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in the Neoclassical Growth Model**

**by
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Population, Impartiality and Sustainability in the Neoclassical Growth Model

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Abstract

A resolution is offered to Koopmans' (1965, 1967a) "paradox of the indefinitely postponed splurge" – i.e. the incompatibility of undiscounted utilitarianism and population weighting in the context of the infinite-horizon neoclassical growth model with exponential population growth. The resolution builds on the conflict between splurging (i.e. dissaving) and sustainability. Consumption paths which contain splurges are not sustained, because they involve reductions in consumption at some point. Thus disallowing unsustained paths removes the incentive to save for a splurge.

A modified utilitarian objective is presented which embodies the commitment to sustainability, as well as impartiality and the golden rule. Maximization over the neoclassical technology yields a monotonically increasing path to the golden rule. The underlying ethical position is described as an intergenerational contract: early generations are willing to sacrifice some consumption to build up the capital stock while future generations are morally obligated to limit consumption to the golden rule.

(JEL D6, D9, E6; keywords: population weighting, impartiality, utilitarianism, neoclassical growth model)

Résumé

On propose une solution au problème d'optimisation d'une fonction de bien-être utilitariste avec un taux d'escompte nul et une pondération temporelle par le niveau de population dans le contexte du modèle de croissance néoclassique. Ce problème fut d'abord soulevé par Koopmans (1965, 1967a). La solution se base sur le conflit entre la tendance à augmenter la consommation au prix d'une réduction du capital – dite la désépargne – et la durabilité. Interdire les sentiers de consommation qui incluent des épisodes de désépargne élimine la tendance à épargner pour de tels épisodes, ce qui est à la source du problème.

On présente une fonction utilitariste modifiée qui répond aux critères de durabilité, d'équité et de la règle d'or. La maximisation de cette fonction sous contrainte de la technologie néoclassique donne une solution monotone qui mène à la règle d'or. Cette fonction peut être assimilée à un contrat intergénérationnel : les générations présentes consentent à sacrifier de la consommation pour augmenter le stock de capital pourvu que les générations futures se limitent à consommer au niveau de la règle d'or.

(JEL D6, D9, E6; mots clés : pondération par le niveau de population, équité, utilitarisme, modèle de croissance néoclassique)

I. Introduction

In one of the most influential papers in modern economics, Koopmans (1965) sought to delineate the scope for ethical considerations in the choice of an optimal growth path under an infinite planning horizon. Employing a utilitarian objective, Koopmans identified time discounting of utility and the possibility of weighting utility by population size as the major points of ethical concern. Most people, including many economists, believe that discounting the utilities of future generations by a positive rate of social time preference is unethical because it discriminates against those generations *a priori*. In addition, many find it compelling to weight by population size when utility is defined in terms of a representative individual enjoying average consumption. This practice ensures impartiality among individuals, as each individual receives equal weight in the objective. In contrast, the alternative practice of not weighting by population means that individuals in more populous generations receive less weight than those in less populous generations, as each generation receives an equal weight.

The challenge of finding an optimal growth path under an undiscounted utilitarian objective and infinite horizon was rigorously treated for the first time by Ramsey (1928). The problem is that the undiscounted objective does not converge over an infinite horizon for many consumption paths. Ramsey sought to address this problem with the device of minimizing aggregate deviations from bliss, defined as the maximum attainable utility level. Koopmans (1965) generalized this device by substituting the golden rule for bliss, where the golden rule is defined as the maximum level of utility which can be sustained indefinitely by the given technology. Simultaneously von Weizsacker (1965) provided an alternative solution to the convergence problem in the form of the overtaking criterion, which redefined optimality in terms of the infinite sequence of finite-horizon problems.

The results of these experiments were mixed. In particular, Koopmans (1965) showed that, given constant-returns-to-scale technology, substitutability of capital and labour, concave utility, and constant exponential population growth, an optimal growth path exists only if utility is not weighted by population. Thus the twin ethical objectives of zero utility discounting and population weighting appear incompatible.

The reason for this negative result is the strength of the incentive for saving under population weighting. Not only does saving yield a stream of returns from the increased capital stock, but it also affords the opportunity for a splurge of consumption from dissaving at some future date. The more people partake in the dissaving, the greater the payoff in terms of aggregate utility. But with constant exponential population growth, any population size can be arranged simply by delaying the time of the splurge. In the interim, the planner must maintain a rate of investment at least as great as the rate of population growth in order to maintain or increase the per capita level of the stock. This situation yields two contradictory effects. On the one hand, the payoff of the splurge, and therefore the incentive to save for it, can be made arbitrarily large by arranging for an arbitrarily large population to enjoy it. On the other hand, the splurge is never taken, since it always pays to delay it further. In the limit, everything is saved, a phenomenon which Koopmans calls “the paradox of the indefinitely postponed splurge” (1967a, p.8).

A separate problem is that, under an infinite horizon, the undiscounted utilitarian objective yields a quasi-ordering; i.e. the objective cannot rank all feasible growth paths notwithstanding the Ramsey-Koopmans device or von Weizsacker’s overtaking criterion. Koopmans obtains the mitigating result, in the case without population weighting, that any path which cannot be ranked is infinitely worse than any path which can be ranked. Thus it is still possible to obtain an optimal path in this case. Nonetheless, most economists agree that it is

desirable to represent social preferences with a complete ordering, so that all feasible paths can be ranked.

In light of this problem, and in light of the proven existence of a solution without population weighting, the literature has focused on the challenge of finding a complete, ethical ordering *in the absence of population growth*. Earlier papers, particularly Koopmans (1960) and Diamond (1965), argued that such an ordering does not exist. More recently, Svensson (1980) and Fleurbaey and Michel (2003) have proven that an ordering does exist but the proofs rely on non-constructive elements such as the axiom of choice and the concept of free ultrafilters. As a consequence, the ordering cannot be represented explicitly.

In contrast, the problem of an undiscounted optimum with population weighting has not come up again in the literature, no doubt due to a widespread belief that Koopmans' (1965) treatment was definitive. At present, most economists apparently regard this dilemma as unfortunate but unavoidable, as in the statement by Koopmans that “ethical principles ... need mathematical screening to determine whether in given circumstances they are capable of implementation” (1967b, p. 125). Nonetheless, the concept of impartiality remains compelling, and therefore this topic must be regarded as an important challenge for research.

In a related development, Chichilnisky (1996) has proposed an ordering which balances the interests of present and future generations through a requirement that neither play a dictatorial role. This ordering is represented by a weighted objective composed of discounted utilitarianism and long-run average utility. This approach has the indisputable merits of completeness and balance between present and future. However, it does not satisfy impartiality between either individuals or generations and therefore it is not entirely satisfactory.

The present paper approaches the problem of an undiscounted optimum with population weighting by noting the key role of sustainability. For this purpose, a sustainable path is defined

as one in which consumption is non-decreasing over time. Splurging results in a path which cycles around the golden rule, which is not sustainable. This observation points to a potential solution. Sustainability is regarded by many today as a compelling normative principle. The fact that the conventional utilitarian objective with population weighting yields solutions which do not satisfy sustainability indicates that it does not reflect everything that is valued. The present paper proposes a revised utilitarian objective which (i) embodies an explicit preference for sustainability, (ii) makes the golden rule an explicit target for policy, and (iii) determines an optimal transition path to the golden rule. The ethical position underlying the revised objective is characterized as an intergenerational contract, in which early generations are willing to sacrifice some of their consumption in order to build up the capital stock while later generations are obligated to claim no more than the golden rule. This contractual ethic replaces the conventional Pareto principle: more is not always better under the revised objective.

The paper is organized as follows. Sections II and III introduce notation for the familiar neoclassical technology and for the undiscounted utilitarian objective with and without population weighting. Section III then presents an alternative proof of Koopman's incompatibility result. Section IV demonstrates the importance of sustainability. When growth paths are pre-screened for sustainability, a solution is obtained under population weighting. Section V characterizes splurging in terms of the phenomenon of cycling around the golden rule. Section VI brings together the three ethical precepts of the intergenerational contract – impartiality, golden rule and sustainability – in the revised objective. Optimization under this constraint is shown to yield the same solution as optimization of the standard objective with a sustainability constraint. Finally, section VII shows how the introduction of a parameter for inequality aversion can be used to address concerns that early generations may be called upon to shoulder too great a burden of saving for the well-being of future generations.

II. Neoclassical Technology

The model, which follows Koopmans (1965), is cast in continuous time. All quantities are real valued. At time $t \geq 0$ there are $N(t)$ people alive, each enjoying the per capita level of consumption $c(t)$, which yields instantaneous utility according to the function $u(c(t))$. This function is assumed to be strictly concave, continuously differentiable, and bounded above, with $u(0) = 0$, $u'(c) > 0$ and $\lim_{c \rightarrow 0} u'(c) = \infty$ (Inada condition).¹

Population dynamics are summarized by the equation $N(t) = e^{nt}$, such that initial population is normalized to unity.

The productive inputs are capital $K(t)$ and labour. For simplicity, labour is assumed equal to the population $N(t)$. Output is produced according to a production function, $F(K, N)$, which is assumed to be strictly concave, continuously differentiable and linearly homogeneous. Each input is essential to production and exhibits positive but diminishing marginal product; i.e.

$$F(0, N) = 0 \quad F(K, 0) = 0$$

$$F_K > 0 \quad F_{KK} < 0$$

$$F_N > 0 \quad F_{NN} < 0$$

where the subscript notation has the standard interpretation as a partial derivative.

At each moment in time, output is divided between aggregate consumption $C(t)$ and investment, which gives rise to a law of motion for the capital stock

$$\dot{K}(t) = F(K(t), N(t)) - C(t) \tag{1}$$

where the dot notation is used to indicate the time derivative. Capital is assumed not to depreciate.

¹ Rather than $u(0) = 0$ and the Inada condition, Koopmans assumes $\lim_{c \rightarrow 0} u(c) = -\infty$.

Per-capita variables are indicated by lowercase; i.e. $k \equiv K/N$ and $c \equiv C/N$. Since F is linearly homogeneous, per capita output is defined

$$f(k) \equiv F\left(\frac{K}{N}, 1\right) = \frac{F(K, N)}{N}.$$

It is a simple matter to convert the law of motion (1) to per-capita form:

$$\dot{k}(t) = f(k(t)) - nk(t) - c(t). \quad (2)$$

The golden rule, associated with Phelps (1961) and others, is defined as the highest level of consumption (alternatively highest level of utility) which can be sustained indefinitely, i.e. in a steady state. From (2), the steady state is characterized by

$$c(t) = f(k(t)) - nk(t).$$

Choosing k to maximize the right-hand side of this expression yields golden rule capital, k^* ; i.e.

$$k^* \text{ solves the first-order condition } f'(k^*) - n = 0.$$

III. Population Weighting in the Social Objective

Koopmans (1965) considers an undiscounted utilitarian objective with and without population weighting, as in respectively

$$W_N(T) = \int_0^T e^{nt} u(c(t)) dt \quad \text{and} \quad W(T) = \int_0^T u(c(t)) dt,$$

where T represents a finite planning horizon.

The absence of time discounting reflects impartiality. But the absence of time discounting is not enough; the planner must also decide whether impartiality is a matter of neutrality among population units, as in W_N , or among time periods, as in W . The equal weighting of time periods in W implies a diminishing weighting of population units, as total population grows over time.

Since a benevolent planner cares more about people than time, W_N is regarded as more compelling.

The discussion of impartiality would be more natural in discrete time, with integer-valued N . In that case, population units would correspond with individuals and time periods with generations. The argument is that impartiality among individuals is more compelling than impartiality among generations. In social choice theory, this argument is based upon the concept of a permutation – a simple rearrangement of allocations among individuals. The principle of anonymity requires that an alternative be considered socially indifferent to a permutation of itself.²

Impartiality is also linked with the choice of planning horizon. Assuming society will last indefinitely, the choice of an infinite time horizon is most appropriate, as impartiality entails that the interests of all individuals should be taken into account. In this light, the objectives become respectively

$$W_N(\infty) = \int_0^{\infty} e^{nt} u(c(t)) dt \quad \text{and} \quad W(\infty) = \int_0^{\infty} u(c(t)) dt .$$

But of course, these forms diverge for many consumption paths. The Ramsey-Koopmans device partially solves this problem by substituting deviations from a value u^* , representing bliss (Ramsey) or the golden rule (Koopmans). The objectives are now

$$W_N^*(\infty) = \int_0^{\infty} e^{nt} [u(c(t)) - u^*] dt \quad W^*(\infty) = \int_0^{\infty} [u(c(t)) - u^*] dt .$$

² Fleurbaey and Michel (2003) provide a discussion of the compatibility of standard axioms with different types of permutations.

$W_N^*(\infty)$ and $W^*(\infty)$ provide quasi-orderings of the set of feasible consumption paths rather than orderings.³ Clearly, the ability to rank a consumption path depends on the convergence of the objective. In the case of $W^*(\infty)$, Koopmans (1965) shows that any divergent path diverges to $-\infty$ (Proposition B, i.e. $\lim_{T \rightarrow \infty} W^*(T) = -\infty$) and therefore any convergent path (i.e. a path for which $\lim_{T \rightarrow \infty} W^*(T)$ exists) is preferred to any divergent path. Furthermore, an optimal path exists. In the special case of $k(0) = k^*$, the constant golden-rule path (k^*, c^*) is optimal, yielding u^* in each period. In the more general case of $k(0) \neq k^*$, the golden rule emerges as a saddle point of the optimization problem, and the optimal path coincides with the stable branch leading to a steady state at the golden rule (Proposition C). Cass (1965) arrives at the same conclusion independently, and Dutta (1991) generalizes this result to other economic environments.

In contrast, Koopmans shows that an optimal path does not exist under $W_N^*(\infty)$ for the neoclassical technology. Given any feasible path, society can do better by reducing consumption and saving more in every period. The form of the proof (Proposition K) is to show that $W_N^*(\infty)$ is not bounded from above on the set of feasible consumption paths. The result can also be demonstrated by the limiting behaviour of the solution to the related finite-horizon optimization problem. This approach will prove useful in later sections, and so it will be presented here.

Under finite T , $W_N(T)$ can be used instead of $W_N^*(T)$, since u^* is a constant. The planner's problem is to maximize $W_N(T)$ subject to (2) and suitable terminal conditions; i.e.

$$\max_{\{c(t)\}_0^T} \int_0^T e^{nt} u(c(t)) dt \quad \text{subject to:}$$

³ A quasi-ordering is defined as a reflexive and transitive set of binary relations on the set of feasible consumption paths. An ordering is a complete quasi-ordering.

$$(i) \quad \dot{k}(t) = f(k(t)) - nk(t) - c(t) \quad (P1)$$

$$(ii) \quad k(0), T \text{ given}$$

$$(iii) \quad k(T) \geq 0.$$

The problem is amenable to solution by optimal control theory, with c as the control variable, k as the state, and a co-state variable denoted λ . The corresponding Hamiltonian function is

$$H = e^{nt}u(c) + \lambda[f(k) - nk - c]$$

where the time subscripts have been suppressed. Pontryagin's Maximum Principle yields necessary conditions which, in the present context, are also sufficient:

$$(i) \quad \max_{\{c\}} H$$

$$(ii) \quad \dot{\lambda} = -\frac{\partial H}{\partial k} \Rightarrow \dot{\lambda} = -\lambda[f'(k) - n] \quad (3)$$

$$(iii) \quad \dot{k} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{k} = f(k) - nk - c$$

$$(iv) \quad \lambda(T)k(T) = 0$$

A solution of (3) will be an interior solution with respect to the control c , since H is non-linear and continuously differentiable, utility is bounded, and c is unconstrained above.

Therefore, condition (3.i) can be replaced with

$$\frac{\partial H}{\partial c} = 0 \Rightarrow \lambda = u'(c)e^{nt}. \quad (3.i')$$

Condition (3.iv) is a transversality condition associated with the free terminal stock $k(T)$. But since we know by (3.i') that $\lambda(T)$ must be positive (assuming non-satiation), it follows that $k(T) = 0$ at a solution.

Combining (3.i') and (3.ii) yields a differential equation in c :

$$\dot{c} = \sigma(c)f'(k)c \quad (4)$$

where $\sigma(c) \equiv -u'(c)/u''(c)c > 0$ is the instantaneous elasticity of substitution of $u(c)$. Condition (3.iii) just gives back the law of motion of capital from (2).

Taken together, (2) and (4) give a dynamic system in the variables k and c , which can be analyzed with the help of the phase diagram shown in Figure 1. The demarcation for $\dot{k} = 0$ is given by the equation $c = f(k) - nk$, derived from (2). This demarcation exhibits the standard properties of the neoclassical growth model: namely it intersects the origin, rises to a maximum at the golden rule level of capital, k^* , and then declines to the horizontal axis at \bar{k} . k^* is implicitly defined by the relationship $f'(k) = n$, as shown in section II, while \bar{k} is defined by $f(k) = nk$. \bar{k} is the maximum amount of capital that can be accumulated starting from an initial value $k(0) < \bar{k}$. Below the demarcation, k is increasing, while above the demarcation k is decreasing.

Inspection of (4) reveals that the only possibility for a demarcation corresponding with $\dot{c} = 0$ would be if the production function exhibited capital satiation; i.e. if there existed a value \tilde{k} such that $f'(\tilde{k}) = 0$. In the remainder of the paper, it is assumed that, if such a value exists, it is greater than \bar{k} , the maximum attainable capital level. It follows then that there is no demarcation corresponding with $\dot{c} = 0$ in the feasible region defined by $k \leq \bar{k}$. Therefore $\dot{c} > 0$ according to (4) for any non-zero c and feasible k .

The possibility of a trajectory solving (2) and (4) along the horizontal axis, i.e. with $c = \dot{c} = 0$, depends upon the limiting behaviour of $\sigma(c)c$. If $\lim_{c \rightarrow 0} \sigma(c)c = 0$, then (4) is defined and such a trajectory exists. Such is the case for utility functions that exhibit constant elasticity of substitution, i.e. $\sigma(c) = \sigma \forall c$. Note that this trajectory leads to a steady state at $(\bar{k}, 0)$, since it intersects with the $\dot{k} = 0$ demarcation at this point. Note also that this trajectory would never be a

solution to (P1), owing to the Inada condition and owing to the failure of the transversality condition along this path. Nonetheless, it constitutes a limit of the set of feasible trajectories which solve (2) and (4). If $\lim_{c \rightarrow 0} \sigma(c)c$ is undefined, then this trajectory is not a solution of (2) and (4). However, it is still a solution of (2) and it still constitutes a limit of the set of feasible trajectories for the system. In both cases, this trajectory will be referred to as the null trajectory.

The phase diagram is divided into two regions, above and below the $\dot{k} = 0$ demarcation. The directional arrows indicate the tendency of trajectories in the corresponding regions. Any non-null trajectory that starts below the demarcation must pass through it, moving first in a northeasterly direction, becoming vertical at the point of intersection, and then switching to a northwesterly direction, as indicated by the arrows. Each non-null trajectory reaches the vertical axis ($k = 0$) in finite time.⁴

The fact that each non-null trajectory reaches the vertical axis in finite time means that it satisfies the transversality condition and therefore is the solution to (P1) when T is set equal to that time. For convenience, denote the solution as $S(T) \equiv (k(t), c(t))_{t=0}^T$, or more compactly as $(k_T(t), c_T(t))$. Strict concavity of $u(\cdot)$ and $f(\cdot)$ ensure that the solution is unique.

The concept of a policy is introduced to distinguish between solutions of the optimization problem (P1) and more general paths. Denoted $P(T) \equiv (k(t), c(t))_{t=0}^T$, a policy is consistent with the law of motion (2) but may or may not be consistent with condition (4). If it does satisfy (4), then it is equivalent to the solution $S(T)$. The null trajectory is denoted P_0 .

⁴ Proof: From (2), derive $\ddot{k}(t) = [f'(k(t)) - n]\dot{k}(t) - \dot{c}(t)$. Near the vertical axis, both \dot{k} and \ddot{k} are negative. Thus, capital is diminishing and the rate of diminution is increasing. It follows that the stock of capital is exhausted in finite time.

Three trajectories have been sketched in Figure 1 as examples, starting from an initial capital stock $k(0) < k^*$. The non-crossing property entails that a reduction in initial consumption, $c(0)$, corresponds with higher terminal consumption, $c(T)$, and higher maximum capital, attained just as the trajectory intersects the demarcation. A reduction in $c(0)$ also results in an increase in the time necessary to reach the vertical axis. Thus, trajectory 1 corresponds with longer elapsed time than trajectory 2, which in turn corresponds with longer elapsed time than trajectory 3.

The last point is essential to the main result of this section, and therefore it is stated formally now. The proof relies on the concept of an isochrone, which is discussed at length in the appendix.

Lemma 1: Given $T, \hat{T} > 0$, $c_T(0) < c_{\hat{T}}(0)$ if and only if $T > \hat{T}$.

Proof: Follows from the vertically increasing property of the isochrone map (see the appendix) and the non-crossing property of solutions. ■

The following theorem presents the main result.

Theorem 1:

$$(A) \lim_{T \rightarrow \infty} S(T) = P_0$$

$$(B) \text{ for all } T < \infty, c_T(0) > 0.$$

Proof: (A) Lemma 1.

$$(B) \text{ Inada condition, i.e. } \lim_{c \rightarrow 0} u'(c) = \infty. \blacksquare$$

The theorem establishes the asymptotic approach of $S(T)$ to the null trajectory. In the limit, everything is saved, leading society to the degenerate steady state at $(\bar{k}, 0)$.

The source of the problem is the absence of a $\dot{c} = 0$ demarcation for (P1). In contrast, in the case without population weighting (optimization of $W^*(\infty)$), equation (4) becomes

$$\dot{c} = \sigma(c)[f'(k) - n]c$$

which yields a $\dot{c} = 0$ demarcation corresponding with the condition $f'(k) = n$; i.e. a vertical line at the golden-rule level of capital. To the left of this demarcation, consumption is rising while to the right it declines. In this case, the intersection of the $\dot{k} = 0$ and $\dot{c} = 0$ demarcations yields a steady state at the golden rule, (k^*, c^*) . Further, there is a unique stable branch leading from the initial state $k(0)$ to the steady state. The steady state has the saddle point property, such that the only way to get there is to get on the stable branch at the outset. This stable-branch path is optimal for $T = \infty$.

Koopmans' paradox also explains the result of Marini and Scaramozzino (2000). These authors define an optimal level of the social rate of time preference as that which induces the system to converge to the golden rule over an infinite horizon. Clearly, this is only possible under population weighting if the social objective exhibits time discounting sufficient to offset the growth of population and productivity.⁵ This case is equivalent to maximizing $W^*(\infty)$ in the present paper. As discussed, this choice is not impartial with respect to individuals, and therefore it is less compelling. Marini and Scaramozzino couch their discussion in terms of impartiality among generations without commenting on the treatment of individuals.

IV. Sustainability and the Golden Rule

Koopmans' Paradox does not in any way compromise the existence of the golden rule in the neoclassical economy. Given the appeal of this allocation, one is led to inquire whether there

⁵ Marini and Scaramozzino include labour-augmenting productivity growth in their model, whereas the present paper does not.

is not some coherent and ethically compelling way of getting there, starting from the exogenously given initial value of capital $k(0)$. The present section provides an affirmative answer under the condition of sustainability.

The discussion begins with the general concept of a policy, introduced in the previous section. Some additional vocabulary will prove useful. A finite (infinite) policy is defined over a finite (infinite) horizon. A golden rule policy is an infinite policy which either starts at the golden rule allocation, (k^*, c^*) , or follows a transition path to it and then remains there indefinitely.

A policy may be stationary, monotonic, or cyclical in terms of $k(t)$. A stationary policy starts on the $\dot{k} = 0$ demarcation and stays there. A monotonic policy starts at a non-stationary point $(k(0), c(0))$, follows a transition path to the $\dot{k} = 0$ demarcation, then stays there. A cyclical policy passes through the demarcation, which results in a change of direction ($\dot{k}(t)$ changes sign). In order to cycle again, the policy must cross the demarcation again; otherwise it comes to rest on the demarcation or proceeds to capital exhaustion ($k = 0$).

A cyclical policy can be illustrated on trajectory 1 in Figure 1. Starting at point α , the policy follows the trajectory through β , continues to γ , jumps back down to β , then resumes its path along the trajectory. This pattern could repeat or the policy could proceed in some other fashion. Policies may be discontinuous, as in the present example which jumps from one side of the demarcation to the other, or continuous. Note that the policy just described does not represent a solution to the optimization problem (P1).

A sustainable policy is defined as an infinite policy with non-decreasing consumption (alternatively utility) over time.⁶ Stationary policies are sustainable by definition. Cyclical

⁶ Pezzey (1997) refers to this definition as a sustained policy, reserving the term sustainable for policies in which consumption is always no greater than a level which could be sustained indefinitely from that period forward. In this

policies violate sustainability, as they entail fluctuations in consumption. Some monotonic policies are sustainable. The following lemma yields a useful result for monotonic policies.

Lemma 2: Given an initial state $k(0) < k^*$, a sustainable monotonic policy (i.) starts below the $\dot{k} = 0$ demarcation, and (ii.) is increasing in k on the transition path.

Proof: (i.) By contradiction. A starting point above the demarcation is characterized by $\dot{k} < 0$. Sustainability entails non-decreasing consumption which entails the policy remains above the demarcation and reaches capital exhaustion in finite time. (ii.) $\dot{k} > 0$ below the demarcation. ■

The desirability of policies will be assessed with von Weizsacker's (1965) overtaking criterion, based on the finite objective $W_N(T)$. Consider two policies P and \hat{P} which correspond with consumption paths $c(t)$ and $\hat{c}(t)$ respectively. P is said to overtake \hat{P} if there exists some T^* such that $\int_0^T e^{nt} u(c(t)) dt > \int_0^T e^{nt} u(\hat{c}(t)) dt$ for all $T > T^*$. P is said to be overtaking optimal if it overtakes all other feasible policies.

The appeal of the golden rule seems fundamental to most people. The impartiality embodied in $W_N(T)$ also seems appealing. Yet the failure of (P1) to yield a golden-rule solution suggests an incompatibility between the ethical content of $W_N(T)$ and that of the golden rule. The following result indicates that this incompatibility is removed when candidate policies are pre-screened for sustainability.

Lemma 3: Among sustainable policies, any golden-rule policy overtakes any non-golden rule policy.

view, sustainedness is a stronger concept than sustainability, since the latter is a necessary but not sufficient condition for the former.

Proof: Trivial for stationary policies. Cyclical policies violate sustainability. For monotonic policies, consider separately the cases $k(0) < k^*$ and $k(0) > k^*$.

For $k(0) < k^*$, Lemma 2 indicates that monotonic policies start below the $\dot{k} = 0$ demarcation and are increasing in k on the transition path. Consider two such policies:

$$P = \begin{cases} (k(t), c(t)) & 0 \leq t < T \\ (k^*, c^*) & t \geq T \end{cases}$$

and

$$\hat{P} = \begin{cases} (\hat{k}(t), \hat{c}(t)) & 0 \leq t < \hat{T} \\ (\hat{k}(\hat{T}), \hat{c}(\hat{T})) & t \geq \hat{T} \end{cases}$$

P is a golden-rule policy; i.e. the economy follows an arbitrary increasing path $(k(t), c(t))$, that obeys the law of motion (2), until it reaches the golden rule, (k^*, c^*) , at time T and then remains there. \hat{P} is a non-golden-rule policy. It follows the arbitrary path $(\hat{k}(t), \hat{c}(t))$, increasing in k and non-decreasing in c , that obeys the law of motion (2), until it reaches the $\dot{k} = 0$ demarcation at time \hat{T} and point $(\hat{k}(\hat{T}), \hat{c}(\hat{T}))$, and then remains there. By definition $c^* > \hat{c}(\hat{T})$.

Consider the case with $T > \hat{T}$. P has a W_N value over $[0, T]$ of $\int_0^T e^{nt} u(c(t)) dt$. Over the same interval, \hat{P} has a W_P value of $\int_0^{\hat{T}} e^{nt} u(\hat{c}(t)) dt + \int_{\hat{T}}^T e^{nt} u(\hat{c}(\hat{T})) dt$. Let Δ represent the difference between these two values:

$$\Delta = \int_0^{\hat{T}} e^{nt} [u(c(t)) - u(\hat{c}(t))] dt + \int_{\hat{T}}^T e^{nt} [u(c(t)) - u(\hat{c}(\hat{T}))] dt < \infty$$

As indicated Δ is finite, since T is finite. Now consider the difference on the interval (T, T') , where T' is an arbitrary value greater than T . In particular, the limit of this difference is

$$\lim_{T' \rightarrow \infty} \int_T^{T'} e^{nt} [u(c^*) - u(\hat{c}(\hat{T}))] dt = \infty. \quad (5)$$

Thus whatever the finite value Δ , it is overwhelmed by the expression in (5). It follows, then, that

there exists a value T' such that $\int_T^{T'} e^{nt} [u(c^*) - u(\hat{c}(\hat{T}))] dt = \Delta$, and therefore the golden rule policy

P overtakes the non-golden rule policy \hat{P} according to W_N . The case with $\hat{T} > T$ follows in a similar fashion.

For $k(0) > k^*$, consider the policy

$$P = \begin{cases} (k(t), c^*) & 0 \leq t < T \\ (k^*, c^*) & t \geq T \end{cases}$$

which consists of setting consumption to c^* at the outset. The capital stock declines from its initial high level, reaching k^* at T . Since c^* represents the highest level of consumption that can be sustained indefinitely, P overtakes all other sustainable policies. ■

This result follows directly from the observation that the overtaking criterion ranks policies with different long-run values of c in order of these values.

The contrast between Koopmans' Paradox and Lemma 3 suggests that, as a prescription for policy choices, the golden rule entails a commitment to sustainability which is lacking in $W_N(T)$ on its own. It is thus natural to inquire whether pre-screening for sustainability might prove useful in determining an optimal transition path based on $W_N(T)$. To this end, consider the problem:

$$\max_{\{c(t)\}_0^T} \int_0^T e^{nt} u(c(t)) dt \quad \text{subject to:}$$

$$(i) \quad \dot{k}(t) = f(k(t)) - nk(t) - c(t)$$

$$(ii) \quad k(0), T \text{ given} \quad (P2)$$

$$(iii) \quad k(T) = k^*$$

$$(iv) \quad c(t) \leq c^*$$

This problem is identical to (P1) with the exception of the transversality condition (iii) and the inequality constraint (iv). The new transversality constraint fixes terminal capital at the golden-rule level, k^* . The inequality constraint provides a parsimonious way of imposing sustainability, as it rules out solutions which get to the golden rule in a cyclical fashion.⁷

The inequality constraint (iv) introduces the possibility of a corner solution with respect to $c(t)$. Thus the Maximum Principle becomes

$$\frac{\partial H}{\partial c} = u'(c)e^{nt} - \lambda \geq 0 \quad \text{and} \quad c \leq c^* \quad \text{with complementary slackness},^8$$

plus conditions (3.ii) and (3.iii).

Solutions will be discussed separately for $k(0) < k^*$ and $k(0) > k^*$. In both cases, a minimum time horizon is necessary for a solution, given the law of motion (2), the constraints on consumption ($0 \leq c \leq c^*$), and the amount of capital which must be accumulated or dissipated, i.e. $|k^* - k(0)|$. This is a minor issue, since we are interested in the limiting behaviour of the solutions as T is increased. Thus, it is assumed henceforth that T is sufficiently large for a solution.

For $k(0) > k^*$, the problem requires dissipating capital down to the golden rule level, k^* . This objective is compatible with choosing the maximum permitted consumption, c^* , at the outset, which represents a corner solution of (P2). Let T^* represent the duration of the transition,

⁷ The definition of sustainability is $\dot{c}(t) \geq 0$. It is convenient to use the weaker condition $c(t) \leq c^*$ in (P2) since it is more tractable mathematically and it yields a solution path which satisfies the definition.

⁸ A non-negativity constraint has not been imposed explicitly for c .

defined as the time to reach k^* (which in this case is also the minimum horizon necessary for a solution). For $T > T^*$, the solution is denoted

$$S^*(T) = \begin{cases} (k(t), c^*) & 0 \leq t < T^* \\ (k^*, c^*) & T^* \leq t \leq T \end{cases}$$

The solution is represented in Figure 1 by the horizontal line segment $\delta\theta$, leading leftward from $(k(0), c^*)$ at δ to the golden rule (k^*, c^*) at θ . The limit is denoted $S^*(\infty)$. An interior solution is also feasible but, since it is characterized by lower consumption during the transition, it is dominated by $S^*(T)$.

The more relevant case occurs for $k(0) < k^*$. When the inequality constraint (P2.iv) is non-binding over the whole planning interval, an interior solution is obtained, characterized by equations (2) and (4), plus the terminal conditions $k(T) = k^*$ and $c(T) \leq c^*$. The special case

$c(T) = c^*$ is obtained for a particular value of the horizon, T^* ; i.e. $\frac{\partial H(T^*)}{\partial c} = 0$ and $c(T^*) = c^*$

simultaneously. For $T < T^*$, terminal consumption is strictly less than the golden rule; i.e.

$c(T) < c^*$. Such a case is illustrated in Figure 1 by the segment $\alpha\beta$ of trajectory 1. As T

approaches T^* , the solution shifts upward in the diagram until it is contiguous with segment $\phi\theta$ on trajectory 2.⁹

When $T > T^*$, the inequality constraint binds on the sub-interval $(T^*, T]$ but not on $[0, T^*]$. In this case, a piece-wise solution is obtained:

$$S^*(T) = \begin{cases} (k^*(t), c^*(t)) & 0 \leq t < T^* \\ (k^*, c^*) & T^* \leq t \leq T \end{cases}$$

⁹ This result follows from the vertically increasing property of isochrones, discussed in the Appendix.

The limiting solution is denoted $S^*(\infty)$. Graphically, this solution consists of the segment $\phi\theta$ on trajectory 2 in Figure 1 followed by the golden rule.

The following optimality result is immediate.

Lemma 4: Among sustainable golden-rule policies, $S^*(\infty)$ is overtaking optimal according to $W_N(T)$.

Proof: $S^*(T)$ is the solution to (P2) for any $T > T^*$. ■

A more general optimality result follows.

Theorem 2: Among sustainable policies, $S^*(\infty)$ is overtaking optimal according to $W_N(T)$.

Proof: Follows from Lemmas 3 and 4. ■

V. The Problem of Cycling

The previous section demonstrated the optimality of the monotonic golden-rule policy $S^*(\infty)$, obtained under the assumption of sustainability. The present section demonstrates the importance of sustainability in suppressing cycling. If permitted, cyclical golden-rule policies dominate $S^*(\infty)$, leading to Koopmans' Paradox.

To demonstrate, consider general notation for a golden-rule policy

$$S^T(\infty) = \begin{cases} (k(t), c(t)) & 0 \leq t < T \\ (k^*, c^*) & t \geq T \end{cases}$$

where $(k(t), c(t))$ is the transition path, T is the transition time, and the horizon is infinite.¹⁰ This notation is consistent with the continuous policy $S^*(\infty)$, for which $T = T^*$, and it is also consistent with discontinuous policies which jump to the golden rule at some other value of T .

The transition path and time are generated by the general golden-rule optimization problem:

$$\max_{\{c(t)\}_0^T} \int_0^T u(c(t)) e^{nt} dt \quad \text{subject to:}$$

$$\begin{aligned} \text{(i)} \quad & \dot{k}(t) = f(k(t)) - nk(t) - c(t) \\ \text{(ii)} \quad & k(0), T \text{ given} \\ \text{(iii)} \quad & k(T) = k^* \end{aligned} \tag{P3}$$

which differs from (P1) in the transversality condition and from (P2) in the absence of the sustainability constraint.

For simplicity, this section focuses on the case of $k(0) < k^*$. When $T < T^*$, the problem is identical to (P2), since the sustainability constraint is not binding in that case. The transition path attains k^* but falls short of c^* ; i.e. $c(T) < c^*$. Such a path is illustrated in Figure 1 by the segment $\alpha\beta$ on trajectory 1. Having attained k^* , the policy then jumps to the golden rule (k^*, c^*) and stays there.

When $T = T^*$, the solution is identical to $S^*(\infty)$.

When $T > T^*$, $S^T(\infty)$ is a cyclical policy as illustrated in Figure 1 by the segment $\alpha\beta\gamma$ on trajectory 1.¹¹ At T , the policy jumps down from γ to the golden rule, located at θ , and then stays

¹⁰ In contrast, in the previous section T was used to denote the horizon.

¹¹ Proof: Isochrones are vertically increasing in k - c space.

there. Clearly this policy violates sustainability, since it entails a reduction in consumption between points γ and θ .

It will be useful to compare two policies for which the transition times are arbitrarily close, i.e. $S^T(\infty)$ and $S^{\hat{T}}(\infty)$ where $\hat{T} = T + \Delta T$ and $\Delta T > 0$. The following proposition shows the effect of a small increase in the transition time.

Theorem 3: $S^{\hat{T}}(\infty)$ overtakes $S^T(\infty)$, for any feasible T , according to $W_N(T)$.

Proof: Denote the maximized W_N value for (P3) as $V(T)$; i.e. $V(T) \equiv \max_{\{c(t)\}_0^T} \int_0^T e^{nt} u(c(t)) dt$. The

effect on $V(T)$ of increasing the transition time by ΔT can be approximated with a second-order Taylor series; i.e.

$$V(\hat{T}) - V(T) = V'(T)\Delta T + \frac{1}{2}V''(T)(\Delta T)^2.$$

Following Seierstad (1984), $V'(T) = H(T)$, which in the present context yields

$$V'(T) = e^{nT} [u(c(T)) + u'(c(T))[f(k^*) - nk^* - c(T)]] . \quad (6)$$

It follows from the definition of the golden rule that $f(k^*) - nk^* = c^*$. Therefore, (6) can be re-written as

$$V'(T) = e^{nT} [u(c(T)) + u'(c(T))[c^* - c(T)]] .$$

When $T > T^*$, $c(T) > c^*$, since the solution is cyclical in this case. Thus

$$\begin{aligned} V'(T) &= e^{nT} [u(c(T)) - u'(c(T))[c(T) - c^*]] \\ &= e^{nT} \left[u(c^*) + \int_{c^*}^{c(T)} [u'(v) - u'(c(T))] dv \right] \end{aligned}$$

This expression is strictly positive by virtue of the concavity of $u(\cdot)$.

The second derivative is

$$\begin{aligned}
V''(T) &= H'(T) \\
&= n H(T) + e^{nT} u''(\cdot) \frac{dc(T)}{dT} [c^* - c(T)]
\end{aligned} \tag{7}$$

When $T > T^*$, this expression is also positive, since $u'' < 0$, $\frac{dc(T)}{dT} > 0$ (vertically increasing isochrones), and $c(T) > c^*$. Since both $V'(T)$ and $V''(T)$ are positive in this case, it follows that $V(\hat{T}) - V(T)$ is positive.

This result also holds for T^* . In this case $V'(T^*) = e^{nT^*} u(c^*) > 0$ and

$$V''(T^*) = n e^{nT^*} u(c^*) > 0.$$

When $T < T^*$, $c(T) < c^*$ and therefore

$$V'(T) = e^{nT} \left[u(c^*) + \int_{c(T)}^{c^*} [u'(c(T)) - u'(v)] dv \right]$$

which is strictly positive due to the concavity of $u(\cdot)$. The second derivative is unchanged from (7) but now the second term is negative, which makes the sign of the whole expression unclear. Nonetheless, $V(\hat{T}) - V(T)$ must still be positive for small ΔT owing to the dominance of the first-order effect.

The fact that $V(\hat{T}) - V(T)$ is positive for any T indicates that $V(T)$ can always be increased by increasing the transition time. Now increasing the transition time will also delay arrival at the golden rule. This delay corresponds with a loss of welfare approximately equal to $e^{nT} u(c^*) \Delta T$. Thus the total effect on welfare of delaying the transition time is

$$\begin{aligned}
\Delta W &= V(\hat{T}) - V(T) - e^{nT} u(c^*) \Delta T \\
&= V'(T) \Delta T + \frac{1}{2} V''(T) (\Delta T)^2 - e^{nT} u(c^*) \Delta T
\end{aligned}$$

When $T > T^*$, substitution for $V'(T)$ and simplification yield

$$\Delta W = e^{nT} \left\{ \int_{c^*}^{c(T)} [u'(v) - u'(c(T))] dv \right\} \Delta T + \frac{1}{2} V''(T) (\Delta T)^2$$

which is positive.

When $T = T^*$, the integral term vanishes, leaving only $\frac{1}{2} V''(T) (\Delta T)^2$, which is also positive. The importance of the second-order approximation is now apparent: with a first-order approximation, ΔW becomes zero in the present context, which is misleading.

When $T < T^*$, the expression becomes

$$\Delta W = e^{nT} \left\{ \int_{c(T)}^{c^*} [u'(c(T)) - u'(v)] dv \right\} \Delta T + \frac{1}{2} V''(T) (\Delta T)^2.$$

The integral term is positive, as discussed above, while the second term may be positive or negative. Lemma 4 implies that $\Delta W > 0$ in this case, which circumvents the difficulty of the indeterminacy of the second term.

The positive value of ΔW indicates that society always benefits from delaying the transition time. ■

This result implies that any cyclical golden-rule policy overtakes $S^*(\infty)$, which suggests that the planner should prefer a policy of this type. However, it also follows that there does not exist an optimal policy of this type: whatever the transition date, it will always be desirable to increase it. In the limit, as $T \rightarrow \infty$, $S^T(\infty)$ degenerates to the null policy, which is Koopmans' Paradox.

VI. Sustainability and Social Preferences

In Koopmans (1965), the ethical content of social preferences is embodied in the utilitarian objective function, in particular the concept of impartiality. There is no commitment a

priori to the golden rule as a prescription for policy. Rather, the golden rule enters as a *deus ex machina* to retrieve convergence of the utilitarian objective function by means of the Ramsey-Koopmans device (i.e. converting $W(\infty)$ into $W^*(\infty)$). But this device is not necessary, since, as observed in section III, the solution to the problem without population weighting can be obtained as the limiting case of the solution of $W(T)$. Nonetheless, the solution converges to the golden rule, which is regarded as an agreeable result.

In contrast, in section IV above, the golden rule is given prominence as an explicit prescription for policy. The impartiality ethics of the utilitarian objective are then invoked to find a transitional path to the golden rule. Moreover, it is shown that these two loci of ethical content are rendered perfectly consistent when policies are pre-screened for sustainability. To be specific, under the sustainability constraint, (i.) golden rule policies overtake non-golden rule policies according to the utilitarian objective, and (ii.) an optimal golden rule policy, including transitional path, can be obtained as the solution of a utilitarian maximization problem.

These results are encouraging, as sustainability commands wide support as an ethical principle in itself. However, the implementation of sustainability as a side constraint is controversial, for, if the principle has merit, why would it not be embodied in the social objective function? Pezzey (1997) and Dasgupta (1994) provide opposing views on this question. More generally, the investment of ethical content in a variety of different instruments – golden rule transversality condition, utilitarian objective, sustainability constraint – seems to indicate that we have not adequately refined the social objective to reflect what is truly valued. In effect, the utilitarian objective is not quite right, and we are patching it up to avoid disagreeable solutions.

The present paper proposes a refined version of the social objective which reflects all desired values without patching up. In particular, consider the following variation on the population weighted utilitarian objective:

$$\hat{W}_N^*(\infty) = -\int_0^{\infty} e^{nt} |u(c(t)) - u^*| dt.$$

Like $W_N^*(\infty)$, $\hat{W}_N^*(\infty)$ represents a quasi-ordering on the set of feasible consumption paths. However, unlike $W_N^*(\infty)$, $\hat{W}_N^*(\infty)$ is bounded above, a result analogous to Koopmans' (1965) Proposition A. Inspection reveals that the upper bound is zero, which corresponds with the constant golden rule path c^* . Furthermore, every divergent path diverges to $-\infty$, by construction, a result analogous to Koopmans' Proposition B. Therefore, any convergent path is preferred to any divergent path, and one can focus on the subset of convergent paths to find an optimum.¹²

Maximization of \hat{W}_N^* subject to the neoclassical technology yields a sustainable, monotonic, golden-rule solution without requiring a golden-rule transversality condition or a sustainability side constraint. To demonstrate, consider the associated finite horizon problem:

$$\max_{\{c(t)\}_0^T} -\int_0^T e^{nt} |u(c(t)) - u^*| dt \quad \text{subject to:}$$

- (i) $\dot{k}(t) = f(k(t)) - nk(t) - c(t)$ (P4)
- (ii) $k(0), T$ given
- (iii) $k(T) \geq 0$.

This problem differs from (P1) in the objective only and from (P2) in the objective, the free terminal stock (iii) and the absence of the sustainability side constraint. The problem is amenable to solution by optimal control theory. The Hamiltonian function is

¹² Note also that a necessary condition for a path to converge (i.e. a path for which $\lim_{T \rightarrow \infty} \hat{W}_N^*(T)$ exists) is that $\lim_{T \rightarrow \infty} c(t) = c^*$, which reproduces Koopmans' Lemma 3 (Appendix, p.271).

$$H = -e^{nt} |u(c) - u^*| + \lambda [f(k) - nk - c]$$

and the Maximum Principle once again yields necessary conditions based on (3).

The Hamiltonian is non-differentiable at c^* . If this is an optimal value, then the solution is characterized by conditions (3.ii) and (3.iii) – the latter equivalent to equation (2) – but not equation (4). For an optimal value of consumption less than c^* , condition (3.i) translates to (3.i'), in which case the trajectory is characterized by equations (2) and (4).

It is never optimal to have consumption greater than c^* , since deviations from c^* are bad in themselves and positive deviations bring no offsetting advantages. Formally, an optimal value $c > c^*$ would correspond with a condition

$$\frac{\partial H}{\partial c} = 0 \Rightarrow \lambda = -e^{nt} u'(c) < 0,$$

which indicates a negative shadow value of capital. Clearly this result cannot be optimal, since it indicates an over-accumulation of capital has occurred in previous periods.

For $k(0) > k^*$, the solution to (P4) is to choose consumption equal to c^* at the outset.

Capital diminishes from its initial high level and eventually reaches the steady state at k^* , given enough time. Similarly for $k(0) = k^*$, it is optimal to choose c^* in every period. Denote this solution

$$\hat{S}(T) = \begin{cases} (k(t), c^*) & 0 \leq t < T^* \\ (k^*, c^*) & T^* \leq t \leq T \end{cases}$$

where T^* represents, as above, the transition time to attain golden-rule capital, k^* (equal to 0 when $k(0) = k^*$). This solution is equivalent to $S^*(T)$, from (P2).

For the more realistic case of $k(0) < k^*$, it is only optimal to choose c^* at the outset if T is sufficiently small. This case is illustrated in Figure 2 by the horizontal line segment starting at

α and moving leftward. However, for larger values of T , this policy leads to premature exhaustion of capital. In this case, it is preferable to choose initial consumption $c(0) < c^*$. Consumption then grows according to (4) until either it reaches c^* or capital is exhausted. If it reaches c^* first, it then remains at this level with capital diminishing to zero at T . This case is illustrated in Figure 2 by the trajectories starting at points β and γ , with respectively longer horizons. The lower value of consumption in early periods slows down the dissipation of capital, allowing for a longer horizon. For still larger values of T , it becomes necessary to start with $c(0)$ below the demarcation, in which case capital is accumulated initially and then dissipated. Such trajectories are illustrated by starting values η , π , ψ and ω in the figure. Lemma 1 applies in this context: the lower the value of $c(0)$, the larger the value of T .

This process of lowering $c(0)$ in response to increases in T continues until a threshold value $\hat{T} = T' + T^*$ is reached, where T^* represents, as before, the elapsed time along the golden-rule transition path $\phi\theta$ (both Figures 1 and 2), and T' represents the elapsed time along the horizontal path leading leftward from θ to capital exhaustion (i.e. the time required to fully dissipate all capital, starting at k^* and consuming c^* throughout). T' is implicitly defined by the equation

$$k^* - \int_0^{T'} [f(k(t)) - nk(t) - c^*] dt = 0$$

where the expression in square brackets is the law of motion of capital, (2).

For $T < \hat{T}$, it is better to start with $c(0)$ greater than ϕ , so that consumption arrives at c^* sooner. For $T \geq \hat{T}$, it is optimal to follow the trajectory $\phi\theta$ to the golden rule and then stay there. In this case, the corresponding transversality condition, adapted from (3.iv), is $\lim_{T \rightarrow \infty} \lambda(T) = 0$. It

would never be optimal to choose $c(0)$ below ϕ , since such a path (not shown in the diagram) would take longer to reach c^* and would be characterized by a lower value of c at every time along the transition path.¹³

For $T \geq \hat{T}$, denote the solution

$$\hat{S}(T) = \begin{cases} (k^*(t), c^*(t)) & 0 \leq t < T^* \\ (k^*, c^*) & T^* \leq t \leq T \end{cases}$$

which is identical to $S^*(T)$. The limiting solution is denoted $\hat{S}(\infty)$.

The following optimality result is immediate.

Theorem 4: Among all policies satisfying the law of motion (2), $\hat{S}(\infty)$ is overtaking optimal according to \hat{W}_N^* .

Proof: $\hat{S}(T)$ is the solution to (P4) for any $T > \hat{T}$. ■

\hat{W}_N^* establishes the golden rule utility level, u^* , as a target. Both deviations below u^* and deviations above u^* are considered undesirable, in contrast with conventional utilitarianism for which more is always better. Despite the symmetry, some deviations are more productive than others and are therefore less bad. In particular, when $k(0) < k^*$, deviations of consumption below c^* are necessary to build up the capital stock leading to sustainable increases in future consumption. Thus, though bad in themselves, such deviations are useful, and therefore a good policy under \hat{W}_N^* includes them. In contrast, deviations of consumption above c^* lead to dissipation of capital, and therefore they cannot be sustained. Not only are they bad in themselves, but they cause future consumption to fall below c^* , in some periods at least, which is also bad. Thus, a good policy under \hat{W}_N^* does not include deviations above c^* .

¹³ Proof: negatively sloped isochrones. See the Appendix.

A justification for this ethical position may be offered in the form of an inter-generational contract. Individuals in early generations are willing to sacrifice some of their consumption in order to build up the capital stock and thus provide sustainable increases in consumption for future generations. However, individuals in the future who take consumption greater than c^* are viewed as having broken the contract by arrogating unto themselves resources in excess of what was intended for them. Their action, characterized by Koopmans as a splurge, is considered bad in itself from a social perspective, even though it increases their individual utility.

This view imputes a moral dimension to sustainability which rings true with popular discourse on the topic. Future individuals are morally entitled to consume c^* because it does not impinge on the ability of subsequent individuals to consume the same amount. In contrast, consumption in excess of c^* is not morally justified because it dissipates capital and thus entails a reduction in the consumption of some subsequent individuals.

This contractual ethic supersedes both the strong and weak Pareto principles. On the surface this result may seem obvious since any increase in consumption above c^* reduces the value of $\hat{W}_N^*(\infty)$. However, the proof is in fact more complex as not all consumption paths which rise above c^* are feasible with the neoclassical technology. For the purpose of definition, consider two consumption paths, $c_1 = c_1(t)$ and $c_2 = c_2(t)$, which are both feasible for the given technology. The following definitions are standard.

strong Pareto: If $c_2(t) \geq c_1(t)$ for all $t \geq 0$ and there exists at least one time t' such that $c_2(t') > c_1(t')$, then c_2 is socially preferred to c_1 .

weak Pareto: If $c_2(t) > c_1(t)$ for all $t \geq 0$, then c_2 is socially preferred to c_1 .

Note that satisfaction of strong Pareto entails satisfaction of weak Pareto.

To see the result for $\hat{W}_N^*(\infty)$, assume c_2 cycles repeatedly around the golden rule, such that

$$c_2(t) = \begin{cases} c^* - 1, & 0 \leq t < t_1 \\ c^* + 1, & t_1 \leq t < t_2 \\ c^* - 1, & t_2 \leq t < t_3 \\ c^* + 1, & t_3 \leq t < t_4 \\ \vdots & \vdots \end{cases}$$

The exact mechanism which generates the cycling is as follows. During the first interval, the social planner takes away units from individuals and invests them. The capital stock starts out slightly below k^* at the beginning of the interval and rises slightly above it as successive units are invested. The average marginal product of capital during this interval is equal to the marginal product of k^* , which is n , the growth rate of population. Thus, by compounding the investment returns, the planner has just enough in the second interval to give one additional unit to each individual. The process repeats.

Now suppose that the other path, c_1 , falls short of c_2 by varying amounts, depending whether the deviation in c_2 is above or below c^* . If below, then c_1 falls short of c_2 by a differentially small amount ε . If above, then c_1 falls short of c_2 by the full amount of the deviation, i.e. $c_1 = c^*$. To summarize

$$c_1(t) = \begin{cases} c^* - 1 - \varepsilon, & 0 \leq t < t_1 \\ c^*, & t_1 \leq t < t_2 \\ c^* - 1 - \varepsilon, & t_2 \leq t < t_3 \\ c^*, & t_3 \leq t < t_4 \\ \vdots & \vdots \end{cases}$$

The capital stock is assumed to be identical along both paths. Thus, the differences between c_1 and c_2 are due to wastage, which is assumed costless.

Weak Pareto entails that c_2 is preferred to c_1 , since $c_2(t) > c_1(t)$ for all $t \geq 0$. In contrast, under $\hat{W}_N^*(\infty)$, c_2 is worse than c_1 because the aggregate of deviations from c^* is larger. In c_2 , not only is one group of individuals consuming below the golden rule, but another group is consuming above. These increments above c^* do not make society better off since they involve breaking the intergenerational contract. The violation of weak Pareto entails the violation of strong Pareto by modus tollens (indirect proof), since strong Pareto entails weak Pareto.

While unusual, the violation of the conventional Pareto criteria under $\hat{W}_N^*(\infty)$ is consistent with the golden-rule contract ethics and therefore it is not problematic if one supports those ethics. What is more problematic is that $\hat{W}_N^*(\infty)$ is explicitly defined in terms of the technology, through the golden rule utility level, u^* . It seems persuasive to argue that the validity of an ethical precept should be independent of production possibilities. Yet, it is also conceivable that a rational planner might choose to adjust ethical precepts in response to the contingencies of new technologies. Further reflection is required on this question.

VII. Inequality Aversion

Some economists, including Arrow (1999), have expressed reservations about using an undiscounted objective function in growth problems on the grounds that it imposes unreasonably high rates of saving in early periods.¹⁴ However, it would seem preferable to address this concern by means of an inequality aversion parameter rather than a positive utility discount rate, since the

¹⁴ This concern is only relevant when $k(0) < k^*$.

latter biases solutions against future generations *a priori*. An extended specification of golden-rule contract preferences takes the form

$$\hat{W}_N^*(\infty) = - \int_0^{\infty} e^{nt} |u(c(t)) - u^*|^{\rho} dt, \quad \rho \geq 1$$

where ρ is a curvature parameter related to the elasticity of substitution of utility, ε , as follows:¹⁵

$$\varepsilon = \frac{1}{\rho - 1}.$$

ρ is interpreted as an index of inequality aversion. The lowest permissible value, $\rho = 1$, corresponds with an infinite elasticity of substitution, which is the standard utilitarian case. A larger value translates into a lower elasticity, which favours early generations, as welfare weights are imputed in inverse proportion to the consumption level. To demonstrate, consider the revised Hamiltonian for the optimization problem:

$$H = -e^{nt} |u(c) - u^*|^{\rho} + \lambda [f(k) - nk - c].$$

For an optimal value of consumption less than c^* , the corresponding necessary condition is

$$\frac{\partial H}{\partial c} = e^{nt} \rho [u^* - u(c)]^{\rho-1} u'(c) - \lambda = 0.$$

Comparison with earlier results reveals the welfare weight to be $\rho [u^* - u(c)]^{\rho-1}$, which varies inversely with c when $\rho > 1$.

Further solution, in combination with the other necessary conditions, yields a generalized version of equation (4):

$$\dot{c} = A \sigma(c) f'(k) c \quad \text{where } A = \left[(\rho - 1) \frac{u'(c)c}{u^* - u(c)} \sigma(c) + 1 \right]^{-1}. \quad (4')$$

¹⁵ Population weights are omitted from the calculation of ε , since the operation of substitution is between individuals rather than generations. In contrast, an inter-generational elasticity of substitution would involve population weights.

Inspection of (4') establishes that $0 < A < 1$; therefore consumption grows more slowly than in the case of $\rho = 1$. The slower growth rate is accompanied in the solution by a higher initial consumption value, $c(0)$, and the transition time, T^* , is delayed.

In the limit, as $\rho \rightarrow \infty$, the elasticity of substitution reduces to zero, and $\hat{W}_N^*(\infty)$ translates into Rawls' criterion $\min\{c(t)\}_{t=0}^\infty$. The optimization is then an exercise in maxi-min. The solution is the constant policy $c(t) = c(0)|_{\dot{k}=0} \forall t$; i.e. choose initial consumption on the $\dot{k} = 0$ demarcation – point δ in Figure 2 – and stay there forever. This case is most favourable to the first generation: $c(0)$ is maximized subject to sustainability and saving is just enough to keep the capital-labour ratio constant in the face of population growth; i.e. $c(0) = f(k(0)) - nk(0)$, from (2). However, this policy forsakes growth, and therefore even members of the first generation are not likely to advocate it, assuming they are sufficiently altruistic toward future generations.

VIII. Conclusion

The paper has sought to reconcile the incompatibility of undiscounted utilitarianism with population weighting in the context of the neoclassical growth model under an infinite planning horizon. Koopmans (1965) has identified the source of the problem as the perverse economics of saving for a splurge. In the context of population weighting and constant exogenous population growth, the payoff to a splurge can be made arbitrarily large by waiting for an arbitrarily large population. As a consequence, the incentive to save for a splurge dominates all other considerations. In the limit, everything is saved. And yet the splurge is never taken, since it is always optimal to delay it.

The paper highlights the conflict between splurging and sustainability. A splurge entails dissipation of capital, and therefore it is not compatible with a sustained path of consumption. It

follows that disallowing unsustainable paths removes the incentive to save for a splurge and thus puts investment for productive returns back on centre stage.

A modified utilitarian objective has been presented which embodies the three ethical precepts of impartiality, the golden rule and sustainability. Maximization of the objective involves minimizing the aggregate deviations from the golden rule level of consumption. Both positive and negative deviations are bad, but negative deviations can nonetheless be useful as they allow for capital accumulation and therefore sustainable increases in consumption. In contrast, positive deviations entail capital dissipation, which entails reduced consumption in subsequent periods. An optimal program may include negative deviations but never positive.

For low initial capital, maximization of the modified objective over the neoclassical technology yields a monotonically increasing path to the golden rule, which contrasts favourably with Koopmans' paradox of total saving. Given a sufficiently long horizon, this outcome is equivalent to maximization of the standard objective subject to a sustainability constraint. This equivalence may prove useful in computational problems, as most computer-based algorithms can easily handle a standard constrained optimization problem while the incidence of non-differentiabilities in the modified objective could cause problems.

The ethical position underlying the modified objective is described as an intergenerational contract. Individuals in early generations are willing to sacrifice some of their consumption in order to build up the capital stock, while individuals in future generations are morally obligated to limit their consumption to the golden rule level, since to consume more would impose a cost of reduced consumption on subsequent individuals. An inequality aversion parameter can be introduced to prevent an excessive burden of saving on early generations. This approach respects impartiality, since welfare weights are based on endogenous consumption

levels rather than exogenous calendar time. In contrast, discounted utilitarianism discriminates against future generations *a priori*, as the discount factor is based on calendar time.

The paper has not considered the possibility of technological change. It would be a simple matter to account for labour-augmenting technological change in the standard manner (see Barro and Sala-i-Martin 2004), in which case values would be measured per unit of effective labour, rather than per capita. The results of the paper would then go through without further change. However, while the mathematics would be formally identical, the ethical content would be obscured by such a treatment. For example, what meaning should be attached to utility defined over consumption per effective labour unit? Who actually enjoys this utility? What is the ethical claim of effective labour units? If one remains convinced that a benevolent social planner should care about real people, then this approach must be regarded as problematic. Thus further reflection is required before a compelling treatment of impartiality can be presented in a context of technological change.

Appendix: Isochrones

Several results in the paper can be easily proven with the concept of isochrones. An isochrone, $I(t)$, is the locus of all solution points of (P1) at time t . Formally,

$$I(t) = \{(k_T(t), c_T(t)) | T > 0\}.$$

Define $\kappa(t)$ as the set of k values and $\chi(t)$ as the set of c values contained in $I(t)$; i.e.

$$\kappa(t) \equiv \{k_T(t) | T > 0\} \text{ and } \chi(t) \equiv \{c_T(t) | T > 0\}. I(t) \text{ describes a mapping } \varphi(t) : \kappa(t) \rightarrow \chi(t)$$

provided $t > 0$.¹⁶ For $t, \hat{t} > 0$, the mappings $\varphi(t)$ and $\varphi(\hat{t})$ would be said to cross if they

assigned the same range element to a common domain element, i.e. if, for any $k \in \kappa(\hat{t}) \cap \kappa(t)$,

$$\varphi(k, \hat{t}) = \varphi(k, t).$$

Lemma A1 (non-crossing property of isochrones): Given $t, \hat{t} > 0$, the mappings $\varphi(t)$ and $\varphi(\hat{t})$ do not cross.

Proof: A given solution point cannot be a member of more than one isochrone. ■

It will now be proven that $\varphi(t)$ is continuous and decreasing over its domain. The following lemma provides some preliminary results.

Lemma A2: For any $T, \hat{T} > 0$ and $0 \leq t \leq \min[T, \hat{T}]$,

$$(A) \ c_T(t) > c_{\hat{T}}(t) \text{ and } k_T(t) < k_{\hat{T}}(t) \text{ if and only if } c_T(0) > c_{\hat{T}}(0),$$

$$(B) \text{ there exists } \tilde{T} > 0 \text{ such that } c_T(t) > c_{\tilde{T}}(t) > c_{\hat{T}}(t) \text{ and } k_T(t) < k_{\tilde{T}}(t) < k_{\hat{T}}(t).$$

Proof: (A) If $c_T(0) > c_{\hat{T}}(0)$, it follows from (2) and (4) that $\dot{k}_T(0) < \dot{k}_{\hat{T}}(0)$ and $\dot{c}_T(0) > \dot{c}_{\hat{T}}(0)$;

i.e. at the outset capital grows more slowly and consumption more quickly for $S(T)$ than $S(\hat{T})$.

¹⁶ When $t = 0$, $I(0)$ associates multiple elements of $\chi(0)$ with the single domain value $k(0)$, given exogenously.

These trends are self-perpetuating by virtue of (2) and (4). Thus $\dot{k}_T(t) < \dot{k}_{\hat{T}}(t)$ and $\dot{c}_T(t) > \dot{c}_{\hat{T}}(t)$ for all $t \leq \min[T, \hat{T}]$, which proves sufficiency.

Necessity is proven by contradiction. Consider $S(T)$ and $S(\hat{T})$ such that (i.) $c_T(t) > c_{\hat{T}}(t)$ and $k_T(t) < k_{\hat{T}}(t)$ for all $t \leq \min[T, \hat{T}]$, and (ii.) $c_T(0) < c_{\hat{T}}(0)$. But by the argument above, $c_T(0) < c_{\hat{T}}(0)$ entails $c_T(t) < c_{\hat{T}}(t)$ and $k_T(t) > k_{\hat{T}}(t)$ for all $t \leq \min[T, \hat{T}]$.

(B) Assume without loss of generality that $c_T(0) > c_{\hat{T}}(0)$. Since c is real, there exists an intermediate value, $c_{\tilde{T}}(0)$, associated with a solution $S(\tilde{T})$; i.e. $c_T(0) > c_{\tilde{T}}(0) > c_{\hat{T}}(0)$. The result then follows from (A). ■

Theorem A1:

- (A) $\kappa(t)$ is a convex set,
- (B) $\varphi(t)$ is continuous, and
- (C) $\varphi(t)$ is decreasing in k .

Proof: (A) Lemma A2, Part B,

(B) Lemma A2, Part B,

(C) Lemma A2, Part A. ■

Finally, it will be proven that the isochrone map defined in k - c space by $\varphi(t)$ is vertically increasing in t .

Theorem A2: For any $\hat{t} > t > 0$, $\varphi(\hat{t})$ lies above $\varphi(t)$ in k - c space; i.e. for any $k \in \kappa(\hat{t}) \cap \kappa(t)$, $\varphi(k, \hat{t}) > \varphi(k, t)$.

Proof: Consider two solutions $S(T)$ and $S(\hat{T})$ such that $c_T(T) < c_{\hat{T}}(\hat{T})$. The transversality condition requires $k_T(T) = k_{\hat{T}}(\hat{T}) = 0$, and the non-crossing property of solutions entails

$c_T(0) > c_{\hat{T}}(0)$. It follows from Lemma A2 (A) that $c_T(t) > c_{\hat{T}}(t)$ and $k_T(t) < k_{\hat{T}}(t)$ for all $t \leq \min[T, \hat{T}]$.

Consider the possibility that $T < \hat{T}$. It follows in this case that $k_T(T) = 0 < k_{\hat{T}}(T)$, which is consistent with $S(\hat{T})$ being a solution of (P1). In contrast, $T > \hat{T}$ entails $k_T(T) < k_{\hat{T}}(\hat{T}) = 0$, in which case $S(T)$ cannot be a solution to (P1) since $k_T(T)$ cannot be negative. Therefore, $T < \hat{T}$.

Now $(k_T(T), c_T(T)) \in I(t = T)$ and $(k_{\hat{T}}(\hat{T}), c_{\hat{T}}(\hat{T})) \in I(t = \hat{T})$, which is equivalent by virtue of the transversality condition to the statement $(0, c_T(T)) \in I(t = T)$ and $(0, c_{\hat{T}}(\hat{T})) \in I(t = \hat{T})$. The result then follows from (i.) $c_T(T) < c_{\hat{T}}(\hat{T})$, (ii.) $T < \hat{T}$ and (iii.)

Lemma A1. ■

To summarize, for (P1), the isochrone map represented in k - c space by $\varphi(t)$ is non-crossing and continuous, with c decreasing in k , and the map vertically increasing in t .

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Figure 1

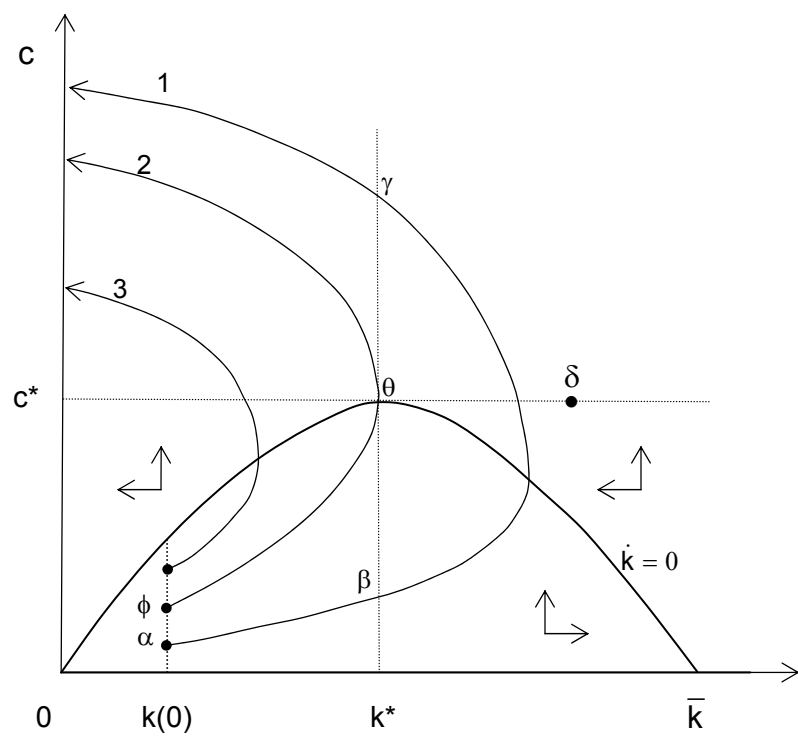


Figure 2

