

# Poristic Loci of Triangle Centers

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## Abstract

The one-parameter family of triangles with common incircle and circumcircle is called a poristic<sup>1</sup> system of triangles. The triangles of a poristic system can be rotated freely about the common incircle. However this motion is not a rigid body motion for the sidelengths of the triangle are changing. Surprisingly many triangle centers associated with the triangles of the poristic family trace circles while the triangle traverses the poristic family. Other points move on conic sections, some points trace more complicated curves. We shall describe the orbits of centers and some other points. Thereby we are able to answer open questions and verify some older results.

*Key Words:* Poristic triangles, incircle, excircle, non-rigid body motion, poristic locus, triangle center, circle, conic section.

Mathematics Subject Classification (AMS 2000): 51M04

## 1 Introduction

The family of poristic triangles has marginally attracted geometers interest. There are only a few articles contributing to this particular topic of triangle geometry: [3] is dedicated to perspective poristic triangles, [12] deals with the existence of triangles with prescribed circumcircle, incircle, and an additional

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<sup>1</sup>The word poristic is deduced from the greek word *porisma*, which could be translated by *deduced theorem*, cf. L. Mackensen: *Neues Wörterbuch der deutschen Sprache*. 13. Auflage, Manuscriptum Verlag, München, 2006.

element. Some more general appearances of porisms are investigated in [2, 4, 6, 9, 16] and especially [7] provides an overview on Poncelet's theorem which is the projective version and thus more general notion of porism.

Nevertheless there are some results on poristic loci, *i.e.*, the traces of triangle centers and other points related to the triangle while the triangle is traversing the poristic family. In [14] some invariant lines, circles, and conic sections have been determined. Also triangles with common circumcircle and nine-point circle have been studied by R. Crane in [5]. The poristic loci of triangle centers have not undergone sincere study. For some centers the loci are given in [11], especially the trace of the Gergonne point is treated in [1]. In [13] a result by Weill is reproduced showing that the centroid  $X_{354}$  of the intouch triangle is fixed while  $\Delta$  traces its poristic family.

For most of the centers listed in [10] the respective poristic locus is unknown. In the following we shall derive these loci, at least for some centers that can easily be accessed with our method. For that purpose we impose a Cartesian coordinate system in Sec. 2 which will henceforth be the system of reference. Subsequently we derive paths of points which are not centers in Sec. 3. Afterwards we pay our attention to triangle centers in Sec. 4. First we focus on the centers on the line  $\mathcal{L}_{1,3}$ ,<sup>2</sup> connecting the incenter  $X_1$  with the circumcenter  $X_3$ . It carries a lot of centers, some of them stay fixed others do not. We only look at the fixed ones. Triangle centers which are located on the incircle or circumcircle naturally trace these circles. None of these remains fixed, except those on  $\mathcal{L}_{1,3}$ . Then we shall derive poristic loci of some triangle centers and focus on those that traverse circles and conic sections. The centers and radii or semiaxes of these poristic orbits are given explicitly.

At this point we shall say a few words about techniques used in this work. Computations are done with Maple. Equations of poristic loci mainly use the framework of resultants. The computation of parametrizations of centers is restricted somehow. This will be clear when we see parametrizations of the circumcircle and incircle describing the vertices of  $\Delta$  and its intouch triangle, respectively. Deriving paths of orthocenters, centroids, circumcenters, midpoints of a pair of triangle centers, as well as paths of triangle centers which appear as intersections of central lines seems to be a very simple task at first glance. But, however, parametrizations become larger and larger and

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<sup>2</sup>Triangle centers are labelled according to C. Kimberlings list [10, 11]. Central lines, *i.e.*, lines joining two centers with Kimberling number  $i$  and  $j$  shall be denoted by  $\mathcal{L}_{i,j}$ .

the computation of equations exceeds memory capacity and cannot be done in an acceptable amount of time. Therefore centers like the incenter of the intouch triangle (which is  $X_{177}$  for the base triangle) cannot be reached with our method.

## 2 Prerequisites

Let  $\Delta$  be a triangle with vertices  $A, B, C$ . We denote its circumcircle and incircle by  $u$  and  $i$ , respectively. The circumcenter and the incenter shall be denoted by  $X_3$  and  $X_1$ , see [10, 11]. The circumradius  $R$ , the radius of the incircle  $r$ , and the distance  $d$  of  $X_1$  and  $X_3$  are related by

$$d^2 = R^2 - 2rR, \quad (1)$$

see for example [11, p. 40]. The incircle and the circumcircle are circles in a special position. To the best of the author's knowledge there is no english word for that. In german we would say: "*Kreise in Schließungslage*".

Any two circles  $u$  and  $i$  define a one-parameter family of triangles all of them having  $u$  for the circumcircle and  $i$  for the incircle provided that Eq. (1) is fulfilled, *cf.* Fig. 1. Any two triangles out of this family are said to form a poristic pair of triangles.

In the following we want to study the traces of centers and other points related to a triangle traversing its poristic family. For that purpose we use Cartesian coordinates in order to represent points in the Euclidean plane. Without loss of generality we can assume that  $X_3 = [0, 0]$  and  $X_1 = [d, 0]$ . The equations of the circumcircle and the incircle are thus

$$u: x^2 + y^2 = R^2, \quad i: (x - d)^2 + y^2 = r^2. \quad (2)$$

Aiming at parametrizations of the traces of centers and other points related to the triangle we assume that the line carrying  $A$  and  $B$  is given by

$$g: x \cos t + y \sin t = r + d \cos t \quad \text{with} \quad t \in [0, 2\pi) \quad (3)$$

since  $[A, B]$  has to be tangent to  $i$ . This allows to parametrize the circular path of points  $A$  and  $B$  in a proper way. Note that these points are the intersections of  $g$  and  $u$  and therefore they are given by

$$\begin{aligned} A &= [r \cos t + d \cos^2 t + W \sin t, r \sin t + d \cos t \sin t - W \cos t], \\ B &= [r \cos t + d \cos^2 t - W \sin t, r \sin t + d \cos t \sin t + W \cos t], \end{aligned} \quad (4)$$

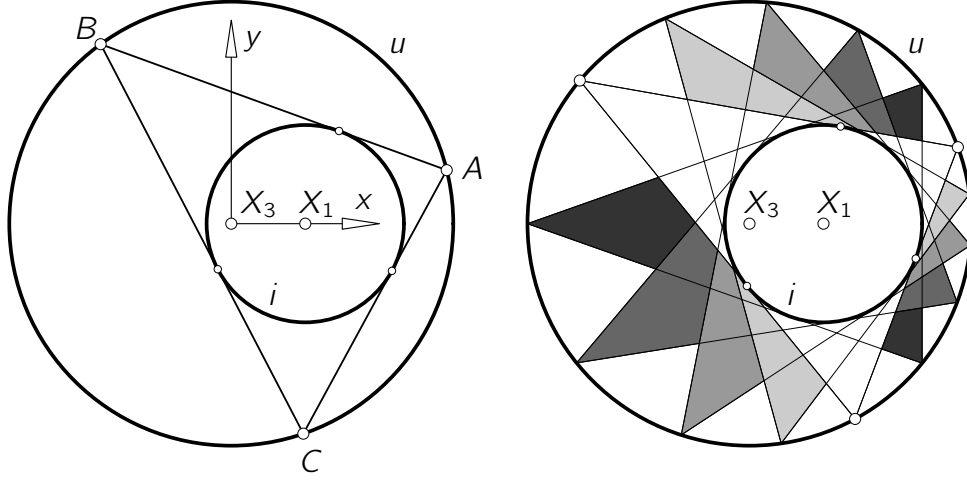


Figure 1: Triangle  $\Delta$  with incircle  $i$ , circumcircle  $u$ , and the Cartesian coordinate system imposed on it.

where  $W = \sqrt{R^2 - (r + d \cos t)^2}$ .<sup>3</sup>

Finally the two tangents from  $A$  and  $B$  to  $i$  which are different from the line  $[A, B]$  intersect in  $\Delta$ 's third vertex  $C \in u$ . This point reads

$$C = \left[ \frac{R(2dR - (R^2 + d^2)c_t)}{R^2 + d^2 - 2dRc_t}, \frac{(d^2 - R^2)Rs_t}{R^2 + d^2 - 2dRc_t} \right], \quad (5)$$

whose trace is now described by the same parameter  $t$ . Here and in the following  $c_t$  and  $s_t$  are short hand for  $\cos t$  and  $\sin t$ , respectively.

The triangle  $\Delta$  and thus any triangle in the pristic family defines some other triangles: The medial triangle  $\Delta_m$  is built by the midpoints of  $\Delta$ 's sides. We denote the anticomplementary triangle by  $\Delta_a$ , the excentral triangle by  $\Delta_e$ , the intouch triangle by  $\Delta_i$ , the tangent triangle by  $\Delta_t$ , the orthic triangle by  $\Delta_o$ , and the extouch triangle by  $\Delta_x$ .

It is elementary to find the vertices of  $\Delta_a$ ,  $\Delta_e$ ,  $\Delta_m$ ,  $\Delta_o$ , and  $\Delta_t$ , if the parametric representation of  $A$ ,  $B$ , and  $C$  is known. One vertex from  $\Delta_i$  is known from the beginning:  $B_{AB} = [rc_t + d, rs_t]$ , the point of contact of  $g$  and  $i$ . The remaining vertices of  $\Delta_i$  can be obtained by reflecting the contact point  $B_{AB}$  of the line  $[A, B]$  with  $i$  in  $[A, X_1]$  and  $[B, X_1]$ , respectively. Though these

<sup>3</sup>Note that  $r$ ,  $R$ , and  $d$  are related via Eq. (1). Sometimes we do not eliminate  $r$  in order to shorten formulae.

operations are elementary we sketch them in order to make any computation traceable.

Finally we point out that the computation of centers  $X_i$  with

$$i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20\}$$

among many others is elementary with these preparations. The shape of the respective poristic loci will be discussed later in Sec. 4. At this point we should confess that the computation of incenters needs normalization of direction vectors. Luckily we have the incenter of  $\Delta$ , but, unfortunately we cannot reach incenters of  $\Delta_e$  and  $\Delta_i$ .

### 3 Traces of some points

We give the answer to the question raised in [11, p. 257] by proving the following:

**Theorem 3.1.** *The trace of the midpoint of any side of a triangle traversing a poristic family is a Limaçon of Pascal.*

*Proof.* The midpoint  $M$  of  $AB$  is given as the arithmetic mean of the coordinate vectors of the two points  $A$  and  $B$  from (4) and reads

$$M = [r \cos t + d \cos^2 t, r \sin t + d \cos t \sin t]. \quad (6)$$

The curve parametrized by (6) is called Limaçon of Pascal, see [15]. Its equations in terms of Cartesian coordinates is obtained by eliminating  $t$  and reads

$$m : (x^2 + y^2)^2 - 2dx(x^2 + y^2) - ((r^2 - d^2)x^2 + r^2y^2) = 0 \quad (7)$$

for variable choices of  $r$  and  $d$  such that Eq. (1) is satisfied.  $\square$

Fig. 2 shows different shapes of this curve: noded, cusped, or without visible singularity. However, independent on the choice of  $R$  and  $d$  the point  $X_3$  is a double point on  $m$  in any case. The quartic curve  $m$  has a cusp at  $U$  exactly if  $|d| = |r|$ . If  $|d| < |r|$   $U$  is an isolated double point.

The quartic  $m$  touches  $i$  twice, *i.e.*, precisely at points  $X_{2446} = [d - r, 0]$  and  $X_{2447} = [d + r, 0]$ . If a midpoint of a side of  $\Delta$  happens to coincide with one

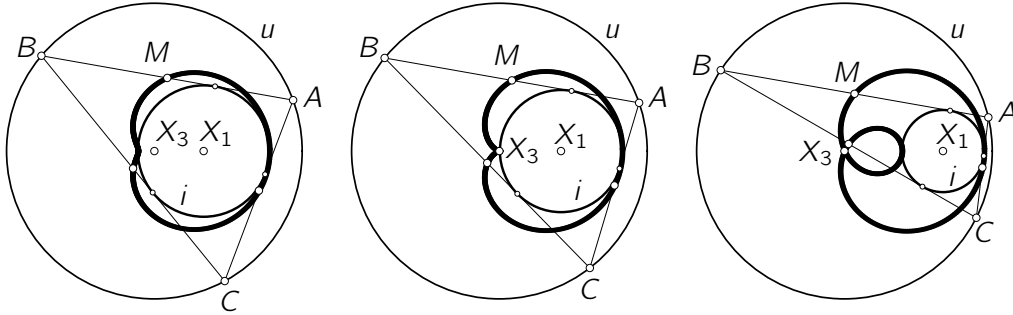


Figure 2: Different shapes of Pascal's limaçon which appears as trace of a side's midpoint.

of these points then  $\Delta$  is isosceles. Obviously there are two isosceles triangles in the family of poristic triangles.

According to Bézout's theorem the total amount of intersection points of  $m$  and  $i$  equals 8. The two real contact points  $X_{2446}$  and  $X_{2447}$  are each of multiplicity two, the remaining four points are the absolute points of Euclidean geometry (a pair of conjugate complex points on the ideal line) each of which has multiplicity two on  $m$  and as a common point of  $m$  and  $i$ . Note that the midpoints of the remaining two sides of  $\Delta$  bound  $M$  on the same curve.

The traces of the excenters (see Fig. 3) of triangles in a poristic family have been studied in [14] with slightly different methods. We observe:

**Theorem 3.2.** *The three excenters of the triangles in a poristic family trace the same circle  $e$ . Its center  $E$  is the reflection of  $X_1$  in  $X_3$  and its radius equals  $2R$ .*

*Proof.* The normals to  $[A, X_1]$  and  $[B, X_1]$  at  $A$  and  $B$ , respectively, intersect at  $\Delta$ 's excenter  $A_3$ , opposite to  $C$ , for these lines are the internal bisectors at  $A$  and  $B$ . An elementary computation using (4) and (5) yields

$$A_3 = [2Rc_t - d, 2Rs_t]$$

which obviously parametrizes the circle  $e$  with equation

$$e: (x + d)^2 + y^2 = 4R^2. \quad (8)$$

Cyclically shifting  $A$ ,  $B$ , and  $C$  yields parametrizations of the loci of the other excenters. These parametrizations annihilate Eq. (8). The center of  $e$  is

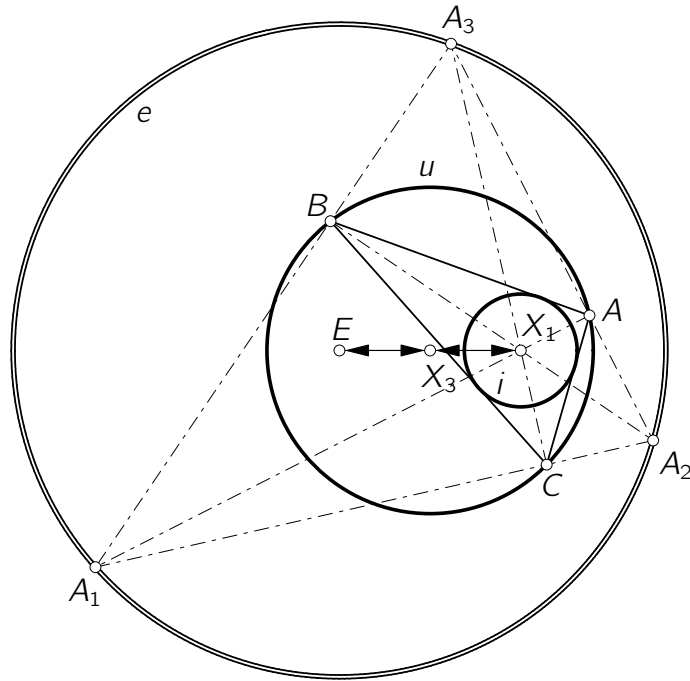


Figure 3: The common path of all three excenters.

$E = [-d, 0]$ , i.e., the point  $X_3$  is the midpoint of  $E$  and  $X_1$ . The radius of  $e$  equals  $2R$ .  $\square$

Note that the point  $E$  is  $\Delta$ 's center  $X_{40}$ , which is frequently called Bevan point (cf. [10, 11]) and it remains fixed while  $\Delta$  goes through the poristic family. The excentral triangles  $\Delta_e$  together with the triangles  $\Delta$  form another poristic family of triangles with common circumcircle  $e$  and nine-point circle  $u$  for  $\Delta$  is the orthic triangle of  $\Delta_e$ , see [5].

Similarly we can show:

**Theorem 3.3.** *The vertices of the tangential triangle  $\Delta_t$  of  $\Delta$  move on an ellipse while  $\Delta$  traverses the poristic family.*

*Proof.* The vertices  $T_A, T_B, T_C$  of  $\Delta_t$  are constructed as the intersections of the tangents of the circumcircle  $u$  at  $A, B, C$ , respectively. The trace of the vertex  $T_C$  opposite to  $C$  is parametrized by

$$T_C = \left[ \frac{2R^3 c_t}{R^2 - d^2 + 2dRc_t}, \frac{2R^3 s_t}{R^2 - d^2 + 2dRc_t} \right].$$

This is an ellipse with center  $[R^2d/(R^2 - 2Rr - r^2), 0]$  and semiaxes  $a = rR^2/(r^2 + 2Rr - R^2)$ ,  $b = R^2/\sqrt{r^2 + 2Rr - R^2}$  which can be seen after implicitization. The traces of  $T_A$  and  $T_B$  have a more complicated parametrization, but, however, they annihilate the same equation.  $\square$

Fig. 4 shows the ellipse appearing as the poristic orbit of the vertices of  $\Delta_t$ .

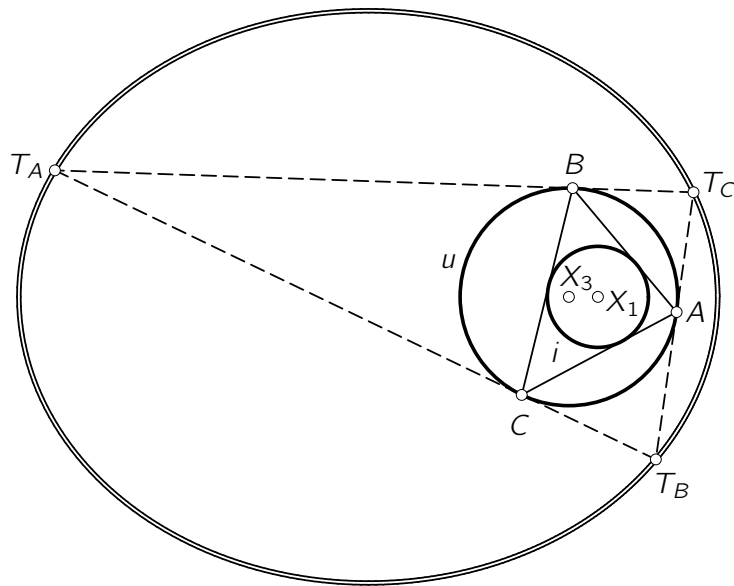


Figure 4: The ellipse traced by the vertices of the tangential triangle.



## 4 Orbits of some centers

### 4.1 Centers on $\mathcal{L}_{1,3}$

On the central line  $\mathcal{L}_{1,3}$  we find the triangle centers  $X_i$  with

$$i \in \{1, 3, 35, 36, 40, 46, 55, 56, 57, 65, 165, 171, 241, 260, 354, 484, 517, 559, 940, 942, 980, 982, 986, 988, 999, 1038, 1040, 1060, 1062, 1082, 1155, 1159, 1214, 1319, 1381, 1382, 1385, 1388, 1402, 1403, 1420, 1429, 1454, 1460, 1466, 1467, 1470, 1482, 1617, 1622, 1697, 1715, 1735, 1754, 1758, 1764, 1771, 1936, 2061, 2077, 2078, 2093, 2095, 2098, 2099, 2223, 2283, 2352, 2446, \dots, 2449, 2556, 2557, 2564, 2565, 2572, 2573, 2646, 2662, 3057, 3072, 3075, 3245, 3256, 3295, 3303, 3304, 3333, 3336, \dots, 3340, 3359, 3361, 3428, 3503, 3513, 3514, 3550, 3576, 3579, 3587, 3601, 3612\},$$

*cf.* [10]. The points  $X_1$  and  $X_3$  are fixed anyway. The circumcenter of  $\Delta$ 's excentral triangle is the point  $X_{40}$  and remains fixed as shown in Th. 3.2. The triangle center  $X_{571}$  is the ideal point of the line  $\mathcal{L}_{1,3}$  and all parallel lines, especially the central lines  $\mathcal{L}_{4,8}$  and  $\mathcal{L}_{5,10}$ .

For some of the centers on  $\mathcal{L}_{1,3}$  we can give their precise position and state:

**Theorem 4.1.** *The triangle centers  $X_i$  of  $\Delta$  with*

$$i \in \{1, 3, 35, 36, 40, 46, 55, 56, 57, 65, 165, 354, 484, 517, 942, 999, 1155, 1159, 1319, 1381, 1382, 1385, 1388, 1420, 1454, 1482, 1697, 2077, 2078, 2093, 2095, 2098, 2099, 2446, 2447, 2646, 3057, 3245, 3256, 3295, 3303, 3304, 3336, \dots, 3340, 3576, 3579, 3587, 3601, 3612\} \quad (9)$$

*remain fixed while  $\Delta$  traverses its poristic family.*

*Proof.* There is nothing to be done for  $X_1 = [d, 0]$ ,  $X_3 = [0, 0]$ , and  $X_{517}$ , the ideal point of  $\mathcal{L}_{1,3}$ . The Bevan point  $X_{40}$  is the circumcenter of  $\Delta_e$  and according to Th. 3.2 it is fixed.

The center  $X_{65} = [d(R+r)/R, 0]$  is the orthocenter of  $\Delta_j$ .  $X_{942} = [d(2R+r)/(2R), 0]$  is the midpoint of  $X_1$  and  $X_{65}$ . The center  $X_{36} = [R^2/d, 0]$  is the inverse of  $X_{942}$  in the incircle and the 1<sup>st</sup> Evans perspecter  $X_{484} = [R(R+2r)/d, 0]$  is the reflection of  $X_1$  in  $X_{36}$ . Then  $X_{35} = [dR/(R+2r), 0]$  is the inverse of  $X_{484}$  in the circumcircle.

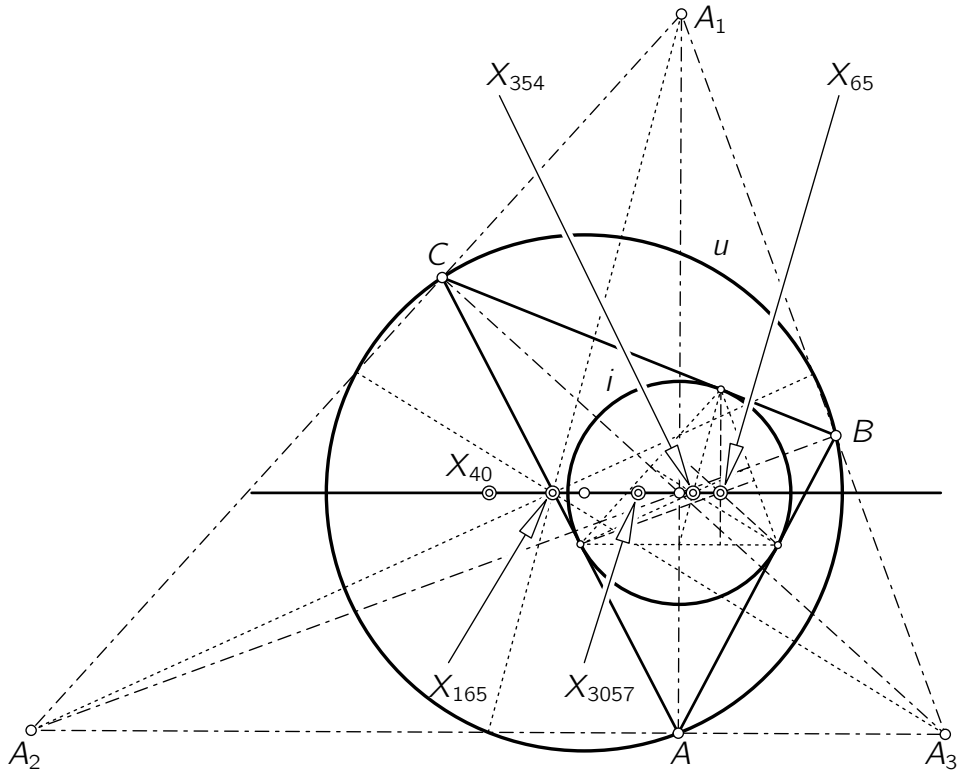


Figure 5: Some centers on the line  $\mathcal{L}_{1,3}$  mentioned in Th. 4.1: the circumcenter  $X_{40}$  of  $\Delta_e$ , the orthocenter  $X_{65}$  of  $\Delta_i$ , the centroid  $X_{165}$  of  $\Delta_e$ , the centroid  $X_{354}$  (Weill point) of  $\Delta_i$ , the de Longchamps point  $X_{3057}$  of  $\Delta_i$ .

The centers  $X_{55} = [dR/(R+r), 0]$  and  $X_{56} = [dR/(R-r), 0]$  are the in- and exsimilicenter of the incircle  $i$  and the circumcircle  $u$ . They are fixed for  $u$  and  $i$  are fixed. As the reflection of  $X_1$  in  $X_{56}$  we find  $X_{46} = [d(R+r)/(R-r), 0]$ .  $X_{57}$  appears as the intersection of  $\mathcal{L}_{1,3}$  and  $\mathcal{L}_{2,7}$  and reads  $X_{57} = [d(2R+r)/(2R-r), 0]$ , which is obviously independent of  $t$ . The center  $X_{165} = [-\frac{1}{3}d, 0]$  is computed as the centroid of  $\Delta_e$ .

The Weill point  $X_{354}$  (cf. [10, 11]) is the centroid of  $\Delta_i$  and therefore  $X_{354} = [d(3R+r)/(3R), 0]$ . The center  $X_{999}$  is the midpoint of centers  $X_1$  and  $X_{57}$  and thus  $X_{999} = [2dR/(2R-d), 0]$ . The Schröder point  $X_{1155} = [R(R+r)/d, 0]$  (cf. [10]) is the inverse of  $X_{55}$  in the circumcircle. The Greenhill point  $X_{1159}$  (see also [10]) is the intersection of  $\mathcal{L}_{1,3}$  and the line parallel to  $\mathcal{L}_{1,5}$  through  $X_7$  and consequently  $M_{1159} = [4d(R+r)/(4R+r), 0]$ .

The Bevan-Schröder point  $X_{1319} = [R(R-r)/d, 0]$  (cf. [10]) is the midpoint

of  $X_1$  and  $X_{36}$ . The center  $X_{1385} = [\frac{1}{2}d, 0]$  is the midpoint of  $X_1$  and  $X_3$ . The triangle center  $X_{1388} = [d(R-2r)/(R-3r), 0]$  is computed as the intersection of  $\mathcal{L}_{1,3}$  and  $\mathcal{L}_{8,1317}$ , with  $X_{1317}$  being the reflection of the Feuerbach point  $X_{11}$  in the incenter  $X_1$ . The points  $X_{1381} = [-R, 0]$  and  $X_{1382} = [R, 0]$  are the common points of the circumcircle and the line  $\mathcal{L}_{1,3}$ .

Now we show that  $X_{1420} = [d(2R-r)/(2R-3r), 0]$  which is thus also fixed and contained in  $\mathcal{L}_{1,3}$ : First we observe that  $X_{1420} = \mathcal{L}_{1,3} \cap \mathcal{L}_{84,104}$ . Now  $X_{84}$  is the reflection of  $X_{1490}$  in  $X_3$  and  $X_{1490} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,9}$ . The triangle center  $X_{104}$  is the circumcircle-antipode of  $X_{100}$  and thus it is the reflection of  $X_{100}$  in  $X_3$  with  $X_{100} = \mathcal{L}_{3,8} \cap \mathcal{L}_{56,145}$ , where  $X_{145}$  is the reflection of  $X_8$  in  $X_1$ .

By the way we obtain  $X_{1454} = [d(R+r)^2/(R^2+Rr-r^2), 0]$  which lies on  $\mathcal{L}_{4,145}$  and  $X_{1482} = [2d, 0]$  is the reflection of the circumcenter in the incenter. We find  $X_{1697} = \mathcal{L}_{1,3} \cap \mathcal{L}_{8,9} = [d(2R-r)/(2R+r), 0]$ . Since  $X_{2077}$  is the inverse of  $X_{40}$  in the circumcircle we have  $X_{2077} = [-R^2/d, 0]$ . Analogously we find  $X_{2078} = [R^2(2R-r)/(d(2R+r)), 0]$  which is the inverse of  $X_{57}$  in the circumcircle.

The triangle center  $X_{2093} = [d(2R+3r)/(2R-r), 0]$  is the reflection of  $X_1$  in  $X_{57}$ . The reflection of  $X_3$  in  $X_{57}$  yields  $X_{2095} = [2d(2R+r)/(2R-r), 0]$ . The reflection of  $X_{56}$  in the incenter  $X_1$  leads to  $X_{2098} = [d(R-2r)/(R-r), 0]$ . The point  $X_{2099}$  can be obtained as reflection of  $X_{55}$  in  $X_1$ .

The centers  $X_{2446} = [d-r, 0]$  and  $X_{2447} = [d+r, 0]$  are each others reflections in  $X_1$ . Moreover they are the intersections of the incircle  $i$  with the line  $\mathcal{L}_{1,3}$ .  $X_{2446}$  is the center closer to  $X_3$ , cf. [10].

Further  $X_{2646} = \frac{1}{2}(X_1 + X_{35}) = [d(R+r)/(R+2r), 0]$ . The center  $X_{3057} = [d(R-r)/R, 0]$  is the de Longchamps point of  $\Delta_i$ . This fact is not mentioned in [10]. There  $X_{3057}$  only appears as the intersection of lines  $\mathcal{L}_{1,3}$  and  $\mathcal{L}_{10,11}$ .

The center  $X_{3245} = [R(R+4r)/d, 0]$  is found as the reflection of  $X_{36}$  in  $X_{484}$ . Now we show that  $X_{3256} = [dR(2R+3r)/(2R^2+Rr+2r^2), 0]$ : First note that  $X_{3256} = \mathcal{L}_{1,3} \cap \mathcal{L}_{100,226}$ . Where  $X_{226}$  is the reflection of  $X_{993}$  in  $X_{1125}$ . The latter point  $X_{1125}$  is the midpoint of  $X_1$  and  $\Delta$ 's Spieker point  $X_{10}$ . The first one,  $X_{993}$ , is the reflection of  $X_1$  in  $X_{63}$ , which is the reflection of  $X_{1478}$  in  $X_{10}$ . The center of the Johnson-Yff circle  $X_{1478}$  (cf. [10]) is given by  $X_{1478} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,36}$ .

Intersecting  $\mathcal{L}_{1,3}$  with  $\mathcal{L}_{4,390}$  gives  $X_{3295} = [2dR/(2R+r), 0]$ , where  $X_{390}$  comes as a byproduct in a very early stage of the computation:  $X_{390}$  is the

reflection of the Gergonne point  $X_7$  in  $X_1$ . We observe  $X_{3303} = \mathcal{L}_{1,3} \cap \mathcal{L}_{12,497} = [3dR/(3R+r), 0]$ , with  $X_{12} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,56}$  and  $X_{497} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,11}$ . Similarly we find  $X_{3304} = \mathcal{L}_{1,3} \cap \mathcal{L}_{11,153} = [3dR/(3R-r), 0]$  with  $X_{153}$  being the reflection of  $X_{20}$  in  $X_{100}$ .

We find the triangle centers  $X_{3336}, \dots, X_{3340}, X_{3361}$  as intersections of  $\mathcal{L}_{1,3}$  with lines  $\mathcal{L}_{7,498}, \mathcal{L}_{7,499}, \mathcal{L}_{7,90}, \mathcal{L}_{7,10}, \mathcal{L}_{7,145}$ , and  $\mathcal{L}_{7,1125}$  and obtain  $X_{3336} = [d(3R+2r)/(3R-2r), 0]$ ,  $X_{3337} = [d(5R+2r)/(5R-2r), 0]$ ,  $X_{3338} = [d(3R+r)/(3R-r), 0]$ ,  $X_{3339} = [d(4R+3r)/(4R-r), 0]$ ,  $X_{3340} = [d(2R+3r)/(2R+r), 0]$ , and  $X_{3361} = [d(4R+r)/(4R-3r), 0]$ , respectively. We remark that  $X_{3338}$  is also the reflection of  $X_1$  in  $X_{3304}$ .

We can easily find the centers  $X_{3576} = \frac{1}{2}(X_1 + X_{165}) = [\frac{1}{3}d, 0]$  and  $X_{3579} = \frac{1}{2}(X_3 + X_{40}) = [-\frac{1}{2}d, 0]$ . The center  $X_{3587} = [-d(2R+r)/(4R+r), 0]$  is the intersection of  $\mathcal{L}_{1,3}$  and  $\mathcal{L}_{84,550}$ , where  $X_{550} = \frac{1}{2}(X_3 + X_{20})$ . The center  $X_{3601} = [d(2R+r)/(2R+3r), 0]$  is also located on  $\mathcal{L}_{9,21}$ , where the Schiffler point  $X_{21}$  can be found as intersection of  $\Delta$ 's Euler line with  $\mathcal{L}_{7,56}$ . Finally  $X_{3612} = [d(R+r)/(R+3r), 0]$  is located on  $\mathcal{L}_{21,90}$ , where  $X_{90} = \mathcal{L}_{1,155} \cap \mathcal{L}_{40,80}$ . The center  $X_{155}$  is the orthocenter of  $\Delta_t$  and  $X_{80} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,214}$  with  $X_{214} = \frac{1}{2}(X_1 + X_{100})$ .  $X_{80}$  can also be found as the reflection of  $X_1$  in the Feuerbach point  $X_{11}$ .  $\square$

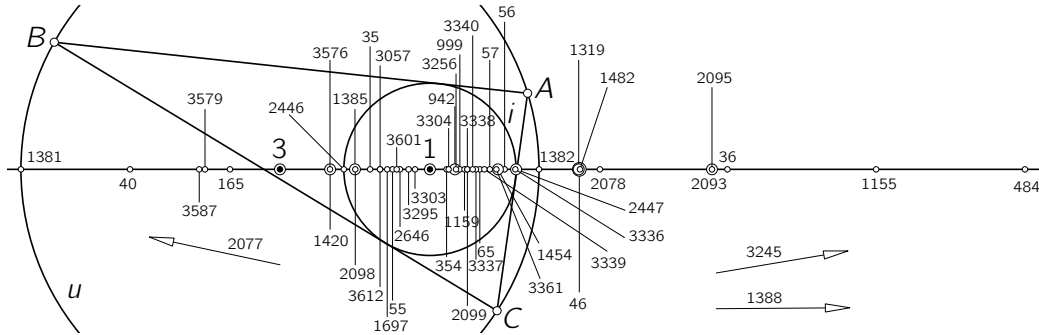


Figure 6: Distribution of fixed centers on  $\mathcal{L}_{1,3}$ .

Fig. 5 shows some triangle centers on the central line  $\mathcal{L}_{1,3}$  which appear as centers of central triangles. Fig. 6 shows the distribution of centers on  $\mathcal{L}_{1,3}$  as described in Th. 4.1.

## 4.2 Centers on the incircle and circumcircle

According to [10] the triangle centers  $X_i$  with

$$i \in \{11, 1314, 1315, 1317, 1354, \dots, 1367, 2446, 2447, 3020, \dots, 3028, 3317, \dots, 3328\}$$

are contained in the incircle. Here we can only verify the following result:

**Theorem 4.2.** *The centers  $X_{2446}$  and  $X_{2447}$  remain fixed while  $\Delta$  is running through the poristic family.*

*Proof.* Actually there is nothing to be done:  $X_{2446} = [d - r, 0]$  and  $X_{2447} = [d + r, 0]$  are the intersections of the incircle  $i$  with the line  $\mathcal{L}_{1,3}$ , see the proof of Th. 4.1.  $\square$

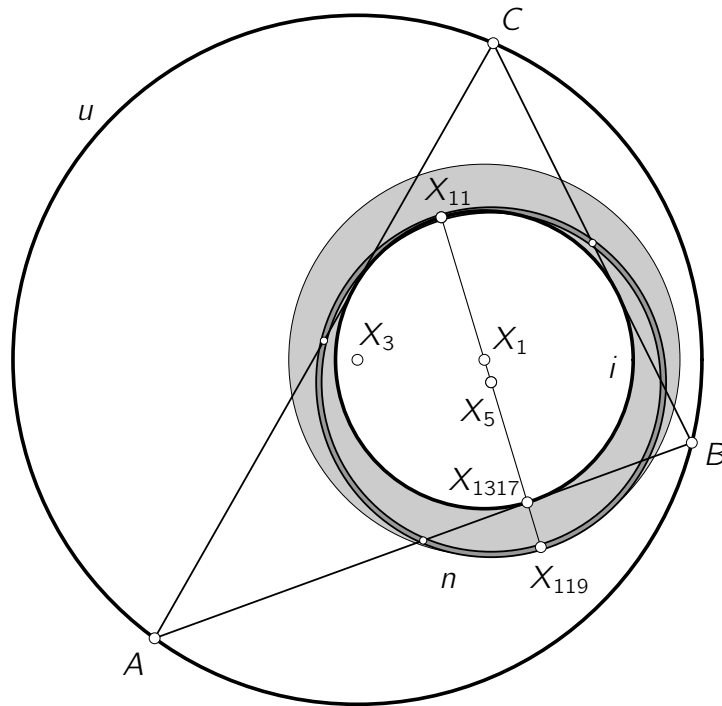


Figure 7: The grey shaded annulus is the locus of all nine-point circles  $n$  of triangles in the poristic family.

The point  $X_{11}$  known as Feuerbach point is the point of contact of the nine-point circle with the incircle. Thus this point moves on the incircle given in

(2). Since the circumradius  $R$  is the same for all triangles in the poristic family the family of corresponding Feuerbach circles consists of congruent circles of radius  $R/2$ . The nine-point circles of the poristic family are in contact with  $i$  and enclose it at any instant. Beside  $X_{11}$  the Feuerbach antipode  $X_{119}$  is the second point of contact of any nine-point circle  $n$  with the outer boundary of their envelope, see Fig. 7. So we can state:

**Theorem 4.3.** *The nine-point circles of the triangles of a poristic family over coat an annulus bounded by the incircle  $i$  and a concentric circle with radius  $R - r$ .*

From this we can deduce the following result:

**Theorem 4.4.** *The poristic locus of  $X_{119}$  is a circle centered at  $X_1$  with radius  $\rho_{119} = R - r$ .*

Among the huge amount of known triangle centers  $X_i$  only those few with indices

$$i \in \{74, 98, \dots, 112, 399, 476, 477, 675, 681, 689, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 645, 747, 753, 755, 759, 761, 767, 769, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 803, 805, 807, 807, 813, 815, 817, 819, 825, 827, 831, 833, 835, 839, 840, 841, 843, 900, 901, 917, 919, 925, 927, 929, \dots, 935, 953, 972, 1286, \dots, 1311, 1381, 1382, 1477, 2222, 2249, 2291, 2365, \dots, 2384, 2687, \dots, 2770, 2855, \dots, 2868, 3222, 3563, 3565\}$$

lie on the circumcircle. Here we have:

**Theorem 4.5.** *Among the triangle centers on the circumcircle  $u$  only the points  $X_{1381}$  and  $X_{1382}$  remain fixed while  $\Delta$  traverses the poristic family.*

*Proof.* We refer to the proof of Th. 4.1 where  $X_{1381} = [-R, 0]$  and  $X_{1382} = [R, 0]$  are mentioned as the intersections of  $u$  with  $\mathcal{L}_{1,3}$ .  $\square$

### 4.3 Centers with circular paths

In the following we describe the orbits of some triangle centers with circular paths. Some of them are points on the circumcircle  $u$ , some lie on the incircle  $i$ . We show:

**Theorem 4.6.** *Let  $\Delta$  be a triangle traversing its poristic family. Then  $\Delta$ 's triangle centers  $X_i$  have circular paths for*

$i \in \{2, 4, 5, 7, 8, 9, 10, 11, 12, 20, 21, 23, 32, 63, 72, 76, 78, 80, 84, 90, 94, 100, 104, 105, 119, 120, 140, 142, \dots, 145, 149, 153, 186, 191, 200, 210, 214, 226, 323, 329, 347, 355, 376, 381, 382, 388, 390, 392, 399, 442, 495, \dots, 499, 501, 546, \dots, 551, 631, 632, 759, 908, 920, 936, 938, 943, 944, 946, 950, 954, 956, 958, 960, 962, 993, 997, 1001, 1004, 1005, 1007, 1125, 1145, 1156, 1158, 1210, 1292, 1317, 1320, 1323, 1324, 1325, 1329, 1376, 1387, 1478, 1479, 1483, 1484, 1490, 1511, 1512, 1519, 1532, 1537, 1538, 1656, 1657, 1698, 1699, 1706, 1737, 1750, 1785, 1837, 1851, 1858, 1898, 1899, 2070, 2071, 2094, 2096, 2478, 2550, 2551, 2886, 2932, 2948, 3036, 3059, 3060, 3085, 3086, 3091, 3110, 3219, 3241, 3243, 3244, 3254, 3305, 3322, 3328, 3358, 3419, 3421, 3434, 3452, 3473, 3474, 3475, 3485, 3486, 3522, 3534, 3543, 3555, 3582, \dots, 3586, 3589, 3600\}$ .

*Each of these centers traces its circular path three times while  $\Delta$  performs one full turn in the poristic family.*

*Proof.* We demonstrate how to prove the above theorem by means of the trace of  $X_2$ :  $X_2$  is the centroid of  $\Delta$  and therefore a parametrization of the poristic orbit of  $X_2$  is given as the arithmetic mean of the coordinate vectors of  $A$ ,  $B$ , and  $C$  from Eqs. (4) and (5), i.e.,  $X_2(t) = \frac{1}{3}(A + B + C)$ . Explicitly we have

$$X_2(t) = \left[ \begin{array}{c} \frac{d(-4d^2c_t^3R^2 + 4d^2c_t^2R - d(R^2 + d^2)c_t + 2R^3)}{3R(R^2 + d^2 - 2dRc_t)} \\ \frac{d^2s_t(R^2 - d^2 + 4dRc_t - 4R^2c_t^2)}{3R(R^2 + d^2 - 2dRc_t)} \end{array} \right]. \quad (10)$$

This parametrization tells us that  $X_2$  traces its path three times. In order to obtain an equation of it and moreover in order to show that the orbit of  $X_2$  is a circle, we eliminate  $t$  by first substituting  $c_t = (1 - u^2)/(1 + u^2)$  and  $s_t = 2u/(1 + u^2)$ . Then we compute the resultant with respect to  $u$  of the two polynomials

$$p_x := \text{den}(x_2(u)) - x \cdot \text{num}(x_2(u)), p_y := \text{den}(y_2(u)) - y \cdot \text{num}(y_2(u)),$$

where  $x_2(u)$  and  $y_2(u)$  are the coordinate functions of  $X_2(u)$  and  $\text{den}(f/g) = g = \text{num}(g/f)$  give the denominator and numerator of a rational expression. This yields

$$2^{30}d^{12}R^{16}(R^2 - d^2)^4(4R^2d^2 - 12xdR^2 + 9y^2R^2 + 9R^2x^2 - d^4)^3$$

and thus

$$c_2 : 9R^2(x^2 + y^2) - 12dR^2x + d^2(4R^2 - d^2) = 0 \quad (11)$$

is an equation of the desired circle. The fact that Eq. (11) appears three times as a factor of the resultant also shows that this circle is traced three times. The latter fact is caused by the so-called improper parametrization of  $c_2$  given in Eq. (11). The circle  $c_2$  is centered at  $M_2 = [\frac{2}{3}d, 0]$  and the radius equals  $\rho_2 = \frac{1}{3}(R - 2r)$ . Note that  $M_2$  is a triangle center of  $\Delta$  (not yet named or labelled, *cf.* [10]) for it is the reflection of  $X_3 = [0, 0]$  in  $X_{3576} = [\frac{1}{3}d, 0]$ .

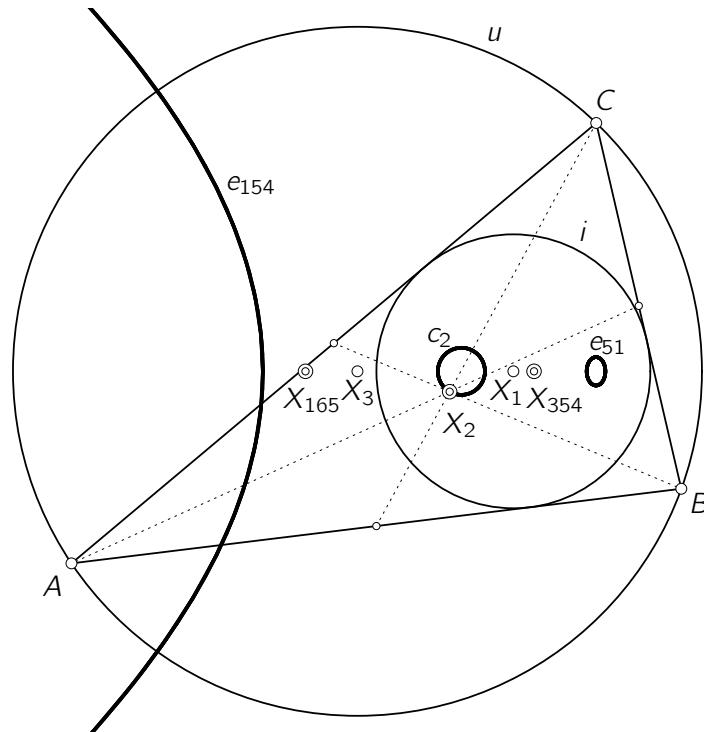


Figure 8: Poristic loci of some centroids:  $X_2$  is the centroid for  $\Delta$ ,  $\Delta_a$ , and  $\Delta_m$  at the same time and moves on a circle  $c_2$ , *cf.* Th. 4.6. The Weill point  $X_{354}$  (centroid of  $\Delta_i$ ) and the centroid  $X_{165}$  of  $\Delta_e$  remain fixed according to Th. 4.1. The centroids  $X_{51}$  and  $X_{154}$  of  $\Delta_o$  and  $\Delta_t$ , respectively, trace conic sections as stated in Th. 4.7.

The method shown so far applies to the orbit of any center listed above. For all other centers we only show how they are related to the vertices of  $\Delta$  and its deduced triangles  $\Delta_a$ ,  $\Delta_e$ ,  $\Delta_i$ ,  $\Delta_o$ ,  $\Delta_m$ ,  $\Delta_t$ , and  $\Delta_x$  in order to find a parametrization of the central orbit.



In the following the poristic path of the center  $X_i$  will be denoted by  $c_i$ . The center and radius of  $c_i$  shall be denoted by  $M_i$  and  $\rho_i$ .

$X_4$  is the orthocenter of  $\Delta$  and thus elementary to find. We have  $M_4 = X_{1482}$  and  $\rho_4 = R - 2r$ . The nine-point center  $X_5$  is the circumcenter of  $\Delta_m$  and  $M_5 = X_1$  and  $\rho_5 = \frac{1}{2}\rho_4$ . The Gergonne point  $X_7$  moves on  $c_7$  with  $M_7 = X_{1159}$  and  $\rho_7 = r\rho_4/(4R + r)$ . This fits to the results given in [8]. For the trace of the Nagel point we have  $M_8 = X_3$  and  $\rho_8 = \rho_4$ . The Mittenpunkt  $X_9$  leads to  $M_9 = [d(2R - r)/(4R + r), 0]$  and  $\rho_9 = 2R\rho_4/(4R + r)$ . The trace of the Spieker point  $X_{10}$  is centered at  $M_{10} = X_{1385}$  and has radius  $\rho_{10} = \frac{1}{2}\rho_4$ . The Feuerbach point is treated earlier, however, it moves on  $i$ . Since  $X_{12} = \mathcal{L}_{1,5} \cap \mathcal{L}_{2,56}$  we find  $M_{12} = X_1$  and  $\rho_{12} = r\rho_4/(R + 2r)$ . The de Longchamps point  $X_{20}$  is the orthocenter of  $\Delta_a$  and we find  $M_{20} = [-2d, 0]$  and  $\rho_{20} = \rho_4$ .

Since the Schiffler point is given by  $X_{21} = \mathcal{L}_{2,3} \cap \mathcal{L}_{7,56}$  we have  $M_{21} = [2Rd/(3R + 2r), 0]$  and  $\rho_{21} = R\rho_4/(3R + 2r)$ . The Far-Out point  $X_{23}$  is the inverse of  $X_2$  in the circumcircle and so we find  $M_{23} = [6R^3/(d(3R + 2r)), 0]$  and  $\rho_{23} = 3R^2/(3R + 2r)$ . The  $3^{rd}$  power point  $X_{32}$  is the intersection of  $\mathcal{L}_{1,4}$  and  $\mathcal{L}_{993,1007}$ . For the latter two points see below. We find  $M_{32} = X_{2099}$  and  $\rho_{32} = r\rho_3/(R + r)$ .

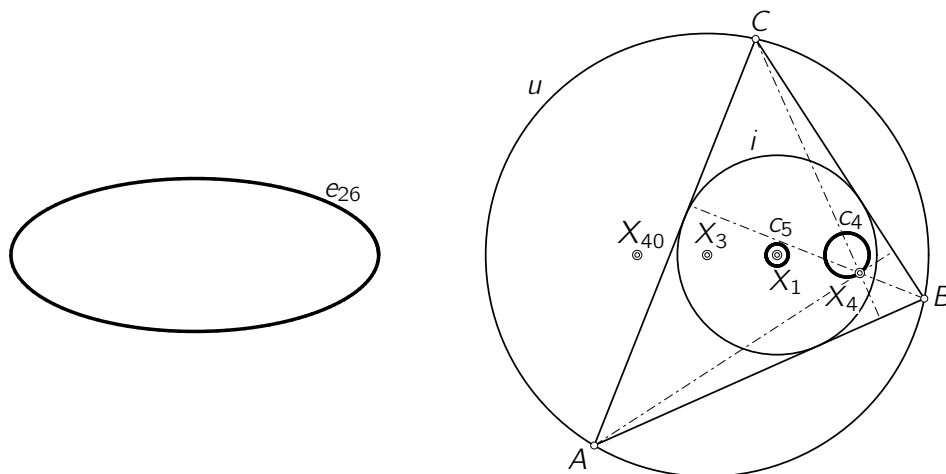


Figure 9: Poristic loci of some circumcenters:  $\Delta$ 's circumcenter  $X_3$  is fixed.  $X_1$  is the circumcenter of  $\Delta_i$ . The circumcenter of  $\Delta_a$  is the orthocenter of  $\Delta$  which is moving on the circle  $c_4$  (cf. Th. 4.6). The nine-point center  $X_5$  is the circumcenter of both,  $\Delta_m$  and  $\Delta_o$ . The circumcenter  $X_{26}$  of  $\Delta_t$  moves on  $e_{26}$  according to Th. 4.7. The Bevan point  $X_{40}$  is the circumcenter of  $\Delta_e$  and is fixed as shown in Th. 3.2.

The center  $X_{63}$  is the reflection of  $X_{1478}$  in  $X_{10}$ . Further  $X_{1478} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,36}$  and therefore  $M_{1478} = X_{2099}$  and  $\rho_{1478} = \rho_{32}$ . Consequently  $M_{63} = [-rd/(R+r), 0]$  and  $\rho_{63} = R\rho_4/(R+r)$ . The point  $X_{72}$  is found as the reflection of  $X_{65}$  in  $X_{10}$  and so  $M_{72} = [-rd/R, 0]$  and  $\rho_{72} = \rho_4$ . The 3<sup>rd</sup> Brocard point  $X_{76}$  is computed as  $X_{76} = \mathcal{L}_{3,98} \cap \mathcal{L}_{4,69}$ , with  $X_{98}$  being the reflection of  $X_8$  in  $X_{3416}$  and  $X_{3416}$  being the reflection of  $X_6$  in  $X_{10}$ . The center  $X_{69}$  is the symmedian point of  $\Delta_a$  and the reflection of  $X_8$  in  $X_{3416}$ . Thus we find  $M_{76} = X_{1482}$  and  $\rho_{76} = \rho_4$ .

We note that  $X_{78} = \mathcal{L}_{1,2} \cap \mathcal{L}_{210,958}$ , where  $X_{210}$  is the centroid of  $\Delta_x$  and  $X_{958} = \mathcal{L}_{1,6} \cap \mathcal{L}_{2,12}$ . This leads to  $M_{78} = [-rd/(R-r)]$ ,  $\rho_{78} = R\rho_4/(R-r)$ ;  $M_{210} = [d(R-r)/(3R), 0]$ ,  $\rho_{210} = \frac{2}{3}\rho_4$ ; and  $M_{958} = [Rd/(2R+r), 0]$ ,  $\rho_{958} = R\rho_4/(2R+r)$ . The center  $X_{80}$  appears as the reflection of  $\Delta$ 's incenter  $X_1$  in  $\Delta$ 's Feuerbach point  $X_{11}$  and therefore  $M_{80} = X_1$  and  $\rho_{80} = 2r$ . We find  $X_{84}$  as the reflection of  $X_{1490}$  in the circumcenter  $X_3$  with  $X_{1490} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,9}$ . So we obtain  $M_{84} = [d(2R-r)/r, 0]$ ,  $\rho_{84} = 2R\rho_4/r$  and  $M_{1490} = [-d(2R-r)/r, 0]$   $\rho_{1490} = \rho_{84}$ .

The trace of  $X_{90} = \mathcal{L}_{1,155} \cap \mathcal{L}_{40,80}$  is centered at  $M_{90} = [d(R-r)^2/(R^2 - 2Rr - r^2)]$  and has radius  $\rho_{90} = 2rR\rho_4/(r^2 - 2Rr - R^2)$ . For the computation of  $X_{40}$ ,  $X_{80}$ , and  $X_{155}$  (the latter being the orthocenter of  $\Delta_t$ ) see the proof of Th. 4.1. Since  $X_{94} = \mathcal{L}_{4,143} \cap \mathcal{L}_{23,98}$  we compute  $X_{143} = \frac{1}{2}(X_5 + X_{52})$  with  $X_{52}$  being the orthocenter of the orthic triangle  $\Delta_o$ . Thus  $M_{143} = [d(R+2r)/R, 0]$  and  $\rho_{143} = \rho_4^2/(4R)$ . Note that  $X_{143}$  is the nine-point center of  $\Delta_o$ , provided that  $\Delta$  is acute. We also have  $M_{94} = X_{1482}$  and  $\rho_{94} = \rho_4$ . The Tarry point  $X_{98}$  is the reflection of the Steiner point  $X_{99}$  in  $X_3$ .  $X_{99}$  is the common point of  $u$  and the Steiner ellipse different from  $A$ ,  $B$ , and  $C$ .

For the computation of  $X_{100}$  and  $X_{104}$  we refer to the proof of Th. 4.1. Then it is easily verified that  $X_{100}$ ,  $X_{104}$  are points on the circumcircle  $u$ . Since  $(X_{105}, X_{1292})$  is a pair of antipodal centers on  $u$ , their poristic locus equals  $u$ . For  $X_{119}$  see Th. 4.4. With  $X_{120} = \frac{1}{2}(X_4 + X_{1292})$  we find  $M_{120} = M_2$  and  $\rho_{120} = \frac{1}{3}\rho_4$ . Now  $X_{140} = \frac{1}{2}(X_3 + X_5)$  and thus  $M_{140} = X_{1385}$  and  $\rho_{140} = \frac{1}{4}\rho_4$ . Note that  $X_{140}$  is also the nine-point center of  $\Delta_m$ .

The Mittenpunkt of  $\Delta_m$  is denoted by  $X_{142}$  and appears as the midpoint of  $X_7$  and  $X_9$  and consequently we have  $M_{142} = [3d(2R+r)/(4R+r), 0]$  and  $\rho_{142} = (2R+r)\rho_4/(2(4R+r))$ .  $X_{144}$  comes along as the reflection of  $X_7$  in  $X_9$  and we find  $M_{144} = [-6rd/(4R+r), 0]$  and  $\rho_{144} = \rho_4(4R-r)/(4R+r)$ . The construction of  $X_{145}$  is already mentioned in the proof of Th. 4.1. We find  $M_{145} = X_{1482}$  and  $\rho_{145} = \rho_4$ . The center  $X_{149}$  appears as the reflection

of  $X_{20}$  in  $X_{104}$  and we observe  $M_{149} = X_{1482}$  and  $\rho_{149} = R + 2r$ .  $X_{153}$  is the reflection of  $X_{20}$  in  $X_{100}$  and we find  $M_{153} = X_{1482}$  and  $\rho_{153} = 3R - 2r$ .

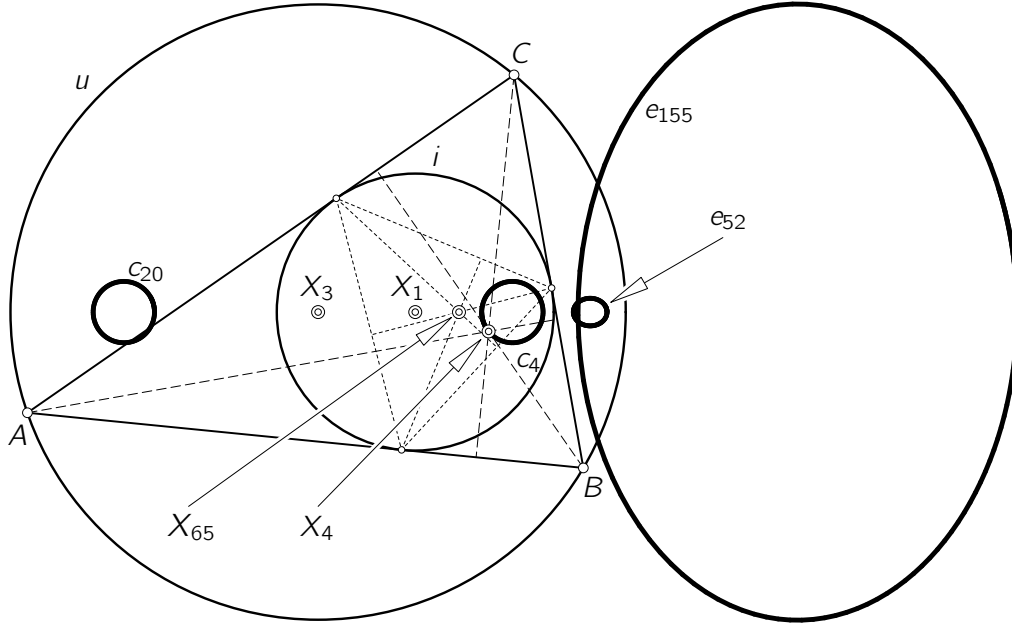


Figure 10: Poristic loci of some orthocenters:  $\Delta$ 's orthocenter  $X_4$  circles along  $c_4$ . The de Longchamps point  $X_{20}$  runs on the circle  $c_{20}$  (cf. Th. 4.6).  $X_1$ ,  $X_3$ , and  $X_{65}$  are the orthocenter  $\Delta_e$ ,  $\Delta_m$ , and  $\Delta_i$ . According to Th. 3.2  $X_{65}$  remains fixed. The orthocenters  $X_{52}$  and  $X_{155}$  of  $\Delta_o$  and  $\Delta_t$  travel along conic sections  $e_{52}$  and  $e_{155}$ , respectively, cf. Th. 4.7.

The center  $X_{186}$  is the inverse of  $X_4$  in the circumcircle and so it is no surprise that its poristic path is a circle. It is centered at  $M_{186} = [2R^3/(d(3R+2r)), 0]$  and has radius  $\rho_{186} = \frac{1}{3}\rho_{23}$ . We reflect the incenter  $X_1$  in the Schiffler point  $X_{21}$  and arrive at  $X_{191}$ . This results in  $M_{191} = [d(R-2r)/(3R+2r), 0]$  and  $\rho_{191} = 2\rho_{21}$ . The center  $X_{200}$  is the intersection of  $\mathcal{L}_{1,2}$  with  $\mathcal{L}_{40,64}$ , where  $X_{64}$  is the reflection of  $X_{1498}$  in  $X_3$  and  $X_{1498} = \mathcal{L}_{1,84} \cap \mathcal{L}_{4,6}$ .  $X_{200}$  traces a circle centered at  $M_{200} = [-rd/(2R-r), 0]$  and with radius  $\rho_{200} = 2R\rho_4/(2R-r)$ . Since  $X_{214} = \frac{1}{2}(X_1 + X_{100})$  we have  $M_{214} = X_{1385}$  and  $\rho_{214} = \frac{1}{2}R$ .

$X_{226}$  is the reflection of  $X_{993}$  in  $X_{1125}$ . For the construction of the latter two we refer to the proof of Th. 4.1. So we obtain the data of three poristic traces:  $M_{226} = [d(2R+3r)/(2(R+r)), 0]$ ,  $\rho_{226} = \frac{1}{2}\rho_{32}$ ;  $M_{993} = [dR/(2(R+r)), 0]$ ,  $\rho_{993} = \frac{1}{2}\rho_{63}$ ; and  $M_{1125} = [\frac{3}{4}d, 0]$ ,  $\rho_{1125} = \frac{1}{4}\rho_4$ . Reflecting  $X_{23}$  in  $X_{110}$  gives  $X_{323}$  moving on a circle with center  $M_{323} = [-6R^3/(d(3R+2r)), 0]$  and

$\rho_{323} = R(9R + 4r)/(3R + 2r)$ . Reflecting  $X_{2093}$  in the Spieker center  $X_{10}$  we obtain  $X_{329}$  and then  $M_{329} = [-4rd/(2R - r), 0]$  and  $\rho_{329} = \rho_4$ . With  $X_{347} = \frac{1}{2}(X_2 + X_5)$  we find  $M_{347} = [\frac{5}{6}d, 0]$  and  $\rho_{347} = \frac{5}{12}\rho_4$ .

For the Fuhrmann center  $X_{355} = \frac{1}{2}(X_4 + X_8)$  we find  $M_{355} = X_1$  and  $\rho_{355} = \rho_4$ . Since  $X_{376} = \frac{1}{2}(X_2 + X_{20})$  and  $X_{381} = \frac{1}{2}(X_2 + X_4)$  we find  $M_{376} = [-\frac{2}{3}d, 0]$ ,  $\rho_{376} = \frac{1}{3}\rho_4$  and  $M_{381} = [\frac{4}{3}d, 0]$ ,  $\rho_{381} = \frac{2}{3}\rho_4$ . The reflection of the circumcenter in the orthocenter yields  $X_{382}$  with  $M_{382} = [4d, 0]$  and  $\rho_{382} = 2\rho_4$ .

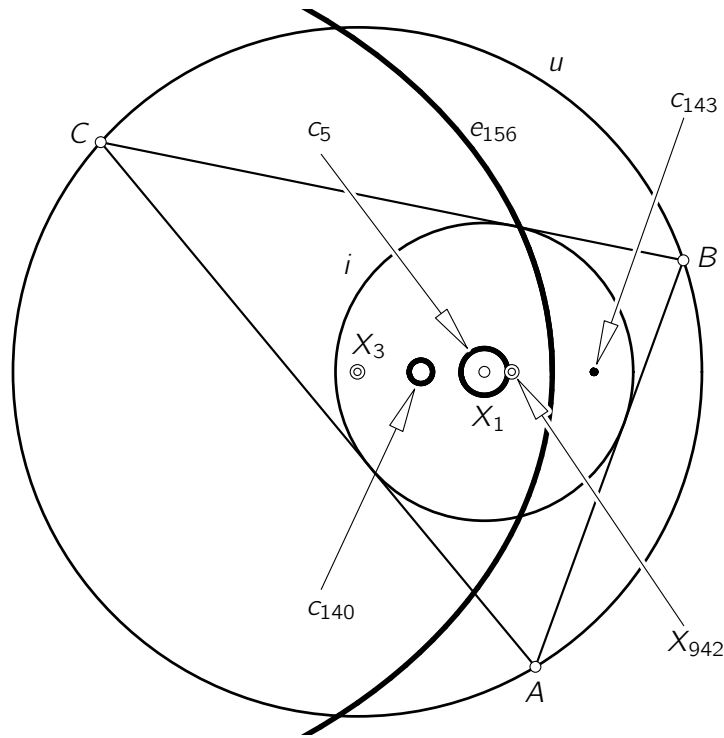


Figure 11: Poristic loci of some nine-point centers:  $X_5$  moves on  $c_5$ . The centers  $X_{140}$  and  $X_{143}$  are the nine-point centers of  $\Delta_m$  and  $\Delta_o$ , respectively. According to Th. 4.6 their poristic loci are the circles  $c_{140}$  and  $c_{143}$ . The nine-point center of  $\Delta_t$  is the point  $X_{156}$ . Its poristic orbit is the conic section  $e_{156}$ , cf. Th. 4.7.  $\Delta$ 's circumcenter  $X_3$  plays a double-role: It is the nine-point center of  $\Delta_a$  and  $\Delta_e$ . The nine-point center of  $\Delta_i$  is the same point for all triangles in the poristic system, i.e.,  $X_{942}$  is fixed, see Th. 4.1.

The center  $X_{388} = \mathcal{L}_{1,4} \cap \mathcal{L}_{7,8}$  runs on a circle with center  $M_{388} = [2d(R + r)/(2R + r), 0]$  and radius  $\rho_{388} = r\rho_4/(2R + r)$ . We reflect the Gergonne point  $X_7$  in the incenter  $X_1$  in order to obtain  $X_{390}$ . So we have  $M_{390} =$

$[2d(2R - r)/(4R + r)]$  and  $\rho_{390} = \rho_7$ . With  $X_{392} = \mathcal{L}_{1,6} \cap \mathcal{L}_{9,11}$  we arrive at  $M_{392} = M_9$  and  $\rho_{392} = \rho_9$ . The Parry reflection point  $X_{399}$  is the reflection of  $X_3$  in  $X_{110}$  therefore we have  $M_{399} = X_3$  and  $\rho_{399} = 2R$ . The complement of the Schiffler point is  $X_{442} = \mathcal{L}_{2,3} \cap \mathcal{L}_{11,214}$  and its trace is centered at  $M_{442} = [2d(R + r)/(3R + 2r), 0]$  and has radius  $\rho_{442} = (R + r)\rho_4/(3R + 2r)$ .

The Johnson midpoint is computed as  $X_{495} = \mathcal{L}_{1,5} \cap \mathcal{L}_{4,390}$  and we derive  $M_{495} = X_1$  and  $\rho_{495} = \frac{1}{2}\rho_{32}$ . For  $X_{496} = \mathcal{L}_{1,5} \cap \mathcal{L}_{36,550}$  we determine  $X_{550} = \frac{1}{2}(X_2 + X_{20})$ . This intermediate result yields  $M_{550} = X_{40}$ ,  $\rho_{550} = \frac{1}{2}\rho_4$  and  $M_{496} = X_1$ ,  $\rho_{496} = r\rho_4/(2\rho_{119})$ . With  $X_{497} = \mathcal{L}_{1,4} \cap \mathcal{L}_{2,11}$ ,  $X_{498} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,12}$ , and  $X_{499} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,11}$  we find  $M_{497} = [2d(R - r)/(2R - r), 0]$ ,  $\rho_{497} = r\rho_4/(2R - r)$ ;  $M_{498} = [d(R + 2r)/(R + 3r), 0]$ ,  $\rho_{498} = r\rho_4/(R + 3r)$ ; and  $M_{499} = X_{1388}$ ,  $\rho_{499} = r\rho_4/(3r - R)$ .

We compute  $X_{501} = \mathcal{L}_{21,214} \cap \mathcal{L}_{36,58}$  with  $X_{58} = \mathcal{L}_{1,21} \cap \mathcal{L}_{3,6}$  which leads to  $M_{501} = M_{21}$  and  $\rho_{501} = \rho_{21}$ . The next five centers are midpoints of centers:  $X_{546} = \frac{1}{2}(X_4 + X_5)$ ,  $X_{547} = \frac{1}{2}(X_2 + X_5)$ ,  $X_{548} = \frac{1}{2}(X_5 + X_{20})$ ,  $X_{549} = \frac{1}{2}(X_2 + X_3)$ , and  $X_{551} = \frac{1}{2}(X_1 + X_2)$ . So we find  $M_{546} = [\frac{3}{2}d, 0]$ ,  $\rho_{546} = \frac{3}{4}\rho_4$ ;  $M_{547} = M_{347}$ ,  $\rho_{547} = \frac{5}{12}\rho_4$ ;  $M_{548} = X_{3579}$ ,  $\rho_{548} = \frac{1}{4}\rho_4$ ; and  $M_{549} = X_{3576}$ ,  $M_{551} = M_{347}$ ,  $\rho_{549} = \rho_{551} = \frac{1}{6}\rho_4$ .  $X_{631}$  is the reflection of  $X_4$  in  $X_{3091}$ . Therefore we have to determine  $X_{3091} = \mathcal{L}_{2,3} \cap \mathcal{L}_{11,153}$ . This gives  $M_{631} = [\frac{2}{5}d, 0]$ ,  $\rho_{631} = \frac{1}{5}\rho_4$  and  $M_{3091} = [\frac{6}{5}d, 0]$ ,  $\rho_{3091} = \frac{3}{5}\rho_4$ . Then  $X_{632}$  appears as the reflection of  $X_{3091}$  in the circumcenter  $X_3$  and we find  $M_{632} = [\frac{3}{5}d, 0]$  and  $\rho_{632} = \frac{3}{10}\rho_4$ . With  $X_{759} = \mathcal{L}_{10,21} \cap \mathcal{L}_{58,65}$  we can verify that  $X_{759}$  travels on  $u$ .

The point Acubens  $X_{908}$  is the intersection of  $\mathcal{L}_{2,7}$  and  $\mathcal{L}_{12,960}$ . So we compute  $X_{960} = \frac{1}{2}(X_1 + X_{72})$ . Since  $X_{908}$  is the reflection of  $X_{1512}$  in  $X_{119}$ , we obtain  $X_{1512}$  as reflection of  $X_{908}$  in  $X_{119}$ . Thus we have  $M_{960} = [d(R - r)/(2R), 0]$ ,  $\rho_{960} = \frac{1}{2}\rho_4$ ;  $M_{908} = [-3rR/d, 0]$ ,  $\rho_{908} = \rho_{119}$ ; and  $M_{1512} = [R(2R - r)/d, 0]$ ,  $\rho_{1512} = \rho_{119}$ . We find  $X_{920} = \mathcal{L}_{1,21} \cap \mathcal{L}_{4,46}$  and therefore we have  $M_{920} = [d(R^2 + r^2)/(R^2 - Rr - r^2), 0]$  and  $\rho_{920} = rR\rho_4/(R^2 - Rr - r^2)$ . If we intersect  $\mathcal{L}_{1,2}$  with the lines  $\mathcal{L}_{3,9}$  and  $\mathcal{L}_{4,7}$  we find  $X_{936}$  and  $X_{938}$ , respectively. The centers and radii of their paths are  $M_{936} = [d(2R - r)/(4R - r), 0]$ ,  $\rho_{936} = 2R\rho_4/(4R - r)$  and  $M_{938} = [4dR/(4R - r), 0]$ ,  $\rho_{938} = r\rho_4/(4R - r)$ . For  $X_{943} = \mathcal{L}_{3,7} \cap \mathcal{L}_{4,12}$  we find  $M_{943} = [4dR(R - r)/(4R^2 + 7Rr + 2r^2), 0]$  and  $\rho_{943} = rR\rho_4/(4R^2 + 7Rr + 2r^2)$ .

The Hofstadter-Trapezoid point  $X_{944}$  is the midpoint in between  $X_{20}$  and  $X_{145}$ . Therefore we have  $M_{944} = X_3$  and  $\rho_{944} = \rho_4$ . The center  $X_{946} = \frac{1}{2}(X_1 + X_4)$  traces a circle with center  $M_{946} = M_{546}$  and radius  $\rho_{946} = \frac{1}{2}\rho_4$ . As intercept

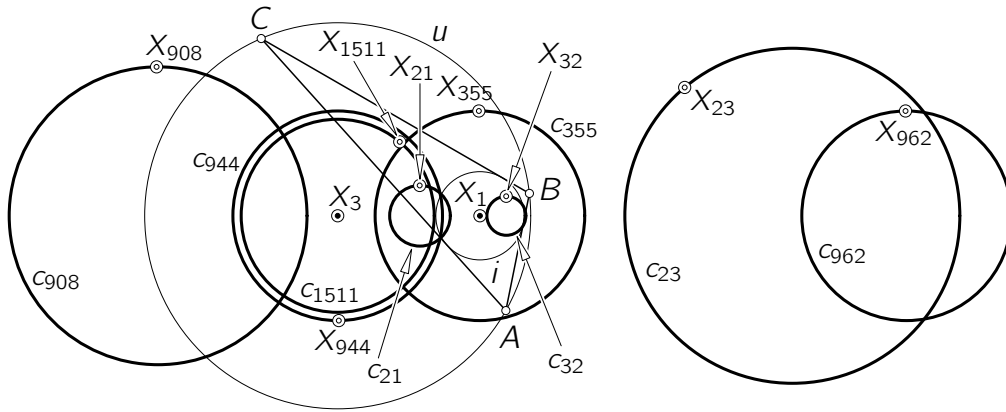


Figure 12: Poristic loci of the Schiffler point  $X_{21}$ , the Far-Out point  $X_{23}$ , the  $3^{rd}$  power point  $X_{32}$ , the Fuhrmann center  $X_{355}$ , the point Acubens  $X_{908}$ , the Hofstadter-Trapezoid point  $X_{944}$ , the Longuet-Higgins point  $X_{962}$ , and the Fermat crosssum  $X_{1511}$ .

of  $\mathcal{L}_{1,4}$  and  $\mathcal{L}_{8,9}$  we obtain the point  $X_{950}$  and  $M_{950} = [d(2R - r)/2R, 0]$  and  $\rho_{950} = r\rho_4/(2R)$ . The central line  $\mathcal{L}_{1,6}$  carries the three centers  $X_{954}$  and  $X_{956}$ , which also lie in the central lines  $\mathcal{L}_{3,7}$  and  $\mathcal{L}_{3,8}$ , respectively. We find  $M_{954} = [4dR/(4R + r), 0]$ ,  $\rho_{954} = Rr\rho_4/(R + r)(4R + r)$  and  $M_{956} = X_3$ ,  $\rho_{956} = \rho_{63}$ . The Longuet-Higgins point is the reflection of the Nagel point  $X_8$  in the orthocenter  $X_4$ . This yields  $M_{962} = M_{382}$  and  $\rho_{962} = \rho_4$ . The midpoint  $X_{997}$  of  $X_1$  and  $X_{200}$  determines  $M_{997} = [d(R - r)/(2R - r), 0]$  and  $\rho_{997} = \frac{1}{2}\rho_{200}$ .

Since  $X_{1001} = \frac{1}{2}(X_1 + X_9)$  we have  $M_{1001} = [3Rd/(4R + r), 0]$  and  $\rho_{1001} = \frac{1}{2}\rho_9$ . For  $X_{1004} = \mathcal{L}_{2,3} \cap \mathcal{L}_{7,100}$  we compute  $M_{1004} = [2Rd(R + r)/(3R^2 - Rr - r^2), 0]$  and  $\rho_{1004} = R(R + r)\rho_4/(3R^2 - Rr - r^2)$ . The centers  $X_{1005}$  and  $X_{1007}$  are located on the Euler line and on the central lines  $\mathcal{L}_{9,100}$  and  $\mathcal{L}_{4,99}$ , respectively. We derive  $M_{1005} = [2Rd(2R - r)/(6R^2 + 5Rr + 2r^2), 0]$ ,  $\rho_{1005} = R(2R - r)\rho_4/(6R^2 + 5Rr + 2r^2)$  and  $M_{1007} = M_2$ ,  $\rho_{1007} = \frac{1}{3}\rho_4$ . The  $3^{rd}$  Ehrmann point  $X_{1145} = \frac{1}{2}(X_8 + X_{100})$  leads to  $M_{1145} = X_3$  and  $\rho_{1145} = \rho_{119}$ . The center  $X_{1156}$  is found as the midpoint of  $X_9$  and  $X_{100}$ . Its circular path is centered at  $M_{1156} = M_{390}$  and has radius  $\rho_{1156} = 9rR/(4R + r)$ . The circumcenter of the extouch triangle  $\Delta_x$  is given by  $X_{1158} = \frac{1}{2}(X_{40} + X_{84})$  and its poristic locus is centered at  $M_{1158} = [d(R - r)/r, 0]$  and has radius  $\rho_{1158} = \frac{1}{2}\rho_{84}$ . The center  $X_{1210} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,950}$  yields  $M_{1210} = [d(2R - r)/(2R), 0]$  and  $\rho_{1210} = \rho_{946}$ .

Since  $X_{1317}$  is the reflection of  $X_{11}$  and  $X_1$  it is easy to find a parametrization of its path which is the incircle. The path of the midpoint  $X_{1320} = \frac{1}{2}(X_{145} +$

$X_{149}$ ) is centered at  $M_{1320} = X_{1482}$  and is congruent to  $u$  for  $\rho_{1320} = R$ . The Fletcher point  $X_{1323}$  is the inverse of the Gergonne point  $X_7$  in the incircle and its trace is centered at  $M_{1323} = [R(2R - r)/(2d), 0]$  and has radius  $\rho_{1323} = \frac{1}{2}r$ . The inverse  $X_{1324}$  of the Spieker point  $X_{10}$  in the incircle moves on a circle centered at  $M_{1324} = [R^3/(rd), 0]$  with radius  $\rho_{1324} = R^2/r$ . The inverse  $X_{1325}$  of the Schiffler point  $X_{21}$  in the incircle has an orbit centered at  $M_{1325} = [2R^2/d, 0]$  with  $\rho_{1325} = R$  for its radius. The center  $X_{1329} = \frac{1}{2}(X_8 + X_{2098})$ , where  $X_{2098}$  is known from Th. 4.1 and its proof, respectively, gives  $M_{1329} = [d(R - 2r)/(2(R - r)), 0]$  and  $\rho_{1329} = \frac{1}{2}\rho_4$ . The exsimilicenter of the circumcircle and the Spieker circle is given by  $X_{1376} = \mathcal{L}_{3,10} \cap \mathcal{L}_{8,56}$ . Its poristic locus is centered at  $M_{1376} = [Rd/(2R - r), 0]$  and has radius  $\rho_{1376} = \frac{1}{2}\rho_{200}$ .  $X_{1387} = \frac{1}{2}(X_1 + X_{11})$  has a circular path centered at  $M_{1387} = X_1$  and  $\rho_{1387} = \frac{1}{2}r$ . From  $X_{1479} = \mathcal{L}_{1,4} \cap \mathcal{L}_{3,11}$  we derive  $M_{1479} = X_{2098}$  and  $\rho_{1479} = 2\rho_{946}$ .

The centers  $X_{1483}$  and  $X_{1484}$  appear as reflections of  $X_5$  in  $X_1$  and  $X_5$  in  $X_{11}$ , respectively. We have  $M_{1483} = M_{1484} = X_1$ ,  $\rho_{1483} = \frac{1}{2}\rho_4$ ,  $\rho_{1484} = \frac{1}{2}\rho_{149}$ . The Fermat crosssum  $X_{1511} = \frac{1}{2}(X_3 + X_{110})$  runs on a circle concentric with  $u$  and thus  $M_{1511} = X_3$  and  $\rho_{1511} = \frac{1}{2}R$ . The construction of  $X_{1519}$  produces a lot of useful byproducts since  $X_{1519}$  is the reflection of  $X_{1532}$  in  $X_{1538}$ , where  $X_{1538}$  is the reflection of  $X_{1512}$  in  $X_{1537}$ . Since  $X_{1537} = \mathcal{L}_{4,145} \cap \mathcal{L}_{11,65}$  and  $X_{1532} = \mathcal{L}_{2,3} \cap \mathcal{L}_{12,946}$  we find:  $M_{1519} = [R(2R - 3r)/d, 0]$ ,  $M_{1532} = [2R(R - r)/d, 0]$ ,  $M_{1537} = X_{1482}$ ,  $M_{1538} = [R(4R - 5r)/(2d), 0]$  and  $\rho_{1519} = \rho_{1532} = \rho_{1537} = \rho_{1538} = \rho_{119}$ . The center  $X_{1656}$  is the intersection of the Euler line with  $\mathcal{L}_{17,18}$ . Without explicitly knowing the latter two points we find  $X_{1656}$  as the reflection of  $X_5$  in  $X_{632}$  and this gives  $M_{1656} = [\frac{4}{5}d, 0]$  and  $\rho_{1656} = \frac{2}{5}\rho_4$ . Reflecting the de Longchamps point  $X_{20}$  in the circumcenter  $X_3$  we find  $X_{1657}$ . Its trace has center  $M_{1657} = [-4d, 0]$  and radius  $\rho_{1657} = 2\rho_4$ .

The poristic locus of the center  $X_{1698} = \mathcal{L}_{1,2} \cap \mathcal{L}_{5,40}$  is the circle with center  $M_{1698} = M_{632}$  and has radius  $\rho_{1698} = \frac{2}{5}\rho_4$ . Since  $X_{1699}$  shows up as the reflection of  $X_{165}$  in the centroid  $X_2$  we find  $M_{1699} = [\frac{5}{3}d, 0]$  and  $\rho_{1699} = \frac{1}{3}\rho_4$ . The exsimilicenter of the Bevan circle and the Spieker circle is the triangle center  $X_{1706}$  which is the reflection of  $X_{2551}$  in the Spieker point  $X_{10}$ . So we compute  $X_{2551} = \mathcal{L}_{4,9} \cap \mathcal{L}_{2,12}$  and find  $M_{2551} = [2d(R - r)/(4R - r), 0]$ ,  $\rho_{2551} = \rho_4(2R - r)/(4R - r)$  and  $M_{1706} = [d(2R + r)/(4R - r), 0]$ ,  $\rho_{1706} = \rho_{936}$ .

The midpoint of  $X_{36}$  and  $X_{80}$  is labelled  $X_{1737}$  and rotates about  $M_{1737} = X_{1319}$  at distance  $\rho_{1737} = r$ . The reflection of  $X_1$  in  $X_{497}$  equals the point  $X_{3586}$ . This enables us to construct  $X_{1750}$  as the reflection of  $X_{3586}$  in the orthocenter

$X_4$ . From that we obtain  $M_{1750} = [d(6R - r)/(2R - r), 0]$ ,  $\rho_{1750} = 2\rho_{200}$  and  $M_{3586} = [d(2R - 3r)/(2R - r), 0]$ ,  $\rho_{3586} = 2\rho_{497}$ . The point  $X_{1785}$  is the inverse of  $X_{946}$  in the incircle. It is circling around  $M_{1785} = X_{1319}$  with  $\rho_{1785} = r$ . For the center  $X_{1837} = \mathcal{L}_{1,5} \cap \mathcal{L}_{4,65}$  we find  $M_{1837} = X_1$  and  $\rho_{1837} = 2\rho_{946}$ . The center  $X_{1851} = \mathcal{L}_{4,8} \cap \mathcal{L}_{25,105}$ , where  $X_{25} = \mathcal{L}_{2,3} \cap \mathcal{L}_{6,51}$  and with  $X_{51}$  being the centroid of  $\Delta_o$  and  $M_{1851} = X_{1482}$ ,  $\rho_{1851} = \rho_4$ . With  $X_{1858} = \mathcal{L}_{1,90} \cap \mathcal{L}_{4,65}$  we get  $M_{1858} = [d(R^2 + r^2)/R^2, 0]$  and  $\rho_{1858} = 2\rho_{950}$ . Reflecting  $X_{65}$  in  $X_{1837}$  gives  $X_{1898}$  and thus  $M_{1898} = X_{3057}$  and  $\rho_{1898} = 4\rho_{496}$ . The point  $X_{1899} = \mathcal{L}_{1,98} \cap \mathcal{L}_{4,51}$  is rotating about  $M_{1899} = X_{1482}$  at distance  $\rho_{1899} = \rho_4$ .

The inverse of  $X_5$  and  $X_{20}$  in the circumcircle yield  $X_{2070}$  and  $X_{2071}$  which are rotating about  $M_{2070} = [4R^3/(d(3R + 2r)), 0]$  at distance  $\rho_{2070} = \frac{2}{3}\rho_{23}$  and  $M_{2071} = [-2R^3/(d(3R + 2r)), 0]$  at distance  $\rho_{2071} = \frac{1}{3}\rho_{23}$ . The reflection of  $X_2$  and  $X_4$  in  $X_{57}$  yields  $X_{2094}$  and  $X_{2096}$ , respectively. From that we conclude that  $M_{2094} = [8d(R + r)/(3(2R - r)), 0]$ ,  $\rho_{2094} = \frac{1}{3}\rho_4$  and  $M_{2096} = [4rd/(2R - r), 0]$ ,  $\rho_{2096} = \rho_4$ . With  $X_{2478} = \mathcal{L}_{2,3} \cap \mathcal{L}_{8,210}$  we find  $M_{2478} = [2d(R - r)/(3R - r), 0]$  and  $\rho_{2478} = \rho_4\rho_{119}/(3R - r)$ . The midpoint  $X_{2550}$  of the Gergonne and Nagel point determines  $M_{2550} = [2d(R + r)/(4R + r), 0]$  and  $\rho_{2550} = 2\rho_{142}$ .

We find  $X_{2886} = \frac{1}{2}(X_1 + X_{3419})$  with  $X_{3419}$  as the reflection of  $X_{55}$  in  $X_{10}$ . This leads to  $M_{2886} = [d(R + 2r)/(2(R + r)), 0]$ ,  $\rho_{2886} = \frac{1}{2}\rho_4$  and  $M_{3419} = [rd/(R + r), 0]$ ,  $\rho_{3419} = \rho_4$ , respectively. The point  $X_{2932}$  is the inverse of  $X_{1145}$  in the circumcircle. It is rotating about  $M_{2932} = X_3$  at distance  $\rho_{2932} = R^2/\rho_{119}$ . The center  $X_{2948}$  comes up as the reflection of  $X_{3448}$  in the Spieker center  $X_{10}$ . For that we determine  $X_{3448}$  as the reflection of  $X_{20}$  in  $X_{74}$  with the latter being  $X_{74} = \mathcal{L}_{20,68} \cap \mathcal{L}_{72,100}$ , where  $X_{68}$  is the reflection of  $X_5$  in  $X_{155}$ . We find  $M_{2948} = X_{40}$  and  $\rho_{2948} = 2R$ .

With  $X_{3036} = \frac{1}{2}(X_8 + X_{11})$  we find  $M_{3036} = X_{1385}$  and  $\rho_{3036} = \frac{1}{2}(3r - R)$ . Then  $X_{3059} = \mathcal{L}_{7,8} \cap \mathcal{L}_{9,55}$  and we get  $M_{3059} = [-dr(R + r)/(R(4R + r)), 0]$  and  $\rho_{3059} = 4\rho_{142}$ . For  $X_{3060} = \mathcal{L}_{2,51} \cap \mathcal{L}_{4,52}$  we find the center and radius of its circular path:  $M_{3060} = [4d(R + 2r)/(3R), 0]$  and  $\rho_{3060} = \frac{4}{3}\rho_{143}$ . We intersect the line  $\mathcal{L}_{1,2}$  with  $\mathcal{L}_{4,12}$  and  $\mathcal{L}_{4,11}$  and get  $X_{3085}$  and  $X_{3086}$ , respectively. The centers and radii of the respective poristic loci are:  $M_{3085} = [2d(R + r)/(2R + 3r), 0]$ ,  $\rho_{3085} = r\rho_4/(2R + 3r)$  and  $M_{3086} = [r(R - 2r)/(2R - 3r), 0]$ ,  $\rho_{3086} = r\rho_4/(2R - 3r)$ . The center  $X_{3110}$  is the inverse of  $X_{3286}$  in the circumcircle and  $X_{3286} = \mathcal{L}_{3,6} \cap \mathcal{L}_{7,21}$ . So we have  $M_{3110} = X_{1385}$  and  $\rho_{3110} = \frac{1}{2}d$ . We compute  $X_{3219} = \mathcal{L}_{2,7} \cap \mathcal{L}_{8,90}$  and find  $M_{3219} = [2d(R - r)/(5R + 2r), 0]$  and



$\rho_{3219} = 3R\rho_4/(5R+2r)$ . The center  $X_{3241} = \frac{1}{2}(X_2 + X_{145})$  moves on a circle centered at  $M_{3241} = M_{381}$  with radius  $\rho_{3241} = \frac{1}{3}\rho_4$ .

We reflect  $X_8$  in  $X_{142}$  and arrive at  $X_{3243}$  with  $M_{3243} = [3d(2R+r)/(4R+r), 0]$  and  $\rho_{3243} = \rho_9$ . The reflection of the Spieker center  $X_{10}$  in the incenter  $X_1$  is named  $X_{3244}$  and circles around  $M_{3244} = M_{546}$  with  $\rho_{3244} = \frac{1}{2}\rho_4$ . Now  $X_{3254}$  is the reflection of the Mittenpunkt  $X_9$  in the Feuerbach point  $X_{11}$  and we get  $M_{3254} = M_{3243}$  and  $\rho_{3254} = 2(R+r)^2/(4R+r)$ . The point  $X_{3305} = \mathcal{L}_{2,7} \cap \mathcal{L}_{210,1001}$  traces a circle with center  $M_{3305} = [d(4R-r)/(7R+r), 0]$  and radius  $\rho_{3305} = 3R\rho_4/(7R+r)$ . The reflection of  $X_{3328}$  in  $X_1$  yields  $X_{3322}$ , where  $X_{3328}$  is computed as the reflection of  $X_{1155}$  in  $X_{1323}$ . Note that  $X_{1155}$  is the reflection of  $X_1$  in  $X_{3245}$ . Now it is easily verified that  $X_{3322}$  and  $X_{3328}$  run on the incircle. The center  $X_{3358} = \frac{1}{2}(X_9 + X_{84})$  determines  $M_{3358} = [d(4R^2 - r^2)/(r(4R+r)), 0]$  and  $\rho_{3358} = 2R\rho_4(2R+r)/(r(4R+r))$ .

The reflection of  $X_8$  in  $X_{3419}$  yields  $X_{3434}$  with circular orbit centered at  $M_{3434} = [2rd/(R+r), 0]$  and radius  $\rho_{3434} = \rho_4$ . We construct  $X_{3452}$  as the intersection of the central lines  $\mathcal{L}_{2,7}$  and  $\mathcal{L}_{5,10}$  and find the center of the circular orbit  $M_{3452} = [d(2R-3r)/(2(2R-r)), 0]$  and the radius  $\rho_{3452} = \frac{1}{2}\rho_4$ . This allows to compute  $X_{3421}$  as the reflection of  $X_1$  in  $X_{3452}$  and we find  $M_{3421} = [-2rd/(2R-r), 0]$  and  $\rho_{3421} = \rho_4$ . From  $X_{3474} = \mathcal{L}_{4,46} \cap \mathcal{L}_{7,55}$  we get  $M_{3474} = [2d(R+2r)/(2R-r), 0]$  and  $\rho_{3474} = \rho_{497}$ . On the central line  $\mathcal{L}_{1,4}$  we find the next four centers: We intersect with  $\mathcal{L}_{8,56}$ ,  $\mathcal{L}_{7,55}$ ,  $\mathcal{L}_{7,21}$ , and  $\mathcal{L}_{8,21}$  and obtain  $X_{3473}$ ,  $X_{3475}$ ,  $X_{3485}$ , and  $X_{3486}$ , respectively. Their poristic orbits are centered at  $M_{3473} = X_{999}$ ,  $M_{3475} = [2d(3R+2r)/(3(2R+r)), 0]$ ,  $M_{3485} = [2d(R+2r)/(2R+3r), 0]$ , and  $M_{3486} = [2Rd/(2R+r), 0]$  and have radii  $\rho_{3472} = \rho_{497}$ ,  $\rho_{3475} = \frac{1}{3}\rho_{388}$ ,  $\rho_{3485} = \rho_{3085}$ , and  $\rho_{3486} = \rho_{388}$ .

The reflection of  $X_{361}$  in the circumcenter  $X_3$  leads to  $X_{3522}$  with  $M_{3522} = [-\frac{2}{5}d, 0]$  and  $\rho_{3522} = \frac{1}{5}\rho_4$ . The center  $X_{3534}$  is the reflection of  $X_{382}$  in  $X_{381}$  and rotates about  $M_{3534} = [-\frac{4}{3}d, 0]$  with  $\rho_{3534} = \frac{2}{3}\rho_4$ . Reflecting  $X_{3534}$  in  $X_5$  we find  $X_{3543}$  and  $M_{3543} = [\frac{10}{3}d, 0]$  and  $\rho_{3543} = \frac{5}{3}\rho_4$ . The Dosa point  $X_{3555}$  is the reflection of  $X_{72}$  in the incenter  $X_1$  and circles about  $M_{3555} = [d(2R+r)/R, 0]$  at distance  $\rho_{3555} = \rho_4$ . On the central line parallel to the Euler line through the Feuerbach point  $X_{11}$  we find  $X_{3582}$  and  $X_{3583}$  by intersecting with  $\mathcal{L}_{1,2}$  and  $\mathcal{L}_{1,4}$ , respectively. This yields circular orbits with centers  $M_{3582} = [R(3R-4r)/(3d), 0]$  and  $M_{3583} = [R(R-4r)/d, 0]$  and radii  $\rho_{3582} = \frac{2}{3}r$  and  $\rho_{3583} = 2r$ , respectively. Since  $X_{3584} = \mathcal{L}_{1,2} \cap \mathcal{L}_{11,547}$  we find  $M_{3584} = [d(3R+4r)/(3(R+2r)), 0]$  and  $\rho_{3584} = \frac{2}{3}\rho_{12}$ . On the central line  $\mathcal{L}_{1,4}$  we find  $X_{3585}$  and  $X_{3586}$  as intersections with  $\mathcal{L}_{5,36}$  and  $\mathcal{L}_{30,57}$ , respectively.

Their poristic paths are centered at  $M_{3585} = [d(R + 4r)/(R + 2r), 0]$  and  $M_{3586} = [d(2R - 3r)/(2R - r), 0]$ . The respective radii are  $\rho_{3585} = 2\rho_{12}$  and  $\rho_{3586} = 2\rho_{497}$ . For  $X_{3589} = \mathcal{L}_{4,5} \cap \mathcal{L}_{8,10}$  we find  $M_{3589} = M_2$  and  $\rho_{3589} = \frac{1}{3}\rho_4$ . Finally the center  $X_{3600} = \mathcal{L}_{1,7} \cap \mathcal{L}_{8,57}$  circles around  $M_{3600} = [2d(2R + r)/(4R - r), 0]$  at distance  $\rho_{3600} = \rho_{938}$ .  $\square$

#### 4.4 Centers moving on conic sections

In this last section we focus on triangle centers that run on conic sections while  $\Delta$  is moving through its poristic family. We shall give the semiaxes and center of the poristic paths only for some prominent centers and in the cases where these (centers and axes) are relatively simple functions in  $R$ ,  $r$ , and  $d$ . We shall skip the lengthy discussion under which circumstances the poristic loci of triangle centers mentioned here are ellipses or hyperbolae. We can show:

**Theorem 4.7.** *The triangle centers  $X_i$  with*

$$i \in \{6, 22, 25, 31, 42, 51, 52, 58, 64, 81, 154, 155, 156, 182, 185, 374, 375, 378, 386, 387, 389, 500, 573, 575, 576, 609, 612, 948, 959, 961, 970, 975, 991, 1012, 1147, 1216, 1350, 1351, 1386, 1486, 1495, 1498, 1658, 1829, 1834, 1836, 1838, 1871, 1900, 1902, 2097, 2334, 2482, 3240, 3242, 3292, 3332, 3581\}$$

*trace conic sections while  $\Delta$  makes a full turn in the poristic family. These conic sections are centered at points on the central line  $\mathcal{L}_{1,3}$ . One of their axes coincides with  $\mathcal{L}_{1,3}$ .*

*Proof.* The center  $X_6$  is the Lemoine point of  $\Delta$ . Its trace has center  $M_6 = [3R^2d/(3R^2 - 2Rr + r^2), 0]$  and major and minor axes are  $a_6 = Rr\rho_4/(3R^2 - 2Rr + r^2)$  and  $b_6 = R\sqrt{r\rho_4}/\sqrt{\rho_{119}(2R^2 - 3Rr - r^2)}$ .

We compute the Exeter point  $X_{22} = \mathcal{L}_{2,3} \cap \mathcal{L}_{51,182}$  with  $X_{182} = \frac{1}{2}(X_3 + X_6)$  and  $X_{51}$  being the centroid of  $\Delta_o$ . We find  $M_{51} = [d(3R + 4r)/R, 0]$  and  $a_{51} = r\rho_4/(3R)$  and  $b_{51} = \rho_4\rho_{908}/(3R)$ . The center  $X_{25}$  is the intersection of  $\mathcal{L}_{2,3}$  and  $\mathcal{L}_{6,51}$ . The 2<sup>nd</sup> Power point  $X_{31}$  is collinear with the incenter  $X_1$  and Schiffler's point  $X_{21}$  and lies on  $\mathcal{L}_{940,1001}$  with  $X_{940} = \mathcal{L}_{1,3} \cap \mathcal{L}_{2,6}$ . We construct  $X_{42}$  as  $\mathcal{L}_{1,2} \cap \mathcal{L}_{35,58}$ , where  $X_{58}$  appears as the intersection of the central lines  $\mathcal{L}_{1,21}$  and  $\mathcal{L}_{3,6}$ . The construction of  $X_{64}$  is explained in the proof of Th. 4.6.

The center  $X_{52}$  is the orthocenter of  $\Delta_o$ . It is moving on an ellipse centered at  $M_{52} = [d(R+4r)/R, 0]$  and with semiaxes  $a_{52} = \rho_4\rho_{119}/R$  and  $b_{52} = r\rho_4/R$ . The point  $X_{64}$  is the reflection of  $X_{1498}$  in  $X_3$  and a construction of  $X_{1498}$  is given in the proof of Th. 4.6.

On the central line joining the incenter  $X_1$  with the Schiffler point  $X_{21}$  we find  $X_{81}$  which also lies on  $\mathcal{L}_{2,6}$ .  $X_{154}$  is the centroid of  $\Delta_t$ ,  $X_{155}$  is the orthocenter of  $\Delta_t$ , and  $X_{156} = \frac{1}{2}(X_{26} + X_{155})$  is the nine-point center of  $\Delta_t$ . The center  $X_{185}$  is the Nagel point of the orthic triangle  $\Delta_o$ . Its poristic locus is the ellipse with center  $M_{185} = [-d(R-4r)/R, 0]$ , its semiaxes are  $a_{185} = (2R-r)\rho_4/R$  and  $b_{185} = (R+r)\rho_4/R$ .

The triangle center  $X_{374}$  is the centroid of the pedal triangle of  $X_9$ . Its poristic locus is the ellipse centered at  $M_{374} = [d(R+r)(8R-r)/(3R(4R+r)), 0]$  with semiaxes  $a_{374} = 4\rho_4(R+r)/(3(4R+r))$  and  $b_{374} = 2R\rho_4/(4R+r)$ , respectively. The centroid of the pedal triangle of the Spieker point is denoted by  $X_{375}$ . Its poristic trace has center  $M_{375} = [d(4R+3r)/(6R), 0]$  and its semiaxes are  $a_{365} = (2R+r)\rho_4/(6R)$  and  $b_{375} = (3R-r)\rho_4/(6R)$ .  $X_{378}$  is determined as the reflection of  $X_{22}$  in  $X_3$ . We have  $X_{386} = \mathcal{L}_{1,2} \cap \mathcal{L}_{3,6}$  and  $X_{387} = \mathcal{L}_{1,2} \cap \mathcal{L}_{4,6}$ . With  $X_{389} = \frac{1}{2}(X_3 + X_{52})$  we find an ellipse with  $M_{389} = [d(R+4r)/(2R), 0]$ ,  $a_{389} = \rho_4\rho_{119}/(2R)$ , and  $\rho_{389} = r\rho_4/R$ . The orthocenter of the incentral triangle  $X_{500}$  leads to  $M_{500} = [d(5R+2r)/(2(3R-2r)), 0]$  and  $a_{500} = \rho_{21}$ , and  $b_{500} = \sqrt{r}\rho_{21}/\sqrt{2R}$ . With  $X_{573} = \mathcal{L}_{3,6} \cap \mathcal{L}_{4,9}$  we find  $M_{573} = [-4d(R+r)/(5R+8r), 0]$ ,  $a_{573} = R\rho_4/(5R+8r)$ , and  $b_{573} = \rho_4\sqrt{rR}/\sqrt{20R^2+37Rr+8r^2}$ . The center  $X_{575}$  is the midpoint in between  $X_3$  and  $X_{576}$ , where  $X_{576}$  is the reflection of  $X_{182}$  in  $X_6$ . The triangle center  $X_{609}$  is the intersection of the central lines  $\mathcal{L}_{1,32}$  and  $\mathcal{L}_{6,36}$ .

The center  $X_{612}$  is found as intersection of  $\mathcal{L}_{1,2}$  and  $\mathcal{L}_{6,210}$ . We find the next three centers and thereby the parametrizations of their poristic paths as intersection of central lines:  $X_{948} = \mathcal{L}_{1,4} \cap \mathcal{L}_{6,7}$ ,  $X_{959} = \mathcal{L}_{1,573} \cap \mathcal{L}_{2,65}$ , and  $X_{961} = \mathcal{L}_{2,12} \cap \mathcal{L}_{6,959}$ . The center of the Apollonius circle is found as  $X_{970} = \mathcal{L}_{3,6} \cap \mathcal{L}_{5,10}$ . Its poristic trace is centered at  $M_{970} = [-d(R+4r)/(2r), 0]$  and the semiaxes are  $a_{970} = \rho_4\rho_{119}/(2r)$  and  $b_{970} = \frac{1}{2}\rho_4$ . Again three centers are found as intersections of central lines:  $X_{975} = \mathcal{L}_{1,2} \cap \mathcal{L}_{9,58}$ ,  $X_{991} = \mathcal{L}_{1,7} \cap \mathcal{L}_{3,6}$ , and  $X_{1012} = \mathcal{L}_{2,6} \cap \mathcal{L}_{1,84}$ . The point  $X_{1147}$  is the midpoint of  $X_3$  and  $X_{155}$ .

The center  $X_{1216}$  appears as the reflection of  $X_{389}$  in  $X_{140}$  and its poristic locus is centered at  $M_{1216} = [d(R-4r)/(2R), 0]$  and the semiaxes are  $a_{1216} = (2R-r)\rho_4/(2R)$  and  $b_{1216} = (R+r)\rho_4/(2R)$ . The points  $X_{1350}$  and  $X_{1351}$  are found as reflections of  $X_6$  in  $X_3$  and  $X_{1350}$  in  $X_{182}$ , respectively.  $X_{1386}$

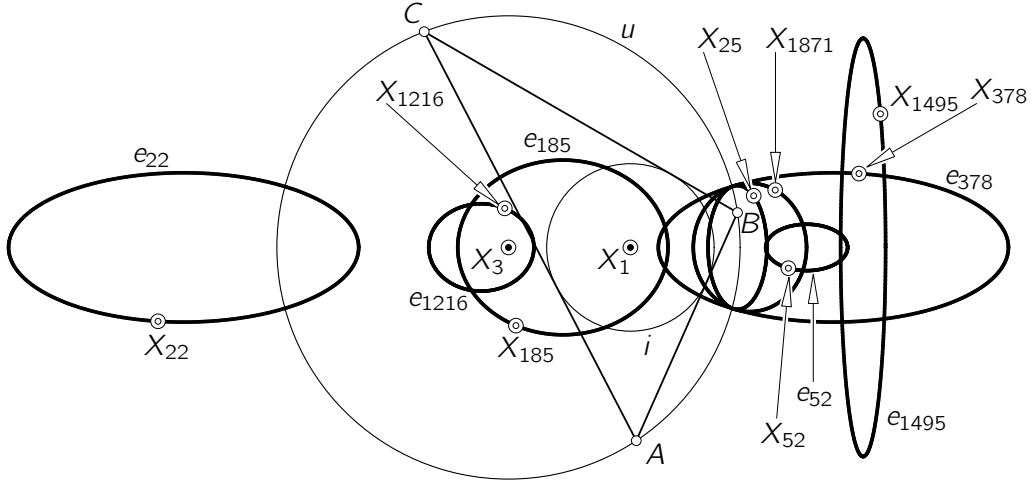


Figure 13: Some ellipses being poristic loci of triangles, *cf.* Th. 4.7.

is the midpoint of  $X_1$  and  $X_6$ . The perspector of  $\Delta_t$  and  $\Delta_i$  is the center  $X_{1486}$ . The triangle center  $X_{1495} = \frac{1}{2}(X_{23} + X_{110})$  moves on an ellipse with center  $M_{1495} = [3R^3/(d(3R + 2r)), 0]$ ,  $a_{1495} = Rr/(3R + 2r)$ , and  $b_{1495} = R(3R + r)/(3R + 2r)$ . We find  $X_{1498} = \mathcal{L}_{1,84} \cap \mathcal{L}_{4,6}$  and  $X_{1658} = \frac{1}{2}(X_3 + X_{26})$ . Then we find three centers by intersecting central lines:  $X_{1834} = \mathcal{L}_{4,6} \cap \mathcal{L}_{12,42}$ ,  $X_{1836} = \mathcal{L}_{4,65} \cap \mathcal{L}_{5,46}$ , and  $X_{1838} = \mathcal{L}_{1,4} \cap \mathcal{L}_{5,1214}$ . The center  $X_{1214}$  lies on  $\mathcal{L}_{1,3}$  and on  $\mathcal{L}_{7,464}$ , where  $X_{464} = \mathcal{L}_{63,69}$ .

On the central line  $\mathcal{L}_{4,8}$  we find the centers  $X_{1829}$ ,  $X_{1871}$ , and  $X_{1900}$  by intersecting with central lines  $\mathcal{L}_{1,25}$ ,  $\mathcal{L}_{5,1848}$ , and  $\mathcal{L}_{25,35}$ , respectively. This yields  $M_{1829} = [d(R^2 + 3Rr - r^2)/R^2, 0]$ ,  $a_{1829} = 2r\rho_4/R$ ,  $b_{1829} = \rho_4$ ;  $M_{1871} = [d(3R^2 + 5Rr - r^2)/(R(2R + r)), 0]$ ,  $a_{1871} = (R + 3r)\rho_4/(2R + r)$ ,  $b_{1871} = \rho_4$ ; and  $M_{1900} = [d(R^2 + 7Rr - r^2)/(R(R + 2r)), 0]$ ,  $a_{1900} = 4\rho_{12}$ ,  $b_{1900} = \rho_4$ . Reflecting  $X_{1829}$  in  $X_4$  we arrive at  $X_{1902}$  with  $M_{1902} = [d(3R^2 - 3Rr + r^2)/R^2, 0]$ ,  $a_{1902} = 2\rho_4\rho_{119}/R$ , and  $\rho_{1902} = \rho_4$ .

The triangle center  $X_{2097}$  is the reflection of  $X_6$  in  $X_{57}$  and  $X_{2482} = \frac{1}{2}(X_2 + X_{99})$ . We obtain  $X_{2334}$  as the common point of the central lines  $\mathcal{L}_{1,210}$  and  $\mathcal{L}_{6,210}$ . The midpoint of  $X_{69}$  and  $X_{145}$  is identified as center  $X_{3242}$ . The point  $X_{3292}$  is constructed as the reflection of  $X_{1495}$  in  $X_{110}$  and its poristic trace is centered at  $M_{3292} = [3R^3/(d(3R + 2r)), 0]$  and its semiaxes are  $a_{3292} = Rr/(3R + 2r)$  and  $b_{3292} = R(3R + r)/(3R + 2r)$ . We find  $X_{3332} = \mathcal{L}_{1,7} \cap \mathcal{L}_{4,6}$ . Finally the center  $X_{3581} \in \mathcal{L}_{3,6}$  lies on the Euler line and we find  $M_{3581} = [6R^3/(d(3R + 2r)), 0]$ ,  $a_{3581} = 2R(3R + r)/(3R + 2r)$ , and  $b_{3581} = 2Rr/(3R + 2r)$ .  $\square$

Fig. 4.4 shows that for certain values of  $R$ ,  $r$ , and  $d$  ellipses, parabolae, and hyperbolae appear as poristic trace of the same center.

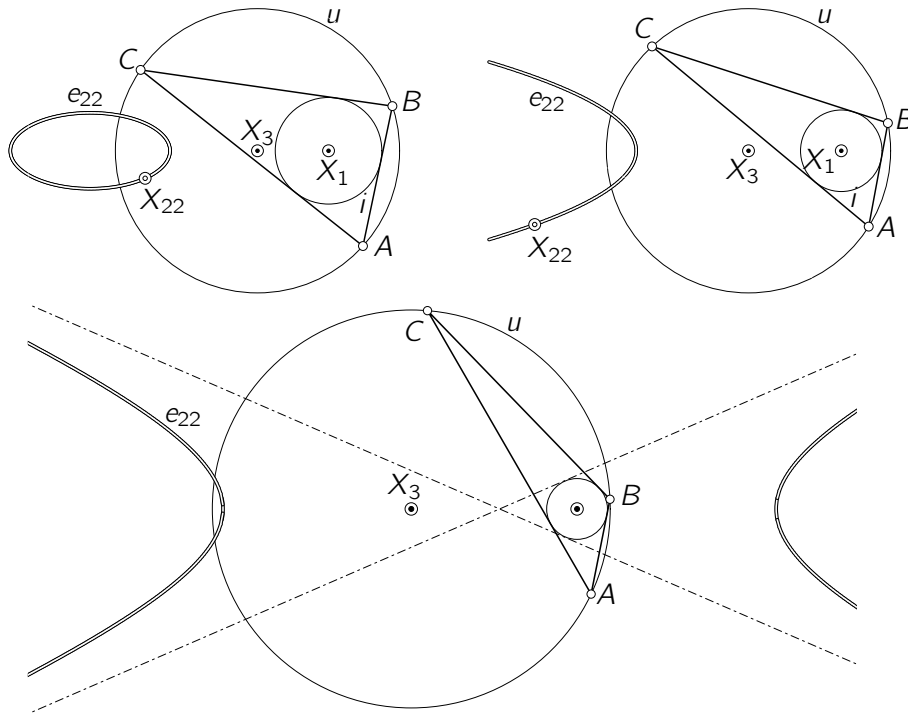


Figure 14: Different shapes of the poristic trace of the Exeter point  $X_{22}$ .

## 5 Final remarks

All the centers mentioned in the proof of Ths. 4.6 and 4.7 are triangle centers for  $\Delta$  since for any fixed triangle  $R$ ,  $r$ , and  $d$  are fixed and so is the relative position of  $M_i$  to  $X_1$  and  $X_3$  on  $\mathcal{L}_{1,3}$ .

The poristic traces of many centers have been parametrized during the computation of the poristic path of all centers mentioned in the theorems. Some of the centers which appear in the construction of centers do not have a conic section for its poristic orbit. The center  $X_{69}$  like many others traces an algebraic curve. In most cases the algebraic degree is larger than 4.

In the previous section we skipped the discussion of the affine type of the poristic paths of the centers investigated there. However, it is easy to show

that the traces of  $X_i$  with  $i \in \{22, 64, 154, 156, 609, 1498, 1658, 2482\}$  can be ellipses and hyperbolae as well.

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