Population Games and Evolutionary Dynamics

1

William H. Sandholm

~

. . .

*

.....

i,

The MIT Press Cambridge, Massachusetts London, England

Contents in Brief

Series Foreword xvii Preface xix 1 Introduction 1 Ι **Population Games** 2 21 **Population Games** 3 Potential Games, Stable Games, and Supermodular Games 53 **Deterministic Evolutionary Dynamics** П 4 119 **Revision Protocols and Evolutionary Dynamics** 5 **Deterministic Dynamics: Families and Properties** 139 **Best Response and Projection Dynamics** 6 177 ш **Convergence and Nonconvergence of Deterministic Dynamics** 7 221 **Global Convergence of Evolutionary Dynamics** Local Stability under Evolutionary Dynamics 271 8 Nonconvergence of Evolutionary Dynamics 9 319 IV **Stochastic Evolutionary Models** 10 Stochastic Evolution and Deterministic Approximation -367 11 **Stationary Distributions and Infinite-Horizon Behavior** 397 12 Limiting Stationary Distributions and Stochastic Stability 451 References 541 Notation Index 565 575 Index

۲. <u>.</u> .

	Serie	es Forev	vord	xvii
	Pref	ace		xix
			<i>s</i> #	
1.	Intr	oductio	on	1
	1.1	Popul	ation Games	2
		1.1.1	Modeling Interactions in Large Populations	2
		1.1.2	Definitions and Classes of Population Games	5
	1.2	Evolu	tionary Dynamics	6
		1.2.1	Knowledge, Rationality, and Large Games	× 7
		1.2.2	Foundations for Evolutionary Dynamics	``, <i>8</i>
		1.2.3	Deterministic Evolutionary Dynamics	9
		1.2.4	Orders of Limits for Stochastic Evolutionary Models	11
		1.2.5	Stationary Distributions and Stochastic Stability	12
	1.3	Rema	rks on History, Motivation, and Interpretation	13
		Notes		15
I	Рор	ulation	n Games	
2	Рор	ulatior	n Games	21
	2.1	Popul	ation Games	22
		2.1.1	Populations, Strategies, and States	22
		2.1.2	Payoffs	23
		2.1.3	Best Responses and Nash Equilibria	23
		2.1.4	Prelude to Evolutionary Dynamics	24
	2.2	Exam	ples	25
		2.2.1	Matching in Normal Form Games	25
		2.2.2	Congestion Games	27
		2.2.3	Two Simple Externality Models	28
	2.3	The G	eometry of Population Games and Nash Equilibria	29
		2.3.1	Drawing Two-Strategy Games	29
		2.3.2	Displacement Vectors and Tangent Spaces	31

i,

.

	2.3.3	Orthogonal Projections	34
	2.3.4	Drawing Three-Strategy Games	36
	2.3.5	Tangent Cones and Normal Cones	38
	2.3.6	Normal Cones and Nash Equilibria	41
2.A	Apper	ndix: Affine Spaces, Tangent Spaces, and Projections	44
	2.A.1	Affine Spaces	44
	2.A.2	Affine Hulls of Convex Sets	45
	2.A.3	Orthogonal Projections	46
	2.A.4	The Moreau Decomposition Theorem	50
	Notes		51
Pote	ential G	ames, Stable Games, and Supermodular Games	53
3.1	Full Pc	otential Games	53
	3.1.1	Full Population Games	54
	3.1.2 *	Definition and Characterization	54
	3.1.3 -	Examples	· 55
	3.1.4	Nash Equilibria of Full Potential Games	58
	3.1.5	The Geometry of Nash Equilibrium in Full Potential Games	64
	3.1.6	Efficiency in Homogeneous Full Potential Games	66
	3.1.7	Inefficiency Bounds for Congestion Games	68
3.2	Potent	ial Games	71
	3.2.1	Motivating Examples	71
	3.2.2	Definition, Characterizations, and Examples	72
	3.2.3	Potential Games and Full Potential Games	76
	3.2.4	Passive Games and Constant Games	77
3.3	Stable	Games	⁻ 79
	3.3.1	Definition	79
	3.3.2	Examples	81
	3.3.3	Invasion	85
	3.3.4	Global Neutral Stability and Global Evolutionary Stability	87
	3.3.5	Nash Equilibrium and Global Neutral Stability in Stable Games	90
3.4	Superr	nodular Games	94
	3.4.1	Definition	94
	3.4.2	Examples	97
	3.4.3	Best Response Monotonicity in Supermodular Games	98
	3.4.4	Nash Equilibria of Supermodular Games	99
3.A	Apper	ndix: Multivariate Calculus	100
	3.A.1	Univariate Calculus	100
	3.A.2	The Derivative as a Linear Map	101
	3.A.3	Differentiation as a Linear Operation	103
	3.A.4	The Product Rule and the Chain Rule	103

3

		3.A.5	Homogeneity and Euler's Theorem	104
		3.A.6	Higher-Order Derivatives	105
		3.A.7	The Whitney Extension Theorem	106
		3.A.8	Vector Integration and the Fundamental Theorem of Calculus	107
		3.A.9	Potential Functions and Integrability	108
	3.B	Appen	dix: Affine Calculus	108
		3.B.1	Linear Forms and the Riesz Representation Theorem	109
		3.B.2	Dual Characterizations of Multiples of Linear Forms	110
		3.B.3	Derivatives of Functions on Affine Spaces	111
		3.B.4	Affine Integrability	113
		Notes		115
II	Det	erminis	tic Evolutionary Dynamics	
4	Rev	ision Pı	rotocols and Evolutionary Dynamics	119
	4.1	The Ba	sic Stochastic Evolutionary Model	120
		4.1.1	Inertia and Myopia	120
		4.1.2	Revision Protocols	121
	4.2	Mean I	Dynamics	122
		4.2.1	Derivation	122
		4.2.2	Target Protocols and Target Dynamics	124
	4.3	Examp	bles	125
		4.3.1	Imitative Protocols and Dynamics	126
		4.3.2	Direct Protocols and Dynamics	128
	4.4	Detern	hinistic Evolutionary Dynamics	129
	4.A	Apper	ndix: Ordinary Differential Equations	131
		4.A.1	Basic Definitions	131
		4.A.2	Existence, Uniqueness, and Continuity of Solutions	155
		4.A.3	Orainary Differential Equations on Compact Convex Sets	100
		Inotes		137
5	Det	erminis	tic Dynamics: Families and Properties	139
	5.1	Inform	ation Requirements for Revision Protocols	141
	5.2	Incenti	ives and Aggregate Behavior	144
	5.3	Familie	es of Evolutionary Dynamics	148
	5.4	Imitati	ve Dynamics	.153
		5.4.1	Definition	153
		5.4.2	Examples	154
		5.4.3	Biological Derivations of the Replicator Dynamic	158
		5.4.4	Extinction and Invariance	160
		5.4.5	Monotone Percentage Growth Rates and Positive Correlation	162
		5.4.6	Rest Points and Restricted Equilibria	164

J

	55	Excess Payoff Dynamics	165
	0.0	5.5.1 Definition and Interpretation	165
		5.5.1 Definition and Interpretation	167
	56	Pairwise Comparison Dynamics	160
	5.0	5.6.1 Definition	107
		5.6.2 Incentives and Aggregate Behavior	170
		5.6.2 Incentives una Aggregate Denavior	170
	57	Multiple Revision Protocols and Hybrid Dynamics	172
	5.7	Notes	175
	_		175
6	Best	Response and Projection Dynamics	177
	6.1	The Best Response Dynamic	178
		6.1.1 Definition and Examples	178
		6.1.2 Construction and Properties of Solution Trajectories	180
		6.1.3 * Incentive Properties	186
	6.2	Perturbed Best Response Dynamics	187
		6.2.1 <i>Revision Protocols and Mean Dynamics</i>	188
		6.2.2 <i>Perturbed Optimization: A Representation Theorem</i>	189
		6.2.3 Logit Choice and the Logit Dynamic	191
		6.2.4 Perturbed Incentive Properties via Virtual Payoffs	196
	6.3	The Projection Dynamic	198
		6.3.1 Definition	198
		6.3.2 Solution Trajectories	200
		6.3.3 Incentive Properties	202
		6.3.4 <i>Revision Protocols and Connections with the Replicator Dynamic</i>	202
	6.A	Appendix: Differential Inclusions	205
		6.A.1 Basic Theory	205
		6.A.2 Differential Equations Defined by Projections	207
	6.B	Appendix: The Legendre Transform	208
		6.B.1 Legendre Transforms of Functions on Open Intervals	208
		6.B.2 Legendre Transforms of Functions on Multidimensional Domains	210
	6.C	Appendix: Perturbed Optimization	212
		6.C.1 Proof of the Representation Theorem	212
		6.C.2 Additional Results	215
		Notes	216
III	Con	vergence and Nonconvergence of Deterministic Dynamics	
7	Glol	oal Convergence of Evolutionary Dynamics	221
	7.1	Potential Games	223
		7.1.1 Potential Functions as Lyapunov Functions	223
		7.1.2 Gradient Systems for Potential Games	228

8

7.2	Stable	Games	232
	7.2.1	The Projection and Replicator Dynamics in Strictly Stable Games	232
	7.2.2	Integrable Target Dynamics	235
	7.2.3	Impartial Pairwise Comparison Dynamics	246
	7.2.4	Summary	248
7.3	Super	modular Games	249
	7.3.1	The Best Response Dynamic in Two-Player Normal Form Games	251
	7.3.2	Stochastically Perturbed Best Response Dynamics	253
7.4	Domir	nance Solvable Games	257
	7.4.1	Dominated and Iteratively Dominated Strategies	258
	7.4.2	The Best Response Dynamic	258
	7.4.3	Imitative Dynamics	259
7.A	Appe	ndix: Limit and Stability Notions for Deterministic Dynamics	260
	7.A.1	ω-Limits and Notions of Recurrence	261
~	7.A.2	Stability of Sets of States	262
7.B	Apper	ndix: Stability Analysis via Lyapunov Functions	262
	7.B.1	Lyapunov Stable Sets	263
	7.B.2	ω-Limits and Attracting Sets	263
	7.B.3	Asymptotically Stable and Globally Asymptotically Stable Sets	265
7.C	Appe	ndix: Cooperative Differential Equations	<u>`</u> 266
	Notes		268
Loca	l Stabi	ility under Evolutionary Dynamics	271
8.1	Non-N	Nash Rest Points of Imitative Dynamics	272
8.2	Local	Stability in Potential Games	273
8.3	Evolu	tionarily Stable States	275
	8.3.1	Single-Population Games	276
	8.3.2	Multipopulation Games	280
	8.3.3	Regular Taylor ESS	281
8.4	Local	Stability via Lyapunov Functions	282
	8.4.1	The Replicator and Projection Dynamics	282
	8.4.2	Target and Pairwise Comparison Dynamics: Interior ESS	283
	8.4.3	Target and Pairwise Comparison Dynamics: Boundary ESS	285
8.5	Linear	rization of Imitative Dynamics	290
	8.5.1	The Replicator Dynamic	291
	8.5.2	General Imitative Dynamics	295
8.6	Linear	rization of Perturbed Best Response Dynamics	297
	8.6.1	Deterministically Perturbed Best Response Dynamics	297
	8.6.2	The Logit Dynamic	298
8.A	Appe	ndix: Matrix Analysis	299
	8.A.1	Rank and Invertibility	299

		8.A.2 Eigenvectors and Eigenvalues	300
		8.A.3 Similarity, (Block) Diagonalization, and the Spectral Theorem	302
		8.A.4 Symmetric Matrices	304
		8.A.5 The Real Jordan Canonical Form	305
		8.A.6 The Spectral Norm and Singular Values	306
		8.A.7 Hines's Lemma	307
	8.B	Appendix: Linear Differential Equations	308
		8.B.1 Examples	308
		8.B.2 Solutions	310
		8.B.3 Stability and Hyperbolicity	312
	8.C	Appendix: Linearization of Nonlinear Differential Equations	313
		Notes	315
9	Nor	convergence of Evolutionary Dynamics	319
	9.1	Conservative Properties of Evolutionary Dynamics	320
		9.1.1 Constants of Motion in Null Stable Games	320
		9.1.2 Preservation of Volume	324
	9.2	Games with Nonconvergent Evolutionary Dynamics	327
		9.2.1 Circulant Games	327
		9.2.2 Continuation of Attractors for Parameterized Games	330
		9.2.3 Mismatching Pennies	333
		9.2.4 The Hypnodisk Game	337
	9.3	Chaotic Evolutionary Dynamics	341
	9.4	Survival of Dominated Strategies	344
		9.4.1 A General Survival Theorem	347
		9.4.2 Examples and Discussion	351
	9.A	Appendix: Three Classical Theorems on Nonconvergent Dynamics	356
		9.A.1 Liouville's Theorem	356
		9.A.2 The Poincaré-Bendixson and Bendixson-Dulac Theorems	358
	9.B	Appendix: Attractors and Continuation	359
		9.B.1 Attractors and Repellors	359
		9.B.2 Continuation of Attractors	361
		Notes	362
w	Sto	hastic Evolutionary Models	
	5.0		

Stochastic Evolution and Deterministic Approximation				
10.1	The Sto	ochastic Evolutionary Process	368	
10.2	Finite-l	Horizon Deterministic Approximation	369	
	10.2.1	Kurtz's Theorem	370	
	10.2.2	Deterministic Approximation of the Stochastic Evolutionary Process	372	
	Stoc 10.1 10.2	Stochastic E 10.1 The Sto 10.2 Finite-J 10.2.1 10.2.2	Stochastic Evolution and Deterministic Approximation 10.1 The Stochastic Evolutionary Process 10.2 Finite-Horizon Deterministic Approximation 10.2.1 Kurtz's Theorem 10.2.2 Deterministic Approximation of the Stochastic Evolutionary Process	

.

	10.3	Extens	ions	376
		10.3.1	Discrete-Time Models	376
		10.3.2	Finite-Population Adjustments	377
	10.A	Appen	dix: The Exponential and Poisson Distributions	378
		10.A.1	Basic Properties	378
		10.A.2	The Poisson Limit Theorem	381
	10.B	Appen	dix: Probability Models and Their Interpretation	382
		10.B.1	Countable Probability Models	383
		10.B.2	Uncountable Probability Models and Measure Theory	384
		10.B.3	Distributional Properties and Sample Path Properties	386
	10.C	Appen	dix: Countable State Markov Chains and Processes	389
		10.C.1	Countable State Markov Chains	389
		10.C.2	Countable State Markov Processes: Definition and Construction	391
		10.C.3	"Countable State Markov Processes: Transition Probabilities	393
		Notes	<u>بد</u>	395
11	Stati	onary	Distributions and Infinite-Horizon Behavior	397
	11.1	Irreduc	cible Evolutionary Processes	399
		11.1.1	Full Support Revision Protocols	399
		11.1.2	Stationary Distributions and Infinite-Horizon Behavior	. 401
		11.1.3	Reversibility	402
	11.2	Station	ary Distributions for Two-Strategy Games	403
		11.2.1	Birth and Death Processes	403
		11.2.2	The Stationary Distribution of the Evolutionary Process	405
		11.2.3	Examples	407
	11.3	Waiting	g Times and Infinite-Horizon Prediction	412
		11.3.1	Examples	412
		11.3.2	Discussion	415
	11.4	Model	Adjustments for Finite Populations	417
		11.4.1	Finite-Population Games	418
		11.4.2	Clever Payoff Evaluation	419
		11.4.3	Committed Agents and Imitative Protocols	421
	11.5	Potenti	al Games and Exponential Protocols	423
		11.5.1	Finite-Population Potential Games	424
		11.5.2	Exponential Revision Protocols	428
		11.5.3	Reversibility and Stationary Distributions	430
	11.A	Appen	dix: Long-Run Behavior of Markov Chains and Processes	434
		11.A.1	Communication, Recurrence, and Irreducibility	434
		11.A.2	Periodicity	436
		11.A.3	Hitting Times and Hitting Probabilities	437
		11.A.4	The Perron-Frobenius Theorem	440

		11.A.5	Stationary Distributions for Markov Chains	. 441
		11.A.6	Reversible Markov Chains	442
		11.A.7	Stationary Distributions and Reversibility for Markov Processes	444
		11.A.8	Convergence in Distribution	445
		11.A.9	Ergodicity	448
		Notes		449
12	Limi	iting St	ationary Distributions and Stochastic Stability	451
	12.1	Definiti	ions of Stochastic Stability	454
		12.1.1	Small Noise Limits	454
		12.1.2	Large Population Limits	455
		12.1.3	Double Limits	458
		12.1.4	Double Limits: A Counterexample	460
	12.2	Expone	ential Protocols and Potential Games	463
		12.2.1 *	Direct Exponential Protocols: The Small Noise Limit	464
		12.2.2	Direct Exponential Protocols: The Large Population Limit	465
		12.2.3	Direct Exponential Protocols: Double Limits	472
		12.2.4	Imitative Exponential Protocols with Committed Agents	472
	12.3	Noisy I	Best Response Protocols in Two-Strategy Games	474
		12.3.1	Noisy Best Response Protocols and Their Cost Functions	474
		12.3.2	The Small Noise Limit	477
		12.3.3	The Large Population Limit	479
	-	12.3.4	Double Limits	480
		12.3.5	Stochastic Stability: Examples	482
		12.3.6	Risk Dominance, Stochastic Dominance, and Stochastic Stability	486
	12.4	Imitativ	ve Protocols in Two-Strategy Games	491
		12.4.1	Imitative Protocols with Mutations	491
		12.4.2	Imitative Protocols with Committed Agents	494
		12.4.3	Imitative Protocols, Mean Dynamics, and Stochastic Stability	496
	12.5	Small N	Joise Limits	499
		12.5.1	Noisy Best Response Protocols and Cost Functions	500
		12.5.2	Limiting Stationary Distributions via Trees	502
		12.5.3	Two-Strategy Games and Risk Dominance	504
		12.5.4	The Radius-Coradius Theorem	510
		12.5.5	Half-Dominance	512
	12.6	Large F	Population Limits	515
		12.6.1	Convergence to Recurrent States of the Mean Dynamic	515
		12.6.2	Convergence to Stable Rest Points of the Mean Dynamic	517
	1 2 .A	Appen	dix: Trees, Escape Paths, and Stochastic Stability	519
		12.A.1	The Markov Chain Tree Theorem	520
		12.A.2	Limiting Stationary Distributions via Trees	521

	12.A.3	Limiting Stationary Distributions via Trees on Recurrent Classes	524
	12.A.4	Radius-Coradius Theorems	526
	12.A.5	Lenient Transition Costs and Weak Stochastic Stability	529
12.B	Append	lix: Stochastic Approximation Theory	531
	12.B.1	Convergence to the Birkhoff Center	532
	12.B.2	Sufficient Conditions for Convergence to Stable Rest Points	533
	Notes		537
References			541
Nota	tion Inde	X	565

Index

*

1

575