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Population Dynamics of Randomly Interacting Self-Oscillators. I

—— *Tractable Models without Frustration* ——

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A class of tractable models for population dynamics with random interactions is proposed which is of the form:

$$\theta_{n+1}^{(j)}=\theta_n^{(j)}+\Omega_j+\frac{1}{2\pi}\sum_{i\in I_j}\tilde{s}_i\tilde{s}_j\sin 2\pi(\theta_n^{(i)}-\theta_n^{(j)})$$

for $j=1, 2, \cdots, N(\gg 1)$, where the $\tilde{s}_i$'s are independent random parameters and I_j is an interacting neighbor of the site j. Theories of phase transitions to mutual synchronization are developed and verified numerically with particular emphasis on a new type of ordered phase characterized by a vanishing order parameter. The frustration effect is also discussed in the context of population dynamics.

§ 1. Introduction

Large assemblies of interacting self-oscillators exhibit a variety of interesting behavior which may be significant in many areas of science. In particular the onset of mutual synchronization in such assemblies or populations has been a subject of much interest to quite a few researchers in recent years.[1)~18)] From a physicist's point of view it is remarkably analogous to second-order phase transitions known to occur in diverse equilibrium systems such as magnets.[19)] Therefore, extensive investigations into the mutual synchronization and related problems in populations of self-oscillators are expected to open a new active area in the field of critical phenomena beyond the scope of conventional statistical mechanics.[20)]

Let us go into the subject in more detail. Suppose that we have a large population of interacting self-oscillators whose native frequencies ω_i are distributed over the population with a density, say, $f(\omega)$. Furthermore, we assume that the typical strength of interaction is adjustable with a single parameter ε. Then, as ε is increased from zero with $f(\omega)$ fixed, we ultimately reach a certain threshold ε_c beyond which a macroscopic number of oscillators are entrained to a common frequency, say, $\tilde{\omega}$, so that the whole population begins to exhibit rhythmic behavior. (By a "macroscopic" number we mean that of $O(N)$, where $N\gg 1$ is the population size.) So far this type of "phase transition" has been investigated numerically and analytically from various points of view using a number of interesting model systems.[1)~18)] For example, some papers are concerned with populations of oscillators under stochastic perturbations.[5),6),9),10),13),14)] While all these previous studies were carried out using systems of differential equations (i.e., continuous-time models), a certain type of discrete-time models have also been proposed and investigated recently.[17),18)]

The purpose of the present paper is to examine how random interactions between the constituent oscillators affect the dynamics of populations. For this purpose, we use discrete-time models of the following form:

$$\theta_{n+1}^{(j)} = \theta_n^{(j)} + \Omega_j + \frac{1}{2\pi} \sum_{i \in I_j} J_{ij} \sin 2\pi (\theta_n^{(i)} - \theta_n^{(j)}) \tag{1}$$

for $j=1, 2, \cdots, N$. In general oscillators are assumed to be placed on a "lattice", and $\theta^{(j)}$ is the phase of an oscillator on the jth site whose interacting neighborhood is denoted by I_j in (1). The parameters $\Omega_1, \cdots, \Omega_N$ are native frequencies (to be more exact, native winding numbers in the discrete-time case) distributed with the density $f(\Omega)$. It is on the interaction parameters J_{ij} that we focus our interest in this paper. They are random variables each obeying a certain distribution law which may most generally vary from one interacting pair (i, j) to another. Such a model is a particular case of a more general one proposed in Ref. 17) (Eq. (2) therein) and the case such that J_{ij} equals ε/N for any pair has already been studied.[17),18)] Note that in the model (1) each oscillator is allowed to possess only one degree of freedom, the phase. Such a simplification of population dynamics models was made perhaps most intensely by Kuramoto[3),5),10)] in the form of differential equations, and several interesting models have been proposed and investigated so far.[7),8),12)~15)]

Let us now explain what motivates us to consider the effect of random interactions in population dynamics. We have two motivations. The first one is very naive, but of practical importance. As discussed extensively by Winfree,[1),2)] biology is one of the disciplines of science to which population dynamics is most relevant. For instance, we may use a population of oscillators to model a living organ consisting of a large number of oscillatory elements (e.g., human small intestine). It would be reasonable to expect that any biological objects inevitably retain more or less random features. This randomness may exist not only in the properties of each element such as the native frequency, but also in the interactions between the elements. In previous models only the former type of randomness is taken into account as the distribution of native frequencies, but it would not be the case that such models are always realistic enough. Note that in the above we mean by randomness "quenched" disorder. Biological systems may also be exposed to time-dependent random perturbations whose effect has been already studied by some authors, as referred to previously.

The second motivation may sound more academic, but is quite exciting (at least to the author). As mentioned earlier, populations of self-oscillators are considerably analogous to magnets which are populations of spins. In statistical mechanics of spin systems it is known that random interactions can yield quite a new type of ordered phase called a "spin glass" phase in which spins are frozen in random directions with a vanishing total magnetization. Recent intensive and extensive studies[21),22)] triggered especially by Edwards and Anderson[23)] have shown that a spin glass phase as well as a transition to it are characterized by some remarkably new properties in comparison with familiar ferromagnetic phases and associated phase transitions in nonrandom magnets. A glance at such a new development in the area of spin systems may tempt us to a question as follows: Can random interactions give rise to a new type of phase analogous to a spin glass in population dynamics as well? If the answer is affirmative, we may term such a phase an *oscillator glass* which is expected to play some new role in the development of population dynamics.

With these motivations in mind we attempt a study of "phase transitions" in populations of randomly interacting self-oscillators using the model (1). As a first step in this direction we confine ourselves in this paper to a particular kind of random interactions as

$$J_{ij}=\tilde{s}_i\tilde{s}_j\,, \tag{2}$$

where $\tilde{s}_1, \cdots, \tilde{s}_N$ are random parameters governed by a common distribution law that are independent not only from one another but also from the Ω_j's. Let the number of interacting neighbors of one site be z. Then, it is convenient to put $\tilde{s}_i=(\varepsilon/z)^{1/2}s_i$ and deal with

$$\theta_{n+1}^{(j)}=\theta_n^{(j)}+\Omega_j+\frac{\varepsilon}{2\pi z}\sum_{i\in I_j}s_is_j\sin 2\pi(\theta_n^{(i)}-\theta_n^{(j)}) \tag{3}$$

for $1\leq j\leq N$, where we use ε as a control parameter while both the distribution of Ω_j, $f(\Omega)$, and that of s_j, $P(s)$, are fixed. The reason for the choice of (2) is simply that it enables us to treat the problem more or less analytically as demonstrated in §§ 2 and 3. We expect, however, that there may be some (e.g., biological) contexts in which interactions of the form (2) are of qualitative use. Unfortunately, random interactions of the type (2) do not produce a truly glass-like phase, though a certain type of ordered phase is generated by them which resembles a spin glass phase only superficially. Nevertheless, we believe that our model (3) works as useful starting point of the study of random population dynamics, and that it will play a considerable instructive role. (We remark that some preliminary results were reported in Ref. 16) on the effect of random interactions in a population of van der Pol oscillators.)

The content of this paper is as follows. In § 2 we discuss a particular case of $P(s)=p\delta(s-1)+(1-p)\delta(s+1)$ which reveals existence of a new type of ordered phase despite its simplicity. Then, in § 3, we proceed to the case of fully general $P(s)$, though the range of interaction is specified to be infinity, i.e., $I_j=\{1, 2, \cdots, N\}$ for any j. Moreover, we assume that the width of distribution of Ω_j is so small that we may invoke a differential-equation approximation for a set of "mean-field" maps. In this case we can derive self-consistent equations for a pair of order parameters whose solutions are compared in § 4 with results of numerical experiments. Section 5 is devoted to a summary of this work as well as discussion on the frustration effect, which our models (3) lack, in the context of population dynamics. Some remarks and further problems are also presented. Finally an appendix is given for § 3.

§ 2. Removable randomness and spurious glass-like phase

First of all we discuss the dynamics of the model (3) for a particular distribution of s as follows:

$$p(s)=p\delta(s-1)+(1-p)\delta(s+1)\,, \tag{4}$$

where δ is the Dirac function and p is a parameter such that $0\leq p\leq 1$. In the area of spin systems a Hamiltonian with this type of interactions is known as the Mattis model.[24] It is easy to check that a set of transformations:

$$\theta^{(j)} \to \phi^{(j)} = \theta^{(j)} - (s_j/4) \qquad (1 \le j \le N) \tag{5}$$

reduces the model (3) with (4) to

$$\phi_{n+1}^{(j)} = \phi_n^{(j)} + \Omega_j + \frac{\varepsilon}{2\pi z} \sum_{i \in I_j} \sin 2\pi(\phi_n^{(i)} - \phi_n^{(j)}) \quad (1 \le j \le N) \tag{6}$$

whose interactions are no longer random. Therefore, it turns out that the randomness of the model (3) with (4) is *removable.*

Let us then consider the behavior of order parameters for phase transitions to mutual synchronization. Needless to say, a most direct measure of mutual entrainment is the ratio of the number of entrained oscillators to N which we mean by R.[3),10)] An ordered phase may be defined by $R > 0$, so that it works as an order parameter. Obviously the two populations, (3) with (4) and (6), have this parameter in common. Another interesting order parameter introduced by Kuramoto[10)] is

$$Z_\theta = \lim_{n \to \infty} \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i'(\theta_n^{(j)} - n\tilde{\Omega})}, \tag{7}$$

where $i' = \sqrt{-1}$ and $\tilde{\Omega}$ is a "frequency" with which a macroscopic number of oscillators are entrained mutually for $\varepsilon > \varepsilon_c$. In the absence of randomness in interactions it is known analytically and numerically for several models[7),10),12)~15),17),18)] that $|Z_\theta| = 0$ for $\varepsilon < \varepsilon_c$ while $|Z_\theta| > 0$ for $\varepsilon > \varepsilon_c$. (Since Z_θ is complex in general, the order parameter is practically $|Z_\theta|$.) To see the effect of random interactions on Z_θ, we first note

$$\langle Z_\theta \rangle = i'(2p-1) Z_\phi ,$$

where $\langle\ \rangle$ means an average with respect to the s_j's and Z_ϕ is the order parameter for (6). We next obtain

$$\langle |Z_\theta - \langle Z_\theta \rangle|^2 \rangle = N^{-1} \langle | e^{i'\pi s/2} - \langle e^{i'\pi s/2} \rangle |^2 \rangle$$

which reveals that in the limit $N \to \infty$, Z_θ equals $\langle Z_\theta \rangle$ with probability unity, so that we may safely put

$$Z_\theta = i'(2p-1) Z_\phi \tag{8}$$

for $N \gg 1$. Now suppose that the model (6) exhibits a phase transition for $\varepsilon = \varepsilon_c$ characterized by the behavior of Z_ϕ as mentioned above. Then, by (8), it follows that the model (3) with (4) also goes into a mutually entrained phase at the same value of ε, ε_c, with the order parameter $|2p-1|$ times that of the model (6). Therefore, randomness causes a decrease in the order parameter. An example of (6) which enjoys a phase transition is in Refs. 17) and 18). It is expected that there are many other examples.

According to (8), the case of $p = 1/2$ is most interesting in which $P(s)$ is symmetric with respect to $s = 0$. In this special case the order parameter Z_θ remains to be zero even for $\varepsilon > \varepsilon_c$. This implies that the phases of entrained oscillators are "frozen" in random directions if we look at them sitting on a frame rotating with the angular velocity $\tilde{\Omega}$. Since Z_θ is an analog of magnetization in spin systems, the above new phase resembles a spin glass phase. Of course, this is so only superficially as will be discussed in § 5. Therefore, in this paper, we call such a phase a spurious glass-like phase (SGP) for convenience. In the next section we will encounter SGP's again for the case of fully general distribution of s_j, though instead some restrictions are introduced on I_j and $f(\Omega)$.

§ 3. Infinite-range interactions under a narrow symmetric distribution of native frequencies

In order to go further we employ infinite-range (or uniform) interactions in (3). Namely we discuss the onset of mutual synchronization in the following model:

$$\theta_{n+1}^{(j)}=\theta_n^{(j)}+\Omega_j+\frac{\varepsilon}{2\pi N}\sum_{i=1}^{N}s_is_j\sin 2\pi(\theta_n^{(i)}-\theta_n^{(j)}) \tag{9}$$

for $j=1, 2, \cdots, N$ where we do not assume any particular form of $P(s)$. Concerning $f(\Omega)$, however, we confine ourselves to the case such that it is symmetric in both sides of a certain $\hat{\Omega}$ with a small "width", which we denote by γ hereafter (γ may be identified with the standard deviation if it exists). The symmetry is assumed only for avoiding inessential complexity. A recipe will be given in the Appendix for the case without such a symmetry.

Let us start our arguments to construct a theory in a self-consistent way as done by Sakaguchi and Kuramoto for populations of continuous-time oscillators with nonrandom uniform interactions.[15] Our basic assumption is expressed by

$$\frac{1}{N}\sum_{j=1}^{N}s_je^{i'2\pi\theta_n^{(j)}}\rightarrow De^{i'2\pi(n\hat{\Omega}+\alpha)} \tag{10}$$

for $n\rightarrow\infty$ where both $D(\geq 0)$ and α are constants. Substituting this into (9) and introducing $\{\psi_n^{(j)}\}$ by

$$\psi_n^{(j)}=\theta_n^{(j)}-n\hat{\Omega}-\alpha \qquad (1\leq j\leq N), \tag{11}$$

we obtain a set of circle maps as follows:

$$\psi_{n+1}^{(j)}=\psi_n^{(j)}+\Delta_j-\frac{\varepsilon s_j}{2\pi}D\sin 2\pi\psi_n^{(j)} \tag{12}$$

for $1\leq j\leq N$ with $\Delta_j=\Omega_j-\hat{\Omega}$. Note that by (10) and (11)

$$\lim_{n\rightarrow\infty}\frac{1}{N}\sum_{j=1}^{N}s_je^{i'2\pi\psi_n^{(j)}}=D\,. \tag{13}$$

All what remains is to evaluate the l.h.s. of the above expression making use of the circle maps to reach a self-consistent equation of D. As will become clear later, D plays a role of an order parameter. For this purpose we first introduce an approximation to the circle maps (12) in which they are replaced by the following differential equations:

$$\frac{d\psi^{(j)}}{dn}=\Delta_j-\frac{\varepsilon s_j}{2\pi}D\sin 2\pi\psi^{(j)} \qquad (1\leq j\leq N). \tag{14}$$

This replacement is valid since Δ_j may be regarded as small by the assumption on $f(\Omega)$. Of course, the control parameter ε must be kept in a correspondingly small range, but this restriction conveys no difficulty for our purpose because ε_c should be $O(\gamma)$. By (14) it follows that the condition of entrainment is given by

$$|\Delta_j| \leq \frac{\varepsilon |s_j|}{2\pi} D \tag{15}$$

under which $\psi^{(j)}$ converges to one of stationary solutions for (14) satisfying the stability condition:

$$s_j \cos 2\pi\psi^{(j)} > 0 . \tag{16}$$

Let us now calculate the l.h.s. of (13). As will be shown in the Appendix, the equality:

$$\lim_{n\to\infty} \frac{1}{N} \sum_{j=1}^{N} s_j e^{i2\pi\phi_n^{(j)}} = \frac{1}{N} \Sigma_{\text{ent}} s_j \cos 2\pi\psi^{(j)} \tag{17}$$

holds, where Σ_{ent} means to sum up over the all of the entrained oscillators. Using (14)~(16) it is easy to find an expression for the r.h.s. of (17) which enables us to arrive at a self-consistent equation of D as follows:

$$D = \frac{\varepsilon D}{\pi} \int_{-\infty}^{\infty} ds P(s) s^2 \int_0^1 dx (1-x^2)^{1/2} f\left(\hat{\Omega} + \frac{\varepsilon D}{2\pi} |s| x\right) . \tag{18}$$

To see how the parameter D is linked to a phase transition to mutual synchronization, we then compute the order parameter R to obtain

$$R = \frac{\varepsilon}{\pi} D \int_{-\infty}^{\infty} ds P(s) |s| \int_0^1 dx f\left(\hat{\Omega} + \frac{\varepsilon D}{2\pi} |s| x\right) , \tag{19}$$

which indicates that the behavior of R is qualitatively the same as that of D. Namely, R is positive if and only if D is so. Therefore, we may use (18) to locate the threshold beyond which a state of macroscopic entrainment stably exists. Unfortunately we have no means to a priori determine the stability of solutions of (18), so that we simply assume a change of stability at the threshold from the trivial solution $D=0$ to a nontrivial one which grows *continuously* from zero. Some numerical results described in the next section show that this is indeed the case. Under the assumption we obtain

$$\varepsilon_c = 4 / \{\langle s^2 \rangle f(\hat{\Omega})\} . \tag{20}$$

We then proceed to consider the behavior of another order parameter Z_k defined by

$$Z_k = \lim_{n\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{i2\pi\phi_n^{(j)}} , \tag{21}$$

which is obviously equivalent to Z in (7) except a minor difference in phase. Let us denote its absolute value by K. What is the relationship between K and D? As shown in the Appendix, Z_k equals $N^{-1}\Sigma_{\text{ent}} \cos 2\pi\psi^{(j)}$, so that we have

$$K=\pm\frac{\varepsilon D}{\pi}\int_{-\infty}^{\infty}dsP(s)s\int_0^1 dx(1-x^2)^{1/2}f\left(\hat{\Omega}+\frac{\varepsilon D}{2\pi}|s|x\right), \tag{22}$$

where the sign of the r.h.s. should be chosen so as for K to be positive. Although (22) looks quite similar to (18), K and D are in general quantitatively very different from each other. This fact is most evident in a SGP. Namely, suppose that $P(s)$ is symmetric with respect to $s=0$. Then, K is zero irrespective of ε or D, so that in this particular case we have a transition to a SGP at $\varepsilon=\varepsilon_c$. In general K as well as D and R grow just beyond ε_c obeying the square root law if $f(\Omega)$ is maximal at $\hat{\Omega}$ with $f'(\hat{\Omega})=0$, as is easily checked.

One of interesting quantities other than the order parameters is a distribution function of the residual phases $\psi^{(j)}$ of entrained oscillators which we denote by $Q(\psi)$. It is useful especially for qualitatively distinguishing a SGP from non SGP's. By (14) and (16) we obtain

$$Q(\psi)=C|\cos 2\pi\psi|\int_0^{\infty}dxxP(hx)f\left(\hat{\Omega}+\frac{\varepsilon D}{2\pi}x\sin 2\pi\psi\right), \tag{23}$$

where $h=\mathrm{sgn}(\cos 2\pi\psi)$ and C is a normalization constant for the interval $0\leq\psi<1$. As this expression implies, the residual phase ψ is confined to the regions $0\leq\psi<1/4$ and $3/4<\psi<1$ if $P(s)=0$ for $s<0$, and to the range $1/4<\psi<3/4$ if $P(s)=0$ for $s>0$. (In fact this is a straightforward consequence of the stability condition (16).) Therefore, no SGP appears for such distributions of s as prohibiting a change of sign. As mentioned earlier, it appears for $\varepsilon>\varepsilon_c$ when $P(s)$ is symmetric at $s=0$. For such a $P(s)$, $Q(\psi)$ acquires periodicity of 1/2, which means that ψ finds itself in the range $1/4<\psi<3/4$ and the remainder with equal weights, in contrast with the above "ψ-confinement" cases. This amounts to $K=0$. Such a broad distribution of ψ over the whole interval is an important characteristic of a SGP.

§ 4. Numerical experiments

We are now ready to compare the theory developed in the foregoing section with results of direct numerical computations. For convenience we follow the evolution of $\tilde{\psi}_n^{(j)}=\theta_n^{(j)}-n\hat{\Omega}$ $(1\leq j\leq N)$, where the $\theta_n^{(j)}$'s are supposed to obey (9) with the Ω_j's given by

$$\Delta_j=\gamma\tan\left(\frac{j}{N}\pi-\frac{N+1}{2N}\pi\right)\qquad(1\leq j\leq N),$$

where $\Delta_j=\Omega_j-\hat{\Omega}$. We choose $N=10^3$ and $\gamma=10^{-3}$ expecting that they are respectively large and small enough to make the theory applicable. The above choice of the Δ_j's is for approximating the Lorentzian distribution $f(\Omega)=(\gamma/\pi)\{(\Omega-\hat{\Omega})^2+\gamma^2\}^{-1}$ for which $\varepsilon_c=4\pi\gamma/\langle s^2\rangle$ by (20). A random number generator was used to get a couple of samples of $s_1,\cdots,s_N$ obeying a Gaussian distribution law:

$$P(s)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(s-a)^2}{2\sigma^2}\right\}. \tag{24}$$

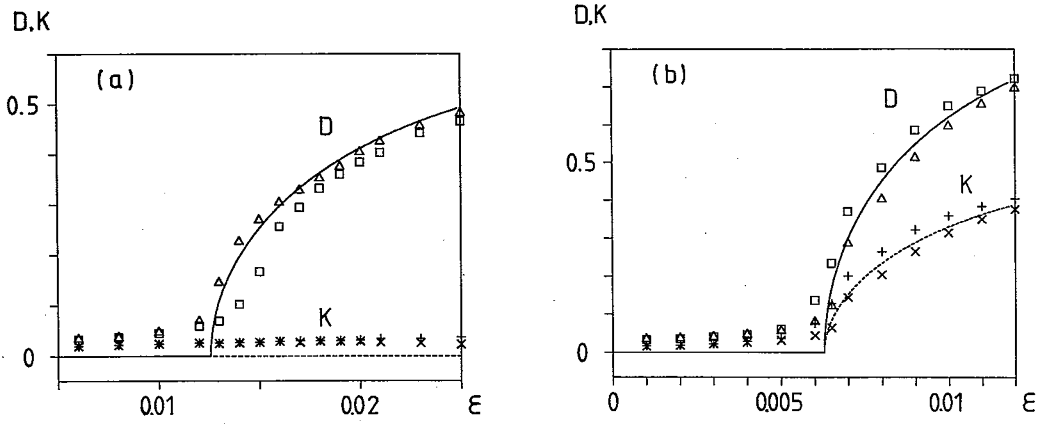

Fig. 1. The order parameters D, K vs ε for the model (9) with the Ω_j's and $P(s)$ given above and in (24), respectively. The real lines as well as the broken one are theoretical results for D and K based on (18) and (22). The experimental results, which were obtained by averaging upon 2^{17} iterations, are displayed for two runs with different samples of $\{s_j\}$, as $(D, K)=(\square, +)$ or $(\triangle, \times)$. The choice of parameters is as follows: $\gamma=10^{-3}$, $\sigma=1$ and $a=0$ ((a)), $=1$((b)).

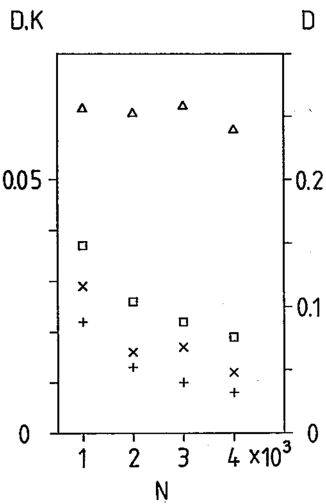

Fig. 2. The N-dependence of the order parameters D and K, where γ, a and σ are the same as in Fig. 1(a). The left ordinate is for $D(\varepsilon=0.008$: $\square)$ and $K(\varepsilon=0.008$: $+$; $\varepsilon=0.016$: $\times)$ while the right one is for $D(\varepsilon=0.016$: $\triangle)$. The initial condition of random-number generation for getting $\{s_j\}$ is equivalent to that for the $(\square, +)$ run in Fig. 1. Note that $\varepsilon=0.008<\varepsilon_c$ while $\varepsilon=0.016>\varepsilon_c$.

The initial condition used is $\tilde{\phi}_0^{(j)}=0$ $(1\leq j\leq N)$, but we confirmed in some test cases that a choice of another initial condition caused no significant change in results.

In Fig. 1 numerical results for the order parameters D and K are displayed in comparison with theoretical curves for them. The latter was obtained from (18) and (22) by numerical integration. The numerical results of D and K stand for long time averages of $|\sum_{j=1}^{N}s_j e^{i'2\pi\tilde{\phi}_n^{(j)}}/N|$ and $|\sum_{j=1}^{N}e^{i'2\pi\tilde{\phi}_n^{(j)}}/N|$, respectively, after an initial transient up to $n=999$. Figure 1(a) shows results for the case $a=0$ and $\sigma=1$. Since $P(s)$ is symmetric concerning $s=0$, the theory predicts the following: As ε is increased, both D and K remain to be zero until $\varepsilon=\varepsilon_c$, and then a phase transition takes place to a SGP in which $D>0$ while $K=0$. Numerical results are given in Fig. 1(a) for two different samples of $\{s_j\}$. Unfortunately the sample dependence of the results turns out not very small even for $N=10^3$, but it is seen that both results are in reasonable agreement with the theory. Although a more careful look at the figure indicates that D and K for $\varepsilon<\varepsilon_c$ as well as K for $\varepsilon>\varepsilon_c$ deviate slightly from zero, additional numerical results presented in

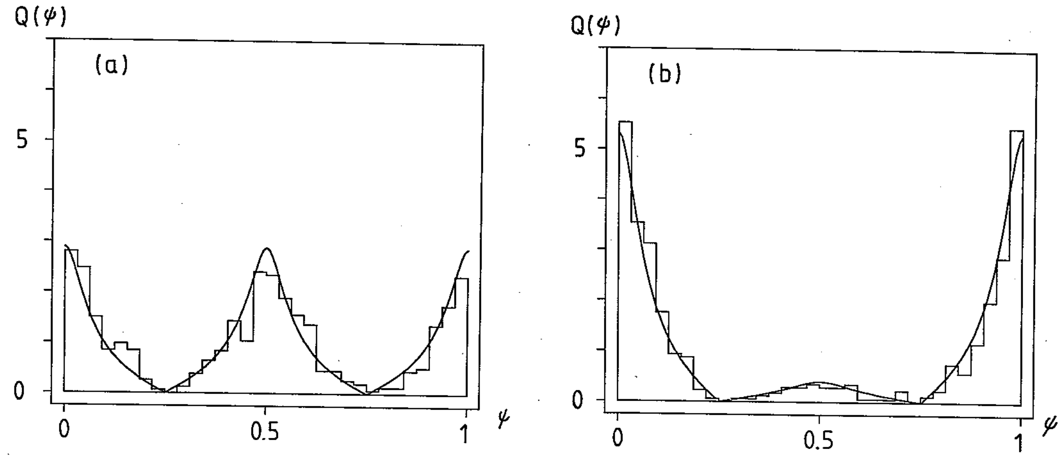

Fig. 3. The distribution of residual phases of entrained oscillators for $\varepsilon>\varepsilon_c$. Besides ε, which is 0.023106 in (a) and 0.011808 in (b), the parameters in (a) and (b) are the same as in Figs. 1(a) and (b), respectively. Theoretical results are drawn by real lines using (23). The histograms show numerical results for $N=10^3$ with the same $\{s_j\}$ as for the (□, +) run in Fig. 1. A criterion of entrainment adopted is that the winding number of ψ averaged upon 10^4 iterations is less than 10^{-4}.

Fig. 2 suggest that it is a finite size (N) effect. Namely, the values of D and K tend to rapidly decrease as N is increased, in the case that they should be zero theoretically. On the other hand, the value of D for $\varepsilon>\varepsilon_c$ is stable against such a change in N. The same effect was found previously in a population with nonrandom interactions.[17),18)] Figure 1(b) is used to display results for the case $a=\sigma=1$. As is in Fig. 1(a), the agreement between the theory and numerical outcomes is at a satisfactory level. In this case K grows beyond the threshold as D does, but randomness in interactions keeps them substantially different from each other. Let us proceed to Fig. 3 where the distribution function $Q(\psi)$ for residual phases of entrained oscillators manifests itself for $a=0$, $\sigma=1$ in (a) and $a=\sigma=1$ in (b) with a qualitatively excellent agreement between the theory and the experiment in each case. In the former the characteristic feature of a SGP should be noticed which is, as discussed in § 3, a highly extended distribution of ψ with equal weights in the range $1/4<\psi<3/4$ and the remainder. On the other hand, such a feature is no longer found in the result for a non SGP presented in Fig. 3(b), and instead we see a remarkably localized distribution.

§ 5. Summary, discussion and further problems

A class of models has been proposed in this paper for populations of self-oscillators whose interaction parameters J_{ij} are randomly chosen. There may be a variety of possibilities for making them random in the class of models expressed in (1), but we have confined ourselves to the case of (2), which has the advantage of analytical tractability. In § 2 we discussed phase transitions in models with probably the simplest interactions of the form (2), and found in particular a new type of ordered phase (SGP) in which $K=0$ despite $R>0$. Next, in § 3, we went a step forward by taking $P(s)$ fully generally (but only for infinite-range interactions and in the limit

$\gamma\to 0$) to develop a self-consistent-theory, a central result of which is the simultaneous equations for the order parameters D and K. The theory predicts emergence of a SGP for $P(s)$ symmetric with respect to $s=0$, and this was numerically verified in § 4 together with other theoretical predictions including the distribution function of residual phases of entrained oscillators.

Let us give a remark. We have found SGP's in § 2 as well as in § 3 when $P(s)$ is an even function. This is not mere coincidence. If $P(s)$ is symmetric concerning $s=0$, sgn(s) and $|s|$ are probabilistically independent from each other. Using this fact and transformations $\theta^{(j)}\to\phi^{(j)}=\theta^{(j)}-(\mathrm{sgn}(s_j)/4)$ in (3), we find

$$\langle|Z_\theta|^2\rangle=N^{-1}$$

which allows us to conclude $Z_\theta=0$ for $N\to\infty$. Therefore, if $P(s)$ is an even function, we may have only a SGP as an ordered phase in (3). We also remark that the theory developed in § 3 is virtually for models of differential equations obtained by replacing $\theta_{n+1}-\theta_n$ to $d\theta/dt$ in (9). Therefore, for the case of § 3, further investigations are necessary to look for any discrete-time effect by making γ much larger.

One of our motivations for the present work was to search for any new type of ordered phase analogous to a spin glass phase in disordered magnetic systems. Recall that our models exhibit, depending on $P(s)$ and the interaction strength ε, one of the following:

(i) a "paramagnetic" phase ($K=R=0$),
(ii) a "ferromagnetic" phase ($K\neq 0$, $R\neq 0$),
(iii) a SGP ($K=0$, $R\neq 0$).

Therefore, if we may identify R with the Edwards-Anderson order parameter[21)~23)] (this would be reasonable considering an analogy between an entrained (i.e., phase-locked) oscillator and a frozen spin), a SGP seems to be a complete counterpart of a spin glass phase as far as the behavior of the order parameters is concerned. This is so, however, only superficially since our models lack *frustration*[25)] which is regarded as one of important ingredients of a spin glass phase.[21),22)] Here we do not intend to give a detailed introduction on the frustration effect, which is now in vogue in the area of spin systems, but instead invoke a simple example useful for illustrating how it may work in the context of population dynamics. Take a system of interacting three oscillators as follows:

$$\theta_{n+1}^{(1)}=\theta_n^{(1)}+\hat{\Omega}+\frac{1}{2}J_1\sin 2\pi(\theta_n^{(2)}-\theta_n^{(1)})+\frac{1}{2}J_3\sin 2\pi(\theta_n^{(3)}-\theta_n^{(1)}),$$

$$\theta_{n+1}^{(2)}=\theta_n^{(2)}+\hat{\Omega}+\frac{1}{2}J_1\sin 2\pi(\theta_n^{(1)}-\theta_n^{(2)})+\frac{1}{2}J_2\sin 2\pi(\theta_n^{(3)}-\theta_n^{(2)}),$$

$$\theta_{n+1}^{(3)}=\theta_n^{(3)}+\hat{\Omega}+\frac{1}{2}J_3\sin 2\pi(\theta_n^{(1)}-\theta_n^{(3)})+\frac{1}{2}J_2\sin 2\pi(\theta_n^{(2)}-\theta_n^{(3)}), \quad (25)$$

where

$$J_i=J\tau_i \quad (J>0,\ \tau_i=1 \text{ or } -1)$$

for $i=1\sim 3$. We put $\Omega_i=\hat{\Omega}$ for all i because $\gamma=0$ corresponds to the zero absolute temperature in terms of a spin system for which the effect of frustration is most conspicuous. Before dealing with (25), let us recall the following: In the case of two oscillators as $\theta_{n+1}^{(1)}=\theta_n^{(1)}+\tilde{\Omega}+\tilde{J}\sin 2\pi(\theta_n^{(2)}-\theta_n^{(1)})$ and the same equation with 1 and 2 interchanged, the phase difference $\phi=\theta^{(1)}-\theta^{(2)}$ has two stationary values, 0 and 1/2, and the former is stable for $\tilde{J}>0$ with the latter unstable, and vice versa if $\tilde{J}<0$ (where $|\tilde{J}|$ is assumed not very large). Namely, the sign of $\tilde{J}$ determines whether the interaction is of a ferromagnetic type ($\tilde{J}>0$) or of an antiferromagnetic one ($\tilde{J}<0$).[10)] Then, as to (25), we may define the frustration function Ψ by

$$\Psi=(1-\tau_1\tau_2\tau_3)/2\,.$$

That is to say, $\Psi=1$ indicates presence of frustration, and $\Psi=0$ does absence of that. As in the case of two oscillators, we are concerned with the phase differences $\phi_1=\theta^{(1)}-\theta^{(3)}$ and $\phi_2=\theta^{(2)}-\theta^{(3)}$. It is easy to derive a pair of evolution equations of ϕ_1 and ϕ_2 from (25), whose stationary solutions (fixed points) are listed in Table I together with their stability for each case of $\{\tau_j\}$.*) It is evident that uniqueness of the stable solution is destroyed by frustration. More specifically, frustration brings about a multi-basin phenomenon in such a way that coexisting attractors possess an identical degree of stability (i.e., they have in common eigenvalues of the Jacobian matrices governing their stability), as is easily realized. This may be regarded as a counterpart of frustration-induced degeneracy of the ground state in a spin system.[25)] Of course, such an effect would become more and more prominent as the system size is increased, and in fully frustrated populations, phases deserving the term "oscillator glass" are expected to appear. Therefore, in a forthcoming paper, we will address ourselves to the study of the case such that in (1) the random parameters J_{ij} are

Table I. Frustration effect in the system of interacting three oscillators, (25). All the stationary solutions of the phase differences ϕ_1 and ϕ_2 (where we assume $-1/2\leq\phi_i<1/2$) are listed together with their stability (○: stable; ×: unstable) for each case of $\{\tau_j\}$. The frustration function Ψ is defined in the text. The stable solutions for the case $\tau_1=\tau_2=\tau_3=-1$ are known as the 120° structure in antiferromagnetic-XY-spin systems on a triangular lattice. (See, e.g., Ref. 26).)

$\tau_1=\tau_2=\tau_3=\tau$			$\tau_1=\tau_2=-\tau_3=\tau$		
ϕ_1, ϕ_2	$\tau=1$	$\tau=-1$	ϕ_1, ϕ_2	$\tau=1$	$\tau=-1$
0, 0	○	×	0, 0	×	×
0, −1/2	×	×	0, −1/2	×	○
−1/2, 0	×	×	−1/2, 0	×	×
−1/2, −1/2	×	×	−1/2, −1/2	×	×
1/3, −1/3	×	○	1/3, 1/6	○	×
−1/3, 1/3	×	○	−1/3, −1/6	○	×
Number of stable solutions	1	2		2	1
Ψ	0	1		1	0

*) The results in Table I are valid for $J<2/(3\pi)$ and equivalent to those for interacting three XY spins (see Ref. 10) for this connection with XY spins). For larger J, however, this equivalence breaks down by period-doubling instability of some of the stable solutions. (This does not happen in a differential-equation version of (25).)

independent of one another obeying the same distribution law. Being different from the frustration-free models (3), more or less frustration can be produced in such a case by adjusting the distribution. Does this really result in a new type of ordered phase and other interesting phenomena?

Appendix

Here we derive useful expressions of

$$Z_d=\lim_{n\to\infty}\frac{1}{N}\sum_{j=1}^{N}s_j e^{i'2\pi\phi_n^{(j)}} \tag{A·1}$$

and

$$Z_k=\lim_{n\to\infty}\frac{1}{N}\sum_{j=1}^{N}e^{i'2\pi\phi_n^{(j)}} \tag{A·2}$$

for general $f(\Omega)$ (not necessarily symmetric) and $P(s)$ where the $\phi_n^{(j)}$'s are assumed to obey (14) with $\Delta_j=\Omega_j-\tilde{\Omega}$. For general $f(\Omega)$, the frequency of entrainment, $\tilde{\Omega}$, must also be determined in a self-consistent way (replace $\hat{\Omega}$ in (10) and (11) by $\tilde{\Omega}$.) We denote $f(\tilde{\Omega}+\Delta)$ by $\tilde{f}(\Delta)$.

Let us start from Z_d. A contribution to it from entrained oscillators, $Z_d^{(e)}$, is given by

$$Z_d^{(e)}=\int_{-\infty}^{\infty}dsP(s)\int_{|\Delta|\le\varepsilon D|s|/2\pi}d\Delta\tilde{f}(\Delta)\left\{\left(s^2-\left(\frac{2\pi\Delta}{\varepsilon D}\right)^2\right)^{1/2}+i'\frac{2\pi\Delta}{\varepsilon D}\right\}. \tag{A·3}$$

On the other hand, nonentrained oscillators give a contribution to Z_d by

$$Z_d^{(n)}=\int_{-\infty}^{\infty}dsP(s)s\int_{|\Delta|>\varepsilon D|s|/2\pi}d\Delta\tilde{f}(\Delta)\int_0^1 d\phi u_{s,\Delta}(\phi)e^{i'2\pi\phi}, \tag{A·4}$$

where $u_{s,\Delta}(\phi)$ is a distribution function of ϕ obtained as a stationary solution of

$$\frac{\partial}{\partial n}u_{s,\Delta}(\phi,n)=-\frac{\partial}{\partial\phi}\left\{\left(\Delta-s\frac{\varepsilon D}{2\pi}\sin 2\pi\phi\right)u_{s,\Delta}(\phi,n)\right\},$$

i.e.,

$$u_{s,\Delta}(\phi)=C'\Big/\left|\Delta-s\frac{\varepsilon D}{2\pi}\sin 2\pi\phi\right| \tag{A·5}$$

with a normalization constant C'. By (A·4) and (A·5) we have

$$Z_d^{(n)}=i'\frac{2\pi}{\varepsilon D}\int_{-\infty}^{\infty}dsP(s)\int_{|\Delta|>\varepsilon D|s|/2\pi}d\Delta\tilde{f}(\Delta)\left\{\Delta-\mathrm{sgn}(\Delta)\left(\Delta^2-\left(\frac{\varepsilon D}{2\pi}s\right)^2\right)^{1/2}\right\}. \tag{A·6}$$

In this way we get an expression of $Z_d=Z_d^{(e)}+Z_d^{(n)}$ by summing up (A·3) and (A·6). Similarly we find

$$Z_k^{(e)}=\int_{-\infty}^{\infty}dsP(s)s^{-1}\int_{|\Delta|\le\varepsilon D|s|/2\pi}d\Delta\tilde{f}(\Delta)\left\{\left(s^2-\left(\frac{2\pi\Delta}{\varepsilon D}\right)^2\right)^{1/2}+i'\frac{2\pi\Delta}{\varepsilon D}\right\} \tag{A·7}$$

and

$$Z_k^{(n)}=i'\frac{2\pi}{\varepsilon D}\int_{-\infty}^{\infty}dsP(s)s^{-1}\int_{|\varDelta|>\varepsilon D|s|/2\pi}d\varDelta\tilde{f}(\varDelta)\left\{\varDelta-\mathrm{sgn}(\varDelta)\left(\varDelta^2-\left(\frac{\varepsilon Ds}{2\pi}\right)^2\right)^{1/2}\right\},\quad(\mathrm{A}\cdot 8)$$

from which we get an expression of $Z_k=Z_k^{(e)}+Z_k^{(n)}$. Then, by (13), we have simultaneous equations Re $Z_d=D$ and Im $Z_d=0$ which determine $\tilde{\Omega}$ and D, where Re and Im stand for a real part and an imaginary one, respectively. Let us now assume as we did in § 3 that $f(\Omega)$ is symmetric with respect to a certain value of Ω, $\hat{\Omega}$, and put $\tilde{\Omega}=\hat{\Omega}$. Then, Im $Z_d=0$ automatically holds together with Im$Z_k=0$ since $\tilde{f}(\varDelta)$ is an even function. We thus arrive at (18) and (22). The analysis presented above is an extension of Sakaguchi and Kuramoto's[15] to our populations with random interactions, (9) in the limit $\gamma\to 0$.

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