_i(w) = \mathcal{N}_i^{\uparrow}(w)$ for all $i \in I$.

In simple (monomodal) classical modal logics, closure of the neighbourhood set $\mathcal{N}(w)$ under supersets corresponds to the axiom[7]

$$\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi). \tag{AM}$$

In the presence of (RE), (AM) entails the monotony rule (RM): if $\Box\varphi$ and $\vdash \varphi \rightarrow \psi$, then $\Box\psi$. Conversely, we may characterize monotonic modal logic by replacing (RE) with (RM). Another equivalent formulation is in terms of disjunction: $\Box\varphi \rightarrow \Box(\varphi \vee \psi)$. We will focus on a multi-modal variant of (AM) in the remainder, but all our observations readily apply to these alternative characterizations.

Semantically, there are various characterizations of monotonic logics with pooling modalities, besides our official characterization obtained by imposing monotony on the models defined in Section 2—this is spelled out in Section 4.3 of [8]. We briefly recall the semantics in terms of the *monotonic semantic clause*, as this will turn out useful for our completeness proof:

Definition 11. Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ and $w \in W$,

0. $\mathfrak{M}, w \not\models_m \bot$,
1. $\mathfrak{M}, w \models_m \varphi$ iff $w \in V(\varphi)$ for all $\varphi \in \mathfrak{P}$,
2. $\mathfrak{M}, w \models_m \neg\varphi$ iff $\mathfrak{M}, w \not\models_m \varphi$,
3. $\mathfrak{M}, w \models_m \varphi \vee \psi$ iff $\mathfrak{M}, w \models_m \varphi$ or $\mathfrak{M}, w \models_m \psi$,

[7] Usually, (AM) is just called (M). We preserve the latter name for its more general counterpart involving pooling modalities.

4. $\mathfrak{M}, w \models_m \Box_M \varphi$ iff there is an $X \in \mathcal{N}_M(w)$ such that $X \subseteq \|\varphi\|_i^{\mathfrak{M}}$

where $\|\varphi\|_i^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models_m \varphi\}$.

As shown in [8], closing the neighbourhood sets of a model $\mathfrak{M}$ under supersets—resulting in the *supplementation* $\mathfrak{M}^{\uparrow}$—is equivalent to using the indirect semantic clause for that model. That is:

Theorem 6. (Corollary 2 of [8]) *Where* $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ *is a model,* $w \in W$*, and* $\varphi \in \mathfrak{L}_\infty^{[\forall]}$*:* $\mathfrak{M}, w \models_m \varphi$ *iff* $\mathfrak{M}^{\uparrow}, w \models \varphi$*.*

In order to axiomatize the logics with pooling modalities over the class of monotonic models, one needs to add all instances of the following schema to **BL**:

$$\Box_M(\varphi \wedge \psi) \to (\Box_M \varphi \wedge \Box_M \psi) \tag{M}$$

Before we prove completeness, we note two properties of these logics. Theorem 7 implies that these logics allow for a slightly simpler axiomatization, skipping the axiom schema (B3). Theorem 8 records a derived rule in the logics with the universal modality, which will turn out useful for our completeness proof.

Theorem 7. *The axiom schema (B3) is derivable from (B1), (B2), (RE), and (M).*

Proof. Suppose that $\Box_M \varphi$ and $\Box_{M \sqcup N \sqcup N'} \varphi$. By the second premise, (RE), and (M), we can derive $\Box_{M \sqcup N \sqcup N'} \top$. By (B2), this gives us $\Box_N \top$. Finally, by (B1), (RE), and the first premise, we derive $\Box_{M \sqcup N} \varphi$. ■

Theorem 8. *Every instance of the schema*

$$(\Box_M \varphi \wedge [\forall](\varphi \to \psi)) \to \Box_M \psi \tag{M[$\forall$]}$$

is derivable in $\mathbf{BL}_\infty^{[\forall]} + \mathbf{M}$ *and* $\mathbf{BL}_\mathbf{f}^{[\forall]} + \mathbf{M}$*.*

Proof. Suppose that $\Box_M \varphi$ and $[\forall](\varphi \to \psi)$. By the second premise and normal modal logic properties, we get $[\forall](\varphi \leftrightarrow (\varphi \wedge \psi))$. By the first premise and (RGE), $\Box_M(\varphi \wedge \psi)$. Finally, by (M) and (MP), $\Box_M \psi$. ■

Theorem 9. *Where* $\mathbf{BL} \in \{\mathbf{BL}_\infty^{[\forall]}, \mathbf{BL}_\mathbf{f}^{[\forall]}, \mathbf{BL}_\infty, \mathbf{BL}_\mathbf{f}\}$*:* $\mathbf{BL} + \mathbf{M}$ *is sound and strongly complete w.r.t. the class of monotonic models. Moreover, it has the finite model property.*

Proof. Soundness is, as usual, safely left to the reader. For completeness and the finite model property, we give the argument for $\mathbf{BL}_\infty^{[\forall]} + \mathbf{M}$. Just as for the base logics, the analogous results for the other three logics are obtained by simplifications of this argument (cf. Section 5.6).

We need to prove that for every (finite) consistent set of formulas Θ, there is a (finite) monotonic model $\mathfrak{M}$ and a world w in its domain such that $\mathfrak{M}, w \models \Theta$. So let us hold fixed such a Θ. We define $\mathfrak{L}_{\Theta}^{[\forall]}$, Γ, and the model $\mathfrak{M}_{\Gamma}^{\Theta}$ in exactly the same way as we did in Section 5. Note that Lemmas 5 , 9, 10, and 11 (and their corollaries) and Facts 4, 6 and 7 are all preserved.

The crucial change occurs in the truth lemma. That is, we now prove the truth lemma given the *indirect* semantic clause, cf. Definition 11:

Proposition 1. *For all* $\Lambda \in \mathsf{MCS}_{\Gamma}^{\Theta}$*, for all* $f \in \mathbb{F}_{\Theta}$*, and all* $\varphi \in \mathfrak{L}_{\Theta}^{[\forall]}$*:*

$$\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models_m \varphi \textit{ iff } \varphi \in \Lambda.$$

The proof is exactly the same as the one for Lemma 12, except for the left-to-right direction of (TL$\Box$). So suppose that $\mathfrak{M}_{\Gamma}^{\Theta}, (\Lambda, f) \models \Box_M \varphi$. We distinguish the same two cases as in the original proof. For Case 1, note that, since each $\mathcal{N}_i(\Lambda, f)$ for $i \in I(M)$ is non-empty, we have

$$\Box_i \top \in \Lambda \text{ for all } i \in I(M) \tag{29}$$

and, hence, $\Box_M \top \in \Lambda$. Moreover, by Corollary 3 and the supposition for this case, $\Lambda \vdash [\forall](\top \leftrightarrow \varphi)$. By (RGE), $\Box_M \varphi \in \Lambda$.

For Case 2, we replace identity ($=$) with the subset-relation ($\subseteq$) in equations (17), (20), and (21). Again putting $\Phi = \{\psi \mid (N, \psi) \in \mathcal{B}\}$, this gives us

$$|\Phi|_{\Gamma}^{\Theta} \subseteq |\varphi|_{\Gamma}^{\Theta}. \tag{30}$$

Applying Lemma 5, we can derive that here is a finite $\mathcal{C} \subseteq \mathcal{B}$ such that, for all $\Delta \in \mathsf{MCS}_{\Gamma}^{\Theta}$, $\Delta \vdash [\forall](\bigwedge_{(N,\psi)\in\mathcal{C}} \psi \to \varphi)$. Consequently,

$$\Lambda \vdash [\forall] \left(\bigwedge_{(N,\psi)\in\mathcal{C}} \psi \to \varphi \right) \tag{31}$$

Let again $K = \bigsqcup_{(N,\psi)\in\mathcal{C}} N$. As before, we have $K \sqsubseteq M$. Using Lemma 5 and (M[$\forall$]), we derive that $\Box_K \varphi \in \Lambda$. Finally, we rely on (29) to derive that $\Box_M \varphi \in \Lambda$, using (B1) for the finite case and (B4) for the infinite case. This completes the proof of Proposition 1.

Let $f \in \mathbb{F}_{\Theta}$ be arbitrary. By Proposition 1,

$$\mathfrak{M}_{\Gamma}^{\Theta}, (\Theta, f) \models_m \Theta \tag{32}$$

Once there, we rely on the equivalence of the semantics in terms of the indirect semantic clause and the one in terms of monotonic models. Let $\mathfrak{M}_\Gamma^{\Theta,\uparrow}$ be the supplementation of $\mathfrak{M}_\Gamma^{\Theta}$. By Theorem 6,

$$\mathfrak{M}_\Gamma^{\Theta,\uparrow}, (\Theta, f) \models \Theta \tag{33}$$

and, by its very definition , $\mathfrak{M}_\Gamma^{\Theta,\uparrow}$ is a monotonic model. So this model serves as a witness to the consistency of Θ, and is moreover finite whenever Θ is finite.[8] ■

One may wonder whether it is not sufficient to add the following single-agent version of (M) to **BL** in order to axiomatize the logic of monotonic models:

$$\Box_i(\varphi \wedge \psi) \to (\Box_i\varphi \wedge \Box_i\psi) \tag{M$_i$}$$

The answer is negative: although such axioms characterize the closure of the $\mathcal{N}_i(w)$ under supersets at the level of *frames*, they do not yield a complete axiomatization.[9] That is, consider a model $\mathfrak{M}_{\mathsf{ex}}$ with two worlds w, w' that verify the same propositional variables, and with the (uniform) neighbourhood functions that put $\mathcal{N}_1(w) = \mathcal{N}_1(w') = \{\{w\}, \{w'\}\}$. Note that the neighbourhood sets of w and w' are not closed under supersets. However, all axioms of the base logics and (M$_1$) are valid at both w and w'. This is so because no formula of the form $\Box_1\varphi$ holds true at either w or w', whence the antecedent of any instance of (M$_1$) is false. However, $\Box_{1,1}\bot$ is true at both w and w', while $\Box_{1,1}\top$ is false at both. Hence, we really need axiom schemata like $\Box_{1,1}(\varphi \wedge \psi) \to \Box_{1,1}\varphi$ in order to obtain all validities obtained by imposing monotony.

This lesson applies to most of the other extensions of the base logics that we will study in this paper. That is, one often needs to add counterparts of the standard (monomodal) axioms for all the operators $\Box_M$, not just for the operators $\Box_i$. This is an immediate consequence of the fact that pooling modalities make the languages strictly more expressive, cf. [8].

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[8] If one would prove completeness in a more direct way, without relying on the alternative semantic characterization, one immediately takes the supplementation of $\mathfrak{M}_\Gamma^{\Theta}$ and proves the original truth lemma for that model. The general outline of such a proof will be analogous to our original proof for $\mathbf{BL}_\infty^{[\forall]}$, but one has to go back and forth a number of times between $\mathfrak{M}_\Gamma^{\Theta}$ and its supplementation.

[9] The positive is well-known, cf. [5, Lemma 2.21].

6.2. Closure Under Intersections

A second well-known constraint on neighbourhood sets is their being closed under intersections. Here we distinguish two cases: closure under finite intersections, and closure under arbitrary intersections. As we will show, while these two conditions cannot be distinguished in $\mathfrak{L}_f^{[\forall]}$ (hence, neither in any of its sublanguages), they yield different logics in the presence of infinitary pooling modalities. In the remainder, we call a model *(finite) intersective* iff for all worlds w in its domain and for all i, $\mathcal{N}_i(w)$ is closed under (finite) intersections.

First some notation. Where $M \in \mathbb{M}_\infty$, let M_f^- be the set

$$\{(i,1) \mid \exists k \in \mathbb{N}^+ : (i,k) \in M\} \cup \{(i,0) \mid i \notin I(M)\} \cup \{(i,\infty) \mid (i,\infty) \in M\}.$$

In words, M_f^- is obtained by replacing every (i,k) for a finite number $k > 0$ with $(i,1)$ in M, leaving the other indices unchanged. We use M_f to denote the set $\{(i,1) \mid (i,k) \in M \text{ for some } k \in \mathbb{N}^+ \cup \{\infty\}\} \cup \{(i,0) \mid i \notin I(M)\}$. Note that

Fact 13. *If $M \in \mathbb{M}_f$, then $M_f^- = M_f$.*

In line with the preceding, we need to distinguish between two axioms:

$$\Box_M \varphi \to \Box_{M_f^-} \varphi \tag{FI}$$

$$\Box_M \varphi \to \Box_{M_f} \varphi \tag{I}$$

Intuitively, (FI) expresses that if you can obtain a certain set $X \subseteq W$ by intersecting any k pieces of information for the index i, then that information is already available in itself, for i. (I) expresses that this is even the case if you get X by intersecting an arbitrary number of pieces of information. In view of Fact 13, (I) reduces to (FI) for languages without infinitary pooling modalities. So we have:

Fact 14. $\mathbf{BL_f^{[\forall]}} + \mathbf{FI} = \mathbf{BL_f^{[\forall]}} + \mathbf{I}$ *and* $\mathbf{BL_f} + \mathbf{FI} = \mathbf{BL_f} + \mathbf{I}$.

Let $\mathfrak{M}^\cap$ ($\mathfrak{M}^{\cap_f}$) denote the model obtained from $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ by closing each neighbourhood set $\mathcal{N}_i(w)$ under (finite) intersections. Let $\mathcal{N}_M^\cap$ ($\mathcal{N}_M^{\cap_f}$) denote the neighbourhood function that corresponds to the pooling profile M, in the model $\mathfrak{M}^\cap$ ($\mathfrak{M}^{\cap_f}$). Also, where $k \in \mathbb{N}^+$, let

$$M^k = \{(i,k) \mid i \in I(M), M(i) \neq \infty\} \cup \{(i,l) \in M \mid l = 0 \text{ or } l = \infty\}.$$

For the proof of completeness of these logics, the following two lemmas are crucial:

Lemma 15. *For all $(\Lambda, f) \in W$: $\mathcal{N}_M^\cap(\Lambda, f) = \mathcal{N}_{M^\infty}(\Lambda, f)$.*

Proof. It holds that: $X \in \mathcal{N}^{\cap}_{M}(\Lambda, f)$ iff for every $i \in I(M)$, there is some $X_i \in \mathcal{N}^{\cap}_{i}(\Lambda, f)$ such that $\bigcap_{i\in I(M)} X_i = X$ iff for every $i \in I(M)$, there is a $\mathcal{X}_i \subseteq \mathcal{N}^{\cap}_{i}(\Lambda, f)$ such that $\bigcap_{i\in I(M)} \bigcap \mathcal{X}_i = X$ iff $X \in \mathcal{N}_{M^\infty}(\Lambda, f)$. ■

Lemma 16. *For all* $(\Lambda, f) \in W$*: if* $X \in \mathcal{N}^{\cap_f}_{M}(\Lambda, f)$*, then there is a* $k \in \mathbb{N}^+$ *such that* $X \in \mathcal{N}_{M^k}(\Lambda, f)$*.*

Proof. Analogous to the proof for the previous lemma; however, in this case the $\mathcal{X}_i$ can be taken to be finite, and hence, since also $I(M)$ is finite, there is a $k \in \mathbb{N}$ such that each $\mathcal{X}_i$ has at most k elements. ■

Theorem 10. *Where* $\mathbf{BL} \in \{\mathbf{BL}^{[\forall]}_{\infty}, \mathbf{BL}^{[\forall]}_{\mathbf{f}}, \mathbf{BL}_{\infty}, \mathbf{BL}_{\mathbf{f}}\}$*:* $\mathbf{BL} + \mathbf{I}$ *is sound and strongly complete w.r.t. the class of intersective models. Moreover, it has the finite model property.*

Proof. We first focus on $\mathbf{BL}^{[\forall]}_{\infty} + \mathbf{I}$. Soundness is safely left to the reader. For completeness and the finite model property, fix a consistent set Θ. Define $\mathfrak{L}^{[\forall]}_{\Theta}$ as before, and let $\Gamma \subseteq \mathfrak{L}^{[\forall]}_{\Theta}$ be maximal such that $\Theta \subseteq \Gamma$ and Γ is consistent.

Let $\mathfrak{M}^{\Theta,\cap}_{\Gamma} = \langle W, \langle \mathcal{N}^{\cap}_{i}\rangle_{i\in I}, V\rangle$. Note that, by Fact 8, $\mathfrak{M}^{\Theta,\cap}_{\Gamma}$ is finite whenever Θ is finite. In order to establish completeness and the finite model property, it suffices to prove the following:

Proposition 2. *For all* $\varphi \in \mathfrak{L}^{[\forall]}_{\infty}$*, all* $\Lambda \in \mathsf{MCS}^{\Theta}_{\Gamma}$*, and all* $f \in \mathbb{F}_{\Theta}$*, the following are equivalent:*

(i) $\mathfrak{M}^{\Theta,\cap}_{\Gamma}, (\Lambda, f) \models \varphi$,

(ii) $\mathfrak{M}^{\Theta}_{\Gamma}, (\Lambda, f) \models \varphi$,

(iii) $\varphi \in \Lambda$.

We prove this by an induction on the complexity of φ. The base case and the induction step for $\neg$, $\vee$, and $[\forall]$ are safely left to the reader. For $\Box_M$, the equivalence of (ii) and (iii) follows from Lemma 12. The implication from (ii) to (i) follows immediately from the induction hypothesis and the fact that, for every $i \in I$ and every world $w \in W$, $\mathcal{N}_i(w) \subseteq \mathcal{N}^{\cap}_{i}(w)$. We now prove that (i) implies (iii).

Suppose that $\mathfrak{M}^{\Theta,\cap}_{\Gamma}, (\Lambda, f) \models \Box_M\varphi$. Hence, $\|\varphi\|^{\mathfrak{M}^{\Theta,\cap}_{\Gamma}} \in \mathcal{N}^{\cap}_{M}(\Lambda, f)$. By the induction hypothesis and by Lemma 15, $\|\varphi\|^{\mathfrak{M}^{\Theta}_{\Gamma}} \in \mathcal{N}_{M^\infty}(\Lambda, f)$ and hence $\mathfrak{M}^{\Theta}_{\Gamma}, (\Lambda, f) \models \Box_{M^\infty}\varphi$.

We now reason exactly as in the proof of Lemma 12, showing that there is a $K \in \mathbb{M}_\infty$ such that $K \sqsubseteq M^\infty$ and $\Lambda \vdash \Box_K\varphi$. By (I), $\Lambda \vdash \Box_{K_f}\varphi$. Note moreover that, since $K \sqsubseteq M^\infty$, $K_f \sqsubseteq M$. We then reason again as in the last part of Lemma 12 to arrive at the conclusion that $\Box_M\varphi \in \Lambda$.

This gives us completeness and the finite model property for $\mathbf{BL}_\infty^{[\forall]} + \mathbf{I}$. For $\mathbf{BL}_\mathbf{f}^{[\forall]} + \mathbf{I}$, note first that we are working with a set $\Theta \subseteq \mathfrak{L}_f^{[\forall]}$, and thus $\mathbb{M}_\Theta \subseteq \mathbb{M}_f$. Since K is a finite union of pooling profiles $N \in \mathbb{M}_\Theta$, it follows that $K \in \mathbb{M}_f$. So we can apply (I) just as before. For the two logics without the universal modality, we have to adapt the proof in the same way as we did for $\mathbf{BL}_\infty$ and $\mathbf{BL}_\mathbf{f}$ (cf. Section 5.6). ∎

THEOREM 11. *Where* $\mathbf{BL} \in \{\mathbf{BL}_\infty^{[\forall]}, \mathbf{BL}_\mathbf{f}^{[\forall]}, \mathbf{BL}_\infty, \mathbf{BL}_\mathbf{f}\}$*:* $\mathbf{BL} + \mathbf{FI}$ *is sound and strongly complete w.r.t. the class of finite intersective models. Moreover, it has the finite model property.*

PROOF. The proof is analogous to that of Theorem 10, but this time relying on Lemma 16 instead of Lemma 15 in the proof of the induction step, for the implication from (i) to (iii). This way, we obtain a $k \in \mathbb{N}$ and a K such that $\Lambda \vdash \Box_K \varphi$ and $K \sqsubseteq M^k$. We then apply (FI) to derive that $\Lambda \vdash \Box_{K_f^-} \varphi$. Observe that $K_f^- \sqsubseteq M$. We can then again reason as in the last part of Lemma 12 to show that $\Box_M \varphi \in \Lambda$. ∎

By Fact 14, it follows that only if we have infinitary pooling modalities at our disposal, the two frame conditions mentioned at the outset of this section yield a different logic. Without such modalities, this distinction remains under the radar.

In classical modal logics with a single modal operator $\Box$, closure under (finite or arbitrary) intersections is axiomatized by the aggregation axiom (AC)[10]:

$$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi) \tag{AC}$$

In the restricted language with only regular sets—no proper pooling profiles—as indexes, completeness with respect to the class of intersective models is obtained by adding all group varians of (AC) to the logic (cf. [9, Theorem 7]). In contrast, to obtain a complete axiomatization for the languages considered here, we need (FI), respectively (I). That (FI) implies (AC) can easily be verified. Suppose that $\Box_i \varphi$ and $\Box_i \psi$. By (B1), this gives us $\Box_{i,i}(\varphi \wedge \psi)$. Then, by (FI), we obtain $\Box_i(\varphi \wedge \psi)$.

There is also a relatively easy way to see that the converse implication, from (AC) to (FI), fails. Let $\mathfrak{M}_{\mathsf{ex}}$ be the model described at the end of Section 6.1. Note that this model is not (finite) intersective, since e.g. $\emptyset \notin \mathcal{N}_1(w)$. Still, any instance of (AC) is true at all worlds, since no formula

[10]Cf. [5, Lemma 2.20]. (AC) is often just called (C).

$\Box_1\varphi$ is true. In contrast, (FI) and (I) fail on $\mathfrak{M}_{\mathsf{ex}}$, since it satisfies $\Box_{1,1}\bot$ but does not satisfy $\Box_1\bot$.

6.3. Regular Models

In the context of classical modal logic, *regular modal logics* are those obtained by requiring both closure under intersection and closure under supersets of the neighbourhood sets. These logics are standardly axiomatized by a combination of the aforementioned axioms (M) and (C) (or variants thereof), or by the single regularity axiom:

$$(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi) \tag{R}$$

Regular modal logics are very close to normal modal logics, in the sense that, when adding necessitation to the regular modal logic, we get a normal modal logic.

As for the case of intersection, we will distinguish between two notions of regularity. In the remainder, let a model $\mathfrak{M} = \langle W, \langle \mathcal{N}_i\rangle_{i\in I}, V\rangle$ be *regular* iff for all $i \in I$ and all $w \in W$, $\mathcal{N}_i(w)$ is closed under arbitrary intersections and under supersets. $\mathfrak{M}$ is *regular*^{f} iff for all $i \in I$ and all $w \in W$, $\mathcal{N}_i(w)$ is closed under finite intersections and under supersets. The axiomatization of logics with pooling modalities, over the class of regular models, is obtained by adding (M) and (I) to our base logics. In view of Theorem 7, the axiom schema (B3) is redundant in the resulting systems.

THEOREM 12. *Where* $\mathbf{BL} \in \{\mathbf{BL}^{[\forall]}_{\infty}, \mathbf{BL}^{[\forall]}_{\mathbf{f}}, \mathbf{BL}_{\infty}, \mathbf{BL}_{\mathbf{f}}\}$*:* $\mathbf{BL}+\mathbf{M}+\mathbf{I}$ *is sound and strongly complete w.r.t. the class of regular models. Moreover, it has the finite model property.*

PROOF. As before we focus on the richest logic, viz. $\mathbf{BL}^{[\forall]}_{\infty}+\mathbf{M}+\mathbf{I}$. Soundness is safely left to the reader. For strong completeness and the finite model property, we combine the adaptations from the two previous sections. We start from the model $\mathfrak{M}^{\Theta,\cap}_{\Gamma}$ constructed in the proof of Theorem 10. However, we now prove that, in the presence of (M), $\mathfrak{M}^{\Theta,\cap}_{\Gamma}$ and $\mathfrak{M}^{\Theta}_{\Gamma}$ are also equivalent if we apply the monotonic semantic clause to both:

PROPOSITION 3. *For all* $\varphi \in \mathfrak{L}^{[\forall]}_{\infty}$*, all* $\Lambda \in \mathsf{MCS}^{\Theta}_{\Gamma}$*, and all* $f \in \mathbb{F}_{\Theta}$*, the following are equivalent:*

(i) $\mathfrak{M}^{\Theta,\cap}_{\Gamma}, (\Lambda, f) \models_m \varphi$,

(ii) $\mathfrak{M}^{\Theta}_{\Gamma}, (\Lambda, f) \models_m \varphi$,

(iii) $\varphi \in \Lambda$.

We prove this by an induction on the complexity of φ. The base case and the induction step for $\neg$, $\vee$, and $[\forall]$ are safely left to the reader. For $\varphi = \Box_M\psi$, the equivalence of (ii) and (iii) follows from Proposition 1. The implication from (ii) to (i) follows immediately from the induction hypothesis and the fact that, for every $i \in I$ and every world $w \in W$, $\mathcal{N}_i(w) \subseteq \mathcal{N}_i^{\cap}(w)$. So it suffices to prove that (i) implies (iii).

Suppose that $\mathfrak{M}_\Gamma^{\Theta,\cap}, (\Lambda, f) \models_m \Box_M\varphi$. Hence, there is an $X \in \mathcal{N}_M^{\cap}(\Lambda, f)$ such that $X \subseteq \|\varphi\|^{\mathfrak{M}_\Gamma^{\Theta,\cap}}$. By the induction hypothesis and Lemma 15, $X \in \mathcal{N}_{M^\infty}(\Lambda, f)$ and $X \subseteq \|\varphi\|^{\mathfrak{M}_\Gamma^{\Theta}}$. Hence, $\mathfrak{M}_\Gamma^{\Theta}, (\Lambda, f) \models_m \Box_{M^\infty}\varphi$.

We now reason exactly as in the proof of Proposition 1, showing that there is a $K \in \mathbb{M}_\infty$ such that $K \sqsubseteq M^\infty$ and $\Lambda \vdash \Box_K\varphi$. Note that in that proof, (M) is called upon. By (I), $\Lambda \vdash \Box_{K_f}\varphi$. As before, we have that $K_f \sqsubseteq M$. We then reason again as in the last part of Lemma 12 to arrive at the conclusion that $\Box_M\varphi \in \Lambda$.

So we have a truth lemma for $\mathfrak{M}_\Gamma^{\Theta,\cap}$ under the monotonic semantic clause. Let $\mathfrak{M}_\Gamma^{\Theta,r}$ be obtained by closing all the neighbourhood sets in $\mathfrak{M}_\Gamma^{\Theta,\cap}$ under supersets. Hence, for every $(\Lambda, f) \in W$ and $i \in I$, $\mathcal{N}_i^r(\Lambda, f) = (\mathcal{N}_i^{\cap}(\Lambda, f))^{\uparrow}$. By Theorem 6, for all φ and all $(\Lambda, f) \in W$:

$$\mathfrak{M}_\Gamma^{\Theta,r}, (\Lambda, f) \models \varphi \text{ iff } \mathfrak{M}_\Gamma^{\Theta,\cap}, (\Lambda, f) \models_m \varphi$$

and, hence, for every $f \in \mathbb{F}_\Theta$, $\mathfrak{M}_\Gamma^{\Theta,r}, (\Gamma, f) \models \Theta$. We now show that $\mathfrak{M}_\Gamma^{\Theta,r}$ is a regular model. In view of its construction, it suffices to prove that each of its neighbourhood functions are closed under arbitrary intersection. So suppose that $\mathcal{X} \subseteq (\mathcal{N}_i^{\cap}(\Lambda, f))^{\uparrow}$ for some $i \in I$ and $(\Lambda, f) \in W$. For every $X \in \mathcal{X}$, let X' be such that $X' \in \mathcal{N}_i^{\cap}(\Lambda, f)$ and $X' \subseteq X$. Let $\mathcal{X}' = \bigcap\{X' \mid X \in \mathcal{X}\}$. Note that $\bigcap\mathcal{X}' \in \mathcal{N}_i^{\cap}(\Lambda, f)$ and $\bigcap\mathcal{X}' \subseteq \bigcap\mathcal{X}$. Hence, $\bigcap\mathcal{X} \in (\mathcal{N}_i^{\cap}(\Lambda, f))^{\uparrow}$. ■

Theorem 13. *Where* $\mathbf{BL} \in \{\mathbf{BL}_\infty^{[\forall]}, \mathbf{BL}_\mathbf{f}^{[\forall]}, \mathbf{BL}_\infty, \mathbf{BL}_\mathbf{f}\}$*:* $\mathbf{BL} + \mathbf{M} + \mathbf{FI}$ *is sound and strongly complete w.r.t. the class of regular*^{f} *models. Moreover, it has the finite model property.*

Proof. Analogous to the proof of Theorem 12, combining the proofs of Proposition 1 and of Theorem 11. ■

Note that, for reasons entirely analogous to those spelled out in the preceding two sections, one cannot obtain sound and complete axiomatizations of pooling logics over regular models by just adding all instances of (R) for each $i \in I$. That is, such axioms will not allow one to derive either (M), or (FI), resp. (I).

6.4. Some Other Frame Conditions

In this subsection, we discuss a number of other frame conditions, showing that completeness and decidability results can be obtained for them by minor amendments to the proofs from preceding (sub)sections. We will gradually move from general to more restricted results. This at once illustrates the power and the limitations of our puzzle piece construction, for the metatheory of pooling modalities.

6.4.1. Definable Frame Conditions Say a condition (C) on neighbourhood sets $\mathcal{N}_i(w)$ is **BL**-*definable* iff there is some φ in the language of **BL** such that, for all models $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ and all $w \in W$, $\mathcal{N}_i(w)$ satisfies (C) if and only if $\mathfrak{M}, w \models \varphi$. In this case, we say that the formula φ *defines* condition (C). One very well-known example of a frame condition that is **BL**-definable is unit-containment, that $W \in \mathcal{N}_i(w)$ for all $w \in W$. This frame condition is defined by the following well-known axiom:

$$\Box_i \top \tag{N}$$

Note that adding (N) to **BL** is equivalent to adding all instances of the form $\Box_M \top$ to **BL**. That is, from the former, one can easily derive the latter by (B1) (for finitary pooling profiles M) and (B4) (for infinitary pooling profiles). Moreover, adding (N) to the base logic simplifies our axiomatization. Let us call the logic thus obtained $\mathbf{BL} + \mathbf{N}$. We have:

Theorem 14. *(B2) and (B3) are derivable from the other axioms and rules of* $\mathbf{BL} + \mathbf{N}$.

Proof. Suppose that for all $i \in I$, we have $\Box_i \top$. By (B1), we can derive for all $M \in \mathbb{M}_f$, $\Box_M \top$. By (B1) and (B4), we can derive $\Box_M$ for every $M \in \mathbb{M}_\infty$. Hence, every instance of (B3) is trivially derivable. (B2) follows from the following stronger property:

$$\vdash_{\mathbf{BL}+\mathbf{N}} \Box_M \varphi \to \Box_{M \sqcup N} \varphi \tag{S}$$

To see why (S) is derivable, suppose $\Box_M \varphi$. In view of the first part of this proof, we can derive $\Box_N \top$ in $\mathbf{BL} + \mathbf{N}$. By (B1), $\Box_{M \sqcup N}(\varphi \wedge \top)$. By (RE), $\Box_{M \sqcup N} \varphi$. ■

In the context of reasoning about possibly conflicting information (evidence, beliefs, norms, etc.), another noteworthy frame condition is (individual or joint) consistency of the neighbourhoods. Here the pooling modalities generate various non-equivalent options for defining frame conditions. First, one may require an individual neighbourhood set $\mathcal{N}_i(w)$ not to generate any n-ary conflicts: there are no $X_1, \ldots, X_n \in \mathcal{N}_i(w)$ such that $X_1 \cap \ldots \cap X_n = \emptyset$.

This frame condition is defined by the axiom $\neg\Box_M\bot$, where M is the pooling profile that puts $M(i) = n$ and $M(j) = 0$ for all $j \neq i$. Alternatively, following Coalition Logic (cf. [8, Section 7.3]), one may require that distinct agents have jointly compatible neighbourhoods: for all $i_1, \dots, i_n \in G$, if $X_1 \in \mathcal{N}_{i_1}(w)$, ..., and $X_n \in \mathcal{N}_{i_n}(w)$, then $X_1 \cap \ldots \cap X_n \neq \emptyset$. This frame condition is defined by the formula $\neg\Box_M\bot$, where $M(i) = 1$ for all $i \in G$ and $M(j) = 0$ for all $j \notin G$. These variants are both covered by the general notion of M-consistency:

$$\emptyset \notin \mathcal{N}_M(w)$$

and defined by the axiom (P_M):

$$\neg\Box_M\bot \qquad (\mathrm{P}_M)$$

Other examples of definable conditions and formulas defining them are given in the second half of Table 3. Of course, not all combinations of definable frame conditions yield an interesting or even sensible logic. For instance, one cannot both have (N) and (NN), or (P) and (NP) on pains of triviality. Also, some (combinations of) axioms will imply others: e.g. if we have (P_M) for some M, and (N) for all $i \in I(M)$, then we can infer (P_i) for all $i \in I(M)$.

We now state a theorem that concerns any combination of such definable frame conditions.

THEOREM 15. *Let* $\mathbf{BL} \in \{\mathbf{BL}_\infty^{[\forall]}, \mathbf{BL}_\mathbf{f}^{[\forall]}, \mathbf{BL}_\infty, \mathbf{BL}_\mathbf{f}\}$. *Let* (C_1), (C_2), ... *be* $\mathbf{BL}$-*definable frame conditions and let* $\varphi_1, \varphi_2, \dots$ *be the respective formulas that express them. Then a sound and strongly complete axiomatization for the class of all (monotonic/intersective/regular) models that satisfy* (C_1), (C_2), ... *is obtained by adding* $\varphi_1, \varphi_2, \dots$ *to* $\mathbf{BL}$ ($\mathbf{BL}+\mathbf{M}/\mathbf{BL}+\mathbf{I}/\mathbf{BL}+\mathbf{R}$).

PROOF. Soundness is routine, as usual. For strong completeness, we focus on the canonical models where the set Θ is a maximal consistent subset of the entire language of the logic in question—so e.g. $\mathfrak{L}_\Theta^{[\forall]} = \mathfrak{L}_\infty^{[\forall]}$ or $\mathfrak{L}_\Theta = \mathfrak{L}_f$. In this case, every maximal consistent set Λ used in the construction will contain all formulas $\varphi_1, \varphi_2, \dots$. By the truth lemma, we know that at every world in these canonical models, each of $\varphi_1, \varphi_2, \dots$ are true. Since these formulas define the respective frame conditions, we know that these canonical models satisfy (C_1), (C_2), ... ■

Theorem 15 is silent about the finite model property. This is slightly less easy to obtain. Let us first explain why, before we outline some strategies to fix this. By our construction, for infinite index sets I and finite Θ, I_Θ will be a proper subset of I. (Recall that I_Θ is the set of indexes that occur in some pooling profile, in some member of $\mathsf{SF}(\Theta)$.) In view of Definition 10, this implies that for all $i \in I \backslash I_\Theta$, and for all $(\Lambda, f) \in W$, $\mathcal{N}_i(\Lambda, f) = \emptyset$. This is so, even if our logic may e.g. have (N_i) as an axiom—simply because that formula is not a member of $\mathfrak{L}_\Theta^{[\forall]}$, and is hence also ignored in the construction of $\mathfrak{M}_\Gamma^\Theta$.

There are various ways one may fix our construction in order to obtain the finite model property for the mentioned extensions. First, in cases where the frame conditions are defined by a finite number of formulas, we may just include all those formulas in the set Θ, thus ensuring that they are "seen" in the construction of I_Θ, $\mathfrak{L}_\Theta^{([\forall])}$, etc. In cases with an infinite number of frame conditions, one may run the whole construction as originally, and tinker around with the constructed model afterwards. Note that we are only constructing a model for Θ, whence the properties of neighbourhood functions $\mathcal{N}_i$ for $i \in I \backslash I_\Theta$ have no impact on the truth of formulas in $\mathfrak{L}_\Theta^{[\forall]}$. In other words: we can change those neighbourhood functions *ad libitum*, making sure we have the right frame conditions in the thus obtained model. This simple strategy allows us e.g. to prove the finite model property for the logic obtained by adding all instances of (N), for all $i \in I$, to $\mathbf{BL}_\infty^{[\forall]}$.

6.4.2. Reflexivity Recall the reflection schema or truth schema T, that is familiar from normal modal logics: $\Box\varphi \to \varphi$. Call a neighbourhood function $\mathcal{N}$ *reflexive* iff, for every $w \in W$ and for every $X \in \mathcal{N}(w)$, $w \in X$. A model $\mathcal{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, W \rangle$ is reflexive if and only if, for every $i \in I$, $\mathcal{N}_i$ is reflexive.

Theorem 16. *A sound and strongly complete axiomatization for the class of all reflexive (monotonic/intersective/regular) models is obtained by adding all instances of the following axiom schema to* $\mathbf{BL}$ $(\mathbf{BL} + \mathbf{M}/\mathbf{BL} + \mathbf{I}/\mathbf{BL} + \mathbf{R})$*:*

$$\Box_M \varphi \to \varphi \tag{T}$$

Moreover, the resulting logics all have the finite model property.

Proof. It suffices to check that, for each of the canonical model constructions given in Sections 5, 6.1, 6.2 and 6.3, the constructed neighbourhood functions are reflexive if the maximal consistent sets are closed under (T). We give the argument for $\mathfrak{M}_\Gamma^\Theta$ first. Let $X_{i,k}^{M,\varphi} \in \mathcal{N}_i(\Lambda, f)$. This implies that $\Box_M \varphi \in \Lambda$. Note that by the construction, $X_{i,k}^{M,\varphi} \supseteq \{(\Lambda', f') \mid \varphi \in \Lambda'\}$. By (T_M) and since $\Box_M \varphi \in \Lambda$, $\varphi \in \Lambda$. Hence, $(\Lambda, f) \in \{(\Lambda', f') \mid \varphi \in \Lambda'\}$ and hence $(\Lambda, f) \in X_{i,k}^{M,\varphi}$.

Note that for monotonic, intersective, and regular models, our proofs basically consisted in showing that if we close the neighbourhood sets of the canonical model under supersets, intersections, resp. supersets and intersections, truth is preserved. Since each of those operations (and their combination) also preserve reflexivity of the neighbourhood functions, it can easily be inferred that all the canonical models constructed in Sections 6.1–6.3 are reflexive when the underlying logic contains (T). ■

Importantly, one cannot get a complete axiomatization of reflexivity by just adding the axioms (T_i), i.e. $\Box_i\varphi \to \varphi$ to the base logic. To see this, note that all axioms $\Box_M\phi \to \phi$ are sound with respect to reflexive frames. The following example of a non-reflexive frame shows that these axioms do not logically follow from $\Box_i\varphi \to \varphi$. We consider a simple case with $I = \{1\}$. Take a model $\mathfrak{M}$ with two worlds, w and v, where all propositional formulas are true at both worlds. Suppose now that $\mathcal{N}_1(w) = \{\{w\}, \{v\}\}$. Since neither $\{w\}$ nor $\{v\}$ correspond to the truth set of any formula φ in this model, $\Box_1\varphi$ is false for every φ at both w and v, and hence (T_1) is trivially valid in this model. However, this model does not validate ($\mathrm{T}_{\{1,1\}}$), since $\Box_{\{1,1\}}\bot$ is true at w and at v. So the model satisfies all formulas of the form $\Box_i\phi \to \phi$ together with (B1)–(B2), but not $\Box_M\phi \to \phi$.

It is not easy to generalize the above theorem to classes of models where some, but not all $i \in I$ are such that $\mathcal{N}_i$ is reflexive. For instance, if we only require reflexivity of $\mathcal{N}_1$ but not of $\mathcal{N}_2$, then $\Box_{\{1,2\}}\varphi \to \varphi$ is not valid. This implies that the reasoning in the first paragraph of our proof does not go through. The investigation of such "mixed" classes of models is left for future work.

6.4.3. Uniformity A model $\mathfrak{M}$ is *uniform* iff for all $i \in I$, for all $w, w' \in W$, $\mathcal{N}_i(w) = \mathcal{N}_i(w')$. Uniform models are e.g. used in the study of evidence-based belief, cf. [1,7].

Theorem 17. *Where* $\mathbf{BL} \in \{\mathbf{BL}_\infty^{[\forall]}, \mathbf{BL}_\mathbf{f}^{[\forall]}\}$, *a sound and strongly complete axiomatization for the class of all uniform (monotonic/intersective/regular) models is obtained by adding all instances of the following axiom schema to* $\mathbf{BL}$ ($\mathbf{BL}+\mathbf{M}/\mathbf{BL}+\mathbf{I}/\mathbf{BL}+\mathbf{R}$)*:*

$$\Box_M\varphi \to [\forall]\Box_M\varphi \tag{U}$$

Moreover, the resulting logics all have the finite model property.

Proof. We only consider uniformity over the base logic; the argument for the monotonic, intersective, or regular models is analogous. It suffices to inspect our construction of $\mathfrak{M}_\Gamma^\Theta$ and check that in the presence of (U), this model will be uniform. Let $X^{M,\varphi}_{i,k} \in \mathcal{N}_i(\Lambda, f)$. Then $\Box_M\varphi \in \Lambda$. In view of the definition of $\mathfrak{L}^{[\forall]}_\Theta$, $[\forall]\Box_M\varphi \in \mathfrak{L}^{[\forall]}_\Theta$. By the uniformity axiom, $[\forall]\Box_M\varphi \in \Lambda$ and, hence, for all $\Delta \in \mathsf{MCS}^\Theta_\Gamma$, $[\forall]\Box_M\varphi \in \Delta$ and so $\Box_M\varphi \in \Delta$. Consequently, for all $\Delta \in \mathsf{MCS}^\Theta_\Gamma$ and all $f' \in \mathbb{F}_\Theta$, $X^{M,\varphi}_{i,k} \in \mathcal{N}_i(\Delta, f')$. ■

Again, we need to add the uniformity axiom (U) for all pooling profiles M; one can again easily make examples showing that just adding this axiom schema for individual indexes will yield an incomplete logic. For instance, let $\mathfrak{M}'$ be just like the model $\mathfrak{M}$ from Section 6.1, except that now $\mathcal{N}_1(w') = \emptyset$. We have: $\mathfrak{M}, w \models \Box_{1,1}\bot$ but $\mathfrak{M}, w \not\models [\forall]\Box_{1,1}\bot$.

Note that we make essential use of the universal modality in order to axiomatize uniformity. In the absence of this modality, the frame condition of uniformity does have an impact. For instance, it makes all instances of the two following schemas valid:

$$(\Box_M\varphi \wedge \Box_N\psi) \to \Box_M(\varphi \wedge \Box_N\psi)$$
$$(\Box_M\varphi \wedge \neg\Box_N\psi) \to \Box_M(\varphi \wedge \neg\Box_N\psi)$$

It is an open problem whether any such validities suffice to obtain a complete axiomatization of uniform models in $\mathfrak{L}_\infty$ or $\mathfrak{L}_f$.

7. Overview of the Results

Table 3 provides an overview of the completeness results mentioned in this paper. As noted, the frame conditions in the second half of the table can be axiomatized for each of the agents independently; those of the first half are required to be valid for all the agents at once. For those first five frame conditions we also established the finite model property.

Taken together, these results show that on the one hand, our method for proving completeness is fairly powerful, allowing us to axiomatize a large class of systems. On the other hand our observations already pointed towards some of its limitations. For instance, upon inspection, it seems one cannot easily apply the canonical model in its present form to prove completeness of logics that satisfy the well-known (iterative) axioms of positive and negative introspection[11]:

$$\Box\varphi \to \Box\Box\varphi \tag{4}$$
$$\neg\Box\varphi \to \Box\neg\Box\neg\varphi \tag{5}$$

[11] For the (4)-axiom, at least another round of copying seems to be required before one can obtain a truth lemma.

Table 3. Overview of the main completeness results from this paper. Any combination of the given frame conditions is axiomatized by adding all associated axioms to **BL**

Class	Frame condition	Axiomatization: **BL**+	
Monotonic	$\mathcal{N}_i = \mathcal{N}_i^{\uparrow}$	$\Box_M(\varphi \wedge \psi) \rightarrow (\Box_M \varphi \wedge \Box_M \psi)$	(M)
Finite intersective	$\mathcal{N}_i = \Cap^f \mathcal{N}_i$	$\Box_M \rightarrow \Box_{M^f_-} \varphi$	(FI)
Intersective	$\mathcal{N}_i = \Cap^\infty \mathcal{N}_i$	$\Box_M \rightarrow \Box_{M^f} \varphi$	(I)
Reflexive	$\forall X \in \mathcal{N}_i(w) : w \in X$	$\Box_M \varphi \rightarrow \varphi$	(T)
Uniform	$\forall w, v : \mathcal{N}_i(w) = \mathcal{N}_i(v)$	$\Box_M \varphi \rightarrow [\forall]\Box_M \varphi$	(U)
Unit-contained	$W \in \mathcal{N}_i(w)$	$\Box_i \top$	(N)
i-consistent	$\emptyset \notin \mathcal{N}_i(w)$	$\neg\Box_i \bot$	(P$_i$)
M-consistent	$\emptyset \notin \mathcal{N}_M(w)$	$\neg\Box_M \bot$	(P$_M$)
Unit-less	$W \notin \mathcal{N}_i(w)$	$\neg\Box_i \top$	(NN)
Null-contained	$\emptyset \in \mathcal{N}_i(w)$	$\Box_i \bot$	(NP)

Likewise, the axiomatization of pooling modalities where only some, but not all the neighbourhood functions satisfy one of the conditions from the first half of Table 3 is an open problem. Note for instance that if we require $\mathcal{N}_1(w)$ to be closed under supersets, this does not entail that also $\mathcal{N}_{\{1,2\}}(w)$ satisfies these properties.

Taking a more high-level perspective, the situation is essentially not much different from that in relational semantics. There is a standard method for proving completeness—i.e. via canonical models—, and there are more or less broadly applicable ways to adapt this method to specific cases, our puzzle construction being one among them. However, completeness is not to be expected in general, and with each new logic, new technical difficulties may arise.

8. Summary and Open Issues

The main contribution of this paper was threefold. First, we proved completeness for a class of logics that feature pooling modalities, considering four different formal languages. Second, we showed that a number of those logics also satisfy the finite model property. Third and perhaps most importantly, we introduced a novel method in proving our results, viz. the puzzle piece construction.

As pointed out in the last section, it remains to be seen how broadly applicable this method is, and thus what other frame conditions can be axiomatized using similar tools. Orthogonally, one may investigate alternative languages with (finitary or infinitary) pooling modalities, as noted in Section 3.4.

A final, more fundamental question is whether one can rephrase our method in terms of a two-step procedure, following common practice in modal logic: (i) show the logic to be complete with respect to a class of quasi-models; (ii) develop a general method for turning every quasi-model into a standard model of the logic. A natural notion of quasi-model would be obtained by combining our notion of a g-model (cf. Definition 8) with the following four constraints that reflect the axioms of the base logics:

(C1) $\mathcal{N}_M(w) \Cap \mathcal{N}_{M'}(w) \subseteq \mathcal{N}_{M \sqcup M}(w)$,

(C2) If $W \in \mathcal{N}_{M \sqcup N}(w)$, then $W \in \mathcal{N}_M(w)$,

(C3) If $X \in \mathcal{N}_M(w)$ and $X \in \mathcal{N}_{M \sqcup M' \sqcup N}(w)$, then $X \in \mathcal{N}_{M \sqcup M'}(w)$,

(C4) $\mathcal{N}_M(w) \subseteq \mathcal{N}_{M^\infty}(w)$.

Step (ii) would then consist in showing that every g-model that satisfies axioms (C1)–(C4) is equivalent to a *pooled* g-model, i.e. one that satisfies the identity in Definition 6.

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Appendix: Finitary Pooling

As explained in Section 3.4, our interpretation of ∞ as "arbitrary pointwise intersection" is crucial for the compactness of the logics that were studied in this paper. At the suggestion of a referee we now briefly discuss a non-compact but natural alternative.

Let $\mathbb{M}_*$ be the set of all multisets of the type $M : I \to \mathbb{N} \cup \{*\}$, such that (i) for at most finitely many $i \in I$, $M(i) > 0$ and (ii) for at least one $i \in I$, $M(i) \neq 0$. Intuitively, $M(i) = *$ means that in the construction of neighbourhoods for M, we can use *any finite* number of neighbourhoods that belong to index i. Analogously, $\mathfrak{L}_*^{[\forall]}$ is defined just as $\mathfrak{L}_\infty^{[\forall]}$, but replacing $\mathbb{M}_\infty$ with $\mathbb{M}_*$.

The following definition should not be surprising in view of our informal reading of the $*$ notation:

Definition 12. Let $\mathcal{X}$ be a set of sets. Then $\Cap^*\mathcal{X}:=\{X_1 \cap \ldots \cap X_n \mid X_1, \ldots, X_n \in \mathcal{X}, n \in \mathbb{N}^+\}$.

Once there, we can simply re-use Definitions 6 and 7 (cf. Section 2.3) to obtain a full-blown semantics for $\mathfrak{L}_*^{[\forall]}$. As before, validity and semantic consequence are defined in the standard way. Let us call the resulting logic $\mathbf{BL}_*^{[\forall]}$.[12]

We first show that this logic is not compact, by an argument that is entirely analogous to the one that illustrates the failure of compactness for **PDL**, cf. Section 3.4. Consider the set $\Theta = \{\Box_{1^*}p, \neg\Box_1 p, \neg\Box_{1,1}p, \neg\Box_{1,1,1}p, \ldots\}$. Note that Θ is inconsistent. Indeed, if a model $\mathfrak{M}$ verifies $\Box_{1^*}p$ at a world w, then there must be some $n \in \mathbb{N}$ such that, for some $X_1, \ldots, X_n \in \mathcal{N}_1(w)$, $X_1 \cap \ldots \cap X_n = \|p\|_{\mathfrak{M}}$. However, this means that for some $n \in \mathbb{N}$, $\mathfrak{M}, w \models \Box_{1^n}p$. But that means at least one of the other members of Θ must be false.

[12]What follows applies, *mutatis mutandis*, also to the variant without the universal modality.

However, every finite subset of this Θ is consistent. To see this, consider the set $\Theta_n = \{\Box_{1^*}p, \neg\Box_1 p, \neg\Box_{1,1}p, \ldots, \neg\Box_{1^n}p\}$. In order to satisfy all members of Θ_n, it suffices to construct a model $\mathfrak{M}$ with $n+1$ distinct neighbourhoods, such that (only) the intersection of all these neighbourhoods yields $\|p\|_{\mathfrak{M}}$.

On the positive side, relying on the other results in this paper, one can easily give a sound and weakly complete axiomatization of $\mathbf{BL}_*^{[\forall]}$. All one has to do is replace ∞ with $*$ everywhere in the axiomatization of $\mathbf{BL}_\infty^{[\forall]}$. To show that the resulting axiomatization is sound, it suffices to observe that the $*$-variant of (B4),

$$\Box_M\varphi \to \Box_{M^*}\varphi$$

is valid given the above semantics.

The argument for weak completeness goes as follows. Where $\varphi \in \mathfrak{L}_*^{[\forall]}$, let φ^∞ be the result of replacing each $*$ in φ with ∞. Note that over finite models, the operations $\Cap^*$ and $\Cap^\infty$ are equivalent. Consequently, over such models, φ and φ^∞ are equivalent for any $\varphi \in \mathfrak{L}_*^{[\forall]}$. Let now $\varphi \in \mathfrak{L}_*^{[\forall]}$ and suppose that $\Vdash \varphi$. Then φ is true in every model and hence *a fortiori* in every finite model. Consequently, φ^∞ is valid in every finite model. By the finite model property, $\Vdash \varphi^\infty$. By completeness, $\vdash \varphi^\infty$. But then, in view of our definition of the axiomatization for $\mathbf{BL}_*^{[\forall]}$, it follows that φ is a theorem in $\mathbf{BL}_*^{[\forall]}$.

The above argument can be re-run for various extensions of $\mathbf{BL}_*^{[\forall]}$. For instance, adding (M) to the logic, we get a weakly complete axiomatization for the language $\mathfrak{L}_*^{[\forall]}$ over monotonic neighbourhood models. Similar remarks apply to the other extensions, with one important exception, viz. closure under intersections. Note that the $*$-variant of our axiom (I) is already sound if we close under finite intersections. For instance, $\Box_{1^*}p \to \Box_1 p$ is valid whenever the neighbourhoods are closed under finite intersections. In sum, the language $\mathfrak{L}_*^{[\forall]}$ cannot distinguish between closure under finite intersection and closure under infinite intersection.

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