PORT-CONTROLLED HAMILTONIAN SYSTEMS: MODELLING ORIGINS AND SYSTEMTHEORETIC PROPERTIES

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Abstract. It is shown that the network representation (as obtained through the generalized bond graph formalism) of non-resistive physical systems in interaction with their environment leads to a well-defined class of (nonlinear) control systems, called port-controlled Hamiltonian systems.
A first basic feature of these systems is that their internal dynamics is Hamiltonian with respect to a Poisson structure determined by the topology of the network and to a Hamiltonian given by the stored energy. Secondly the network representation provides automatically (intrinsically to the notation) to every port-control variable (input) a port-conjugated variable as output. This definition of port-conjugated input and output variables, based on energy considerations, is shown to have important consequences for the observability and controllability properties, as well as the external characterization of port-controlled Hamiltonian systems.

Keywords: Network dynamics, general Poisson structures, gyrators, Hamiltonian equations, observation space, minimal realizations

1 Introduction

In the previous paper [15] it was shown how by using a (generalized) bond graph formalism [7],[2],[3] the dynamics of non-resistive physical systems (pertaining to different domains, i.e. mechanical, electrical, hydraulical, etc.) can be given an intrinsic Hamiltonian formulation of dimensionn equal to the order of the physical system. Here "Hamiltonian" has to be understood in the generalized sense of defining Hamiltonian equations of motion with respect to a general Poisson structure on the state space manifold [6], [4], [15].
In the present paper we formalize the interaction of a non-resistive physical system with its environment by including external ports in the network model, which naturally leads to two conjugated sets of external variables: the inputs represented as generalized flow sources and the outputs which are the conjugated efforts [8]. We show that this yields a well-defined class of (linear and nonlinear) control systems, called port-controlled Hamiltonian systems, which generalizes the notions of Hamiltonian or Poisson control systems as given before [9], [17], [18]. We investigate the relation between observability and controllability properties of these systems and give some preliminary, (i.e. pertaining to the linear case) results on their external characterization.

2 Port controlled Hamiltonian systems

2.1 Network models and port- variables

In this section we briefly recapitulate the basic ingredients of the generalized bond graph formalism for modelling non-resistive physical systems. The bond-graph formalism provides physical insight into modelling and control of (complex) physical systems [7], [2], [22]. The main novelty as compared to [15] lies in the inclusion of external ports.
The energy-storage element, denoted by **C**, represents all the energetic properties of a bond-graph model of a physical system. It constitutes the elemental systems in the bond-graph model and is endowed with an energy function $H_0(x)$, depending on the energy variables $x \in \mathbf{R}^n$ defining its internal state. It interacts with its environment through its (n-dimensional) port by a power flow which may change its energetic state. The power flow at its port is expressed by two vectors of so-called conjugated power variables called "effort" e and "flow" f; defined by the constitutive relation:

$$\left.\begin{aligned} f &= \dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)^t \\ e &= dH_0(x) = \left(\tfrac{\partial H_0}{\partial x_1}(x), \ldots, \tfrac{\partial H_0}{\partial x_n}(x)\right) \end{aligned}\right\} \quad (1)$$

The time-variation of the energy (equal to the power flow at its port) is then the inner product of these two vectors:

$$\frac{dH}{dt} = \sum_{i=1}^{n} e_i . f_i . \tag{2}$$

The energy storage elements may interact dynamically through an element called "gyrator" [1],[2] and denoted by GY. This element describes the dynamic interactions between different physical domains in so-called "conservative systems", i.e. excluding irreversible resistive phenomena implying the thermal domain. The basic interaction is described by the "symplectic gyrator" [1] and relates either the kinetic with the potential domain, or the electrical with the magnetic domain. Its constitutive relation is, if f and e are respectively the n-dimensional flow and effort variable at its port:

$$f = -\mathcal{J}^{\text{symp}} e, \qquad \mathcal{J}^{\text{symp}} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{3}$$

with I_n the n-dimensional identity matrix.
In general, a gyrator element GY is a power-continuous element (i.e. the sum of power flows is equal to zero at its ports) and has a constitutive relation $f = -J(x)e$ with a general skew-symmetric constitutive matrix $J(x)$ satisfying the Jacobi identities [15] for $i, j, k = 1, \ldots, n$

$$\sum_{\ell=1}^{n} J_{\ell j} \frac{\partial J_{ik}}{\partial x_\ell} + J_{\ell i} \frac{\partial J_{kj}}{\partial x_\ell} + J_{\ell k} \frac{\partial J_{ji}}{\partial x_\ell} = 0, \tag{4}$$

(Skew-symmetry of $J(x)$ corresponds to power-continuity, while for an interpretation of the Jacobi-identities (4) we refer to [15]).
Complex systems may now be defined as a set of energy-storage elements interacting through a topology of power flows, called *"junction structure"* [3] [7, chap. 4 and 5]. The edges of the junction structure, called *(power) bonds*, carry a power P which is, according to the "power postulate", equal to the pairing of a flow variable f (a vector in $\mathbf{R}^n$) and an effort e (a vector in $(\mathbf{R}^n)^*$)

$$P = < e, f > . \tag{5}$$

A half-arrow gives the positive orientation of the power P and the flow variable f.
The two basic nodes of a junction structure are called *0-* and *1-junction* [3] [7, chap. 6]; they are power-continuous elements and have dual constitutive relations which is for the 0-junction (permuting e and f for the 1-junction):

$$\left. \begin{array}{l} e_1 = \ldots = e_n \\ \sum_{i=1}^{n} \varepsilon_i \, f_i = 0 \end{array} \right\} \tag{6}$$

with $\varepsilon_i = \pm 1$, depending on the orientation of the power bonds.
The *transformer element*, denoted by $(M)TF$, allows

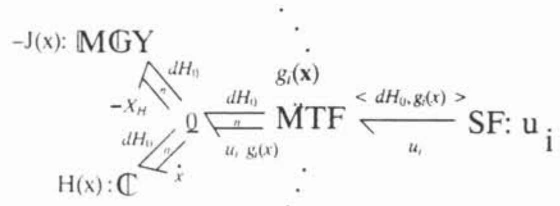

Figure 1

to generalize these sign-weighted constraints on the power variables to any linear relation on the power variables. Its constitutive relation consists of two adjoint maps on the effort e_i and flow f_i variables at its ports indexed by $i = 1, 2$, defined by a linear map $M(x)$ (parametrized by the energy variables x)

$$\left. \begin{array}{l} f_2 = M(x) f_1 \\ e_1 = M^t(x) e_2 \end{array} \right\} \tag{7}$$

The complete network interconnecting the energy-storage elements may contain additionally some gyrators and is finally called a *"generalized junction structure"*.

A set of energy-storage elements (**C**) interconnected by a generalized junction structure constitutes a complete bond-graph model of an *autonomous* conservative physical system. For the sake of simplicity, in the following we shall only consider bond graphs whose efforts at the ports of the energy-storage elements are *independent*. This is equivalent to assume that the space of the energy variables is not constrained to a proper subset of $\mathbf{R}^n$ by the topological constraints induced by the generalized junction structure (if this is not the case then the results presented hereafter remain valid, when restricted to the corresponding subset of $\mathbf{R}^n$ [5]). The *interaction* of this system with its *environment* may be described by the definition of some *ports* in the generalized junction structure, analogously to the definition of the elemental energy storage systems. The *control inputs* may be represented as generalized *flow sources* connected to some ports of the model, with its flow variables equal to the inputs. Indeed an external control leaving the network structure valid has to act through the ports of the system, in order to change the energetic state (and hence the energy) of the system: *the system is port-controlled.* The final bond graph model now becomes as depicted in figure 1 (the flow variables are indicated on the side of the half arrow at each bond, the conjugated effort variables on the opposite side). Finally it is worth to note that the network formalism naturally generates an *output* associated with each port-input control flow, namely the conjugated effort on the port bond [8].

2.2 Hamiltonian equations associated with port-controlled bond graph models.

First the basic definitions of general Poisson structures and the resulting generalized Hamiltonian equations of motion are recalled [4] [6] [11].

Definition 1. *Let M be a smooth (i.e. C^∞) manifold and let $C^\infty(M)$ denote the smooth real functions on M. A Poisson structure on M is a bilinear map from $C^\infty(M)\times C^\infty(M)$ into $C^\infty(M)$, called the Poisson bracket and denoted as:*

$$(F,G) \rightarrow \{F,G\} \in C^\infty(M), \qquad F,G \in C^\infty(M)$$

which satisfies for every $F,G,H \in C^\infty(M)$ the following properties: skew-symmetry

$$\{F,G\} = -\{G,F\} \tag{8}$$

Jacobi identity:

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0 \tag{9}$$

Leibniz rule:

$$\{F,G\cdot H\} = \{F,G\}.H + G\cdot\{F,H\} \tag{10}$$

M together with the Poisson structure is called a Poisson manifold.

Now let M be a Poisson manifold with Poisson bracket $\{,\}$. Then for any $H \in C^\infty(M)$ and arbitrary $x \in M$, we can define the mapping:

$$X_H(x) : C^\infty(M) \rightarrow \mathbf{R}$$

as:

$$X_H(x)(F) = \{F,H\}(x), F \in C^\infty(M) \tag{11}$$

It follows from the bilinearity of the Poisson bracket and (10) that $X_H(x) \in T_xM$ for every $x \in M$ (where T_xM denotes the tangent space to M at the point $x \in M$). Consequently X_H defines a smooth vectorfield on M, called the Hamiltonian vectorfield corresponding to the Hamiltonian function H (and with respect to the Poisson bracket $\{,\}$.) Poisson structures and Hamiltonian equations may also be defined locally (i.e. taking local coordinate functions $x_1,\dots x_n$ of the state space M defined on a neighborhood of some point $x_0 \in M$). It may be shown that locally the Poisson structure is uniquely determined by its *structure matrix* $J(x)$

$$\begin{aligned} J(x) &= [J_{k\ell}(x)]_{k,\ell=1,\dots,n} \\ J_{k\ell}(x) &:= \{x_k, x_\ell\} \end{aligned} \tag{12}$$

satisfying by (8)

$$J_{k\ell}(x) = -J_{\ell k}(x), \qquad k,\ell \in \{1\dots n\} \tag{13}$$

and by (9) for $i,j,k = 1,\cdots,n$

$$\sum_{\ell=1}^{n}\left(J_{\ell j}\frac{\partial J_{ik}}{\partial x_\ell} + J_{\ell i}\frac{\partial J_{kj}}{\partial x_\ell} + J_{\ell k}\frac{\partial J_{ji}}{\partial x_\ell}\right) = 0, \tag{14}$$

The Poisson bracket of two C^∞ functions F and G is then expressed as

$$\{F,G\} = \sum_{k,\ell=1}^{n} \frac{\partial F}{\partial x_k}(x) J_{k\ell}(x) \frac{\partial G}{\partial x_\ell}(x) \tag{15}$$

and the Hamiltonian vectorfield X_H is locally given as

$$\begin{pmatrix} X_H^1(x) \\ \vdots \\ X_H^n(x) \end{pmatrix} = J(x) \begin{pmatrix} \frac{\partial H}{\partial x_1}(x) \\ \vdots \\ \frac{\partial H}{\partial x_n}(x) \end{pmatrix} \tag{16}$$

Considering the bond-graph representation of figure 1, the time-variation of the energy variables (i.e. the generalized flow at the port of the energy storage element $\mathbf{C}$) is, according to the constitutive relation (6) of a 0-junction,

$$\dot{x} = -f_{GY} + \sum_{i=1}^{m} f_i, \tag{17}$$

where f_{GY} is the generalized flow at the port of the gyrator element and f_i is the flow of the bond relating the 0-junction to the transformers.
It may be shown [15], that the flow f_{GY} is a *Hamiltonian vectorfield* defined on the space of the energy variables, endowed with a Poisson structure with structure matrix equal to the (negative) of the junction structure matrix of the generalized junction structure (represented as a gyrator GY) and the energy function $H_0(x)$ of the energy storage elements, i.e.,

$$f_{GY} = -X_{H_0}(x). \tag{18}$$

Since the flow f_i, $i = 1,\dots,m$, is the image of the real-valued input u_i through the transformers constitutive relation defined by the input vectorfield $g_i(x)$, $i = 1,\dots m$:

$$f_i = g_i(x)u_i, \qquad i = 1,\dots,m, \tag{19}$$

the dynamic equations of the energy variables are *Hamiltonian* with *arbitrary* input vectorfields defined by the junction structure

$$\dot{x} = X_{H_0} + \sum_{i=1}^{m} g_i(x)u_i \tag{20}$$

The network structure generates naturally *a port-output* y_i as the conjugated effort to the port input u_i. In the bond graph representation of figure 1 this is expressed by the adjoint map defined by the constitutive relation (7), and yields the output

$$y_i = \langle dH_0, g_i \rangle (x), \qquad i = 1,\dots,m \tag{21}$$

since $\langle dH_0, g_i \rangle$ is the effort conjugated to the flow f_i. The adjoint maps correspond for instance to the adjoint relation on currents and voltages induced by Kirchhoff's current and voltage laws at the ports of an electrical circuit, or to the adjoint state relation

on forces and kinematic relations on velocities in mechanical systems.
Finally the pairing of the port-conjugate input and output variables is equal to the *variation of energy* in the system (since $< dH_0, X_{H_0} >= \{H_0, H_0\} = 0$)

$$\sum_{i=1}^m u_i\, y_i = \sum_{i=1}^m < dH_0, u_i g_i >= \\ < dH_0, \sum_{i=1}^m u_i g_i + X_{H_0} >= \tfrac{d}{dt}\, H_0 \qquad (22)$$

Example 1. 1-dimensional mechanical system:

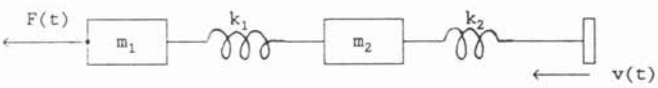

Figure 2

Consider the mass-spring system in figure 2. The energy storage elements define on the energy variables x_{12}, x_{20} (the displacements associated with the springs) and p_1, p_2 (the momenta associated with the point masses) the energy function $H_0(x_{12}, x_{20}, p_1, p_2)$ which is for instance, assuming linear constitutive relations,

$$H_0 = \frac{k_1 x_{12}^2}{2} + \frac{k_2 x_{20}^2}{2} + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \qquad (23)$$

Two inputs are considered: a velocity $u_1 = v$ acting on an extremity of spring 2 and a force $u_2 = F$ exerted on mass 1. The equations of motion of the energy variables derived from the bond graph model are:

$$\dot{x} = J dH_0 + u_1 g_1 + u_2 g_2 \qquad (24)$$

with

$$\begin{aligned} x &= (x_{12}, x_{20}, p_1, p_2)^t \\ dH_0 &= \left(k_1 x_{12}, k_2 x_{20}, \frac{p_1}{m_1}, \frac{p_2}{m_2}\right) \\ g_1 &= (0, -1, 0, 0)^t \\ g_2 &= (0, 0, 1, 0)^t \end{aligned}$$

and

$$J = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

The conjugated port-outputs y_1 and y_2 are the reaction force exerted by the spring 2 and the velocity of the mass 1, i.e.,

$$\begin{aligned} y_1 &= < dH_0, g_1 > \\ &= (k_1 x_{12}, k_2 x_{20}, \frac{p_1}{m_1}, \frac{p_2}{m_2}) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= -k_2 x_{20}. \qquad (25) \\ y_2 &= < dH_0, g_2 > \\ &= (k_1 x_{12}, k_2 x_{20}, \frac{p_1}{m_1}, \frac{p_2}{m_2}) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{p_1}{m_1} \end{aligned}$$

(The same equations are obtained for the electrical analogue of the system, formed by an LC network with a voltage and current source. For the general Hamiltonian formulation of LC-networks we refer to [15]).

Example 2. Rigid body with one input-torque:

Consider a rigid body driven by one input-torque u_1 around an axis with coordinates $g_1 = (b_x, b_y, b_z)^t$ with respect to the principal axes of inertia of the rigid body. The energy variables are the three components of the angular momentum p along the principal axes: $p = (p_x, p_y, p_z)$, and the energy function is the kinetic energy $H_0(p)$:

$$H_0(p) = \tfrac{1}{2}\left(\frac{p_x^2}{J_x} + \frac{p_y^2}{J_y} + \frac{p_z^2}{J_z}\right) \qquad (26)$$

where J_x, J_y and J_z are the principal moments of inertia. The generalized junction structure in the bond graph model is called "Eulerian junction structure" [19], and has following junction structure matrix $J(p)$:

$$J(p) = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix} \qquad (27)$$

The effort variables are:

$$dH_0(p) = \left(\frac{p_x}{J_x} + \frac{p_y}{J_y} + \frac{p_z}{J_z}\right) = (\omega_x, \omega_y, \omega_z) \qquad (28)$$

where $\omega_x, \omega_y, \omega_z$ are the angular velocities around the principal axes. The bond graph model induces then the Euler equations

$$\dot{p} = J(p)\, dH_0(p) + g_1\, u_1, \qquad (29)$$

and the port conjugated output y_1 is the instantaneous velocity around the input-torque axis, i.e.

$$y_1 =< dH_0, g_1 >= b_x \frac{p_x}{J_x} + b_y \frac{p_y}{J_y} + b_z \frac{p_z}{J_z}. \qquad (30)$$

2.3 Comparison with Hamiltonian input-output systems

An alternative definition of Hamiltonian systems with conjugated inputs and outputs was proposed in [9] [10] [16]. An affine Hamiltonian input-output system is defined as a Hamiltonian vectorfield on a symplectic manifold (i.e. a Poisson manifold with *regular* Poisson structure) with a Hamiltonian function $H(x,u)$ which is linear in the inputs:

$$H(x,u) = H_0(x) - \Sigma_{i=1}^{m} u_i \, H_i(x), \tag{31}$$

together with conjugated outputs $\tilde{y}_i$, equal to the interaction Hamiltonians $H_i(x)$:

$$\begin{array}{ll} \dot{x} = X_{H_0}(x) - \Sigma_{i=1}^{m} u_i \, X_{H_i}(x) & \\ \tilde{y}_i = H_i(x) & i = 1, \ldots, m \end{array} \tag{32}$$

The relation with port-controlled Hamiltonian systems (20), (21) is as follows. Take as output for (32) instead of $\tilde{y}_i$ the time-derivative of $\tilde{y}_i$, i.e.

$$\begin{array}{rl} \dot{\tilde{y}}_i = & < dH_i, \, X_{H_0} > (x) \\ & -\Sigma_{j=1}^{m} < dH_i, X_{H_j} > (x) \\ = & < dH_0, -X_{H_i} > (x) \\ & -\Sigma_{j=1}^{m} \{H_i, H_j\}(x) \end{array} \tag{33}$$

We see that *if* $\{H_i, H_j\} = 0$, $i,j = 1, \ldots, m$, then (32) with output $\dot{\tilde{y}}_i$ is a special case of (20), (21) (with $g_i = -X_{H_i}$). (Notice that even if $\{H_i, H_j\} \neq 0$ then one still finds for (32) the energy balance [9]

$$\sum_{i=1}^{m} u_i \dot{\tilde{y}}_i = \frac{dH_0}{dt}, \tag{34}$$

as in the case of port-controlled Hamiltonian systems.) In a similar way the definition of port-controlled Hamiltonian systems generalizes the definition of Poisson input-output systems as proposed in [17]. (Here degenerate Poisson structures are allowed.) Finally we note that systems of the general form (20) (without explicit use of the output equations (21)) have been considered before in the modelling of mechanical control systems with symmetries, see e.g. [18], [20].

Example 3:

The mechanical system of Example 1 may also be defined as a Hamiltonian input-output system with interaction Hamiltonians $H_1 = p_1 + p_2$ and $H_2 = x_{12} + x_{20}$. However, since

$$\{H_1, H_2\} = -1 \neq 0 \tag{35}$$

the outputs become different from (25), namely

$$\begin{array}{l} \dot{\tilde{y}}_1 = \dot{H}_1 = -kx_2 + v(t) \neq -kx_2 = y_1 \\ \dot{\tilde{y}}_2 = \dot{H}_2 = \frac{p_1}{m_1} - F(t) \neq \frac{p_1}{m_1} = y_2 \end{array} \tag{36}$$

Example 4:

The rigid body with one input-torque defined in the angular momenta $p = (p_x, p_y, p_z)$ is endowed with a degenerate Poisson structure and thus can not be defined as a Hamiltonian input-output system. Also regularization of the Poisson structure by also considering the configuration variables (i.e. the state-space $T^*SO(3)$) still does not allow to define the rigid body with 1 input-torque as a Hamiltonian input-output system ([9], page 148.)

3 Observability, controllability and minimal realizability

Consider a port-controlled Hamiltonian system as introduced in Section 2

$$\sum : \begin{array}{rl} \dot{x} = & X_{H_0}(x) + \sum_{j=1}^{m} g_j(x) u_j \\ y_j = & < dH_0, g_j > (x) \\ = & L_{g_j} H_0(x), \qquad j = 1, \ldots, m, \end{array} \tag{37}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for a Poisson manifold M with Poisson bracket $\{,\}$, and $H_0 : M \to \mathbf{R}$ is the internal energy.
For investigating the observability properties of (37) we note that its *observation space* $\mathcal{O}$ (see e.g. [21], [12], Def. 3.29) is spanned by all functions of the form

$$L_{X_1} L_{X_2} \ldots L_{X_k} (L_{g_j} H_0), \qquad j = 1, \ldots, m, \tag{38}$$

with X_i, $i \in \underline{k}$, in the set $\{X_{H_0}, g_1, \ldots, g_m\}$, $k = 0, 1, 2, \cdots$. Since $L_{X_{H_0}} H_0 = 0$ the functions (38) with $L_{g_j} H_0$ replaced by $L_{X_{H_0}} H_0$ are zero and trivially contained in $\mathcal{O}$. It follows that $\mathcal{O}$ is also spanned by all functions

$$L_{X_1} L_{X_2} \ldots L_{X_k} H_0, \qquad k = 1, 2, \ldots \tag{39}$$

with X_i, $i \in \underline{k}$, in the set $\{X_{H_0}, g_1, \ldots, g_m\}$. This suggests to consider an associated system given as

$$\sum_a : \begin{array}{l} \dot{x} = X_{H_0}(x) + \sum_{j=1}^{m} g_j(x) u_j \\ y = H_0(x) \end{array} \tag{40}$$

with the *single* output function $y = H_0(x)$ equal to the internal energy. Indeed the observation space $\mathcal{O}_a$ of the latter system is spanned by all functions (39) *and* the function H_0, and thus by $\mathcal{O}$ and H_0.
The physical interpretation of the close relation between $\sum$ and $\sum_a$ is rather immediate. By measuring the inputs u_j and y_j for $\sum$ one also measures the *power* $\sum_{j=1}^{m} u_j y_j$ at the external ports. Since the system is power-continuous one may reconstruct, by *integration*, from the measurement of the power the internal energy $H_0(x)$, up to a constant however. Conversely by differentiation one obtains from $y = H_0(x)$ for

$\sum_a$ the external power, and thus, since one knows $u_j, \quad j = 1, \ldots, m$, also $y_j, \quad j = 1, \ldots, m$, of $\sum$.

With regard to *controllability* properties we note (see [21], [12], Prop. 3.31) that the observation space $\mathcal{O}$ of (37) is alternatively given as the span of all functions

$$L_{Z_1} L_{Z_2} \ldots L_{Z_k} H_0, \qquad \ell = 1, 2, \ldots \tag{41}$$

with $Z_i, \quad i = 1, \ldots, \ell$, in the accessibility algebra $\mathcal{C}$ of $\sum$, which is spanned by all vectorfields of the form

$$[X_k, [X_{k-1}, [\ldots [X_2, X_1] \ldots]]], \quad k = 1, 2, \ldots \tag{42}$$

where $X_i, \quad i \in \underline{k}$, is in the set $\{f, g_1, \ldots, g_m\}$. In general this yields a rather complicated relation between $\mathcal{O}$ and $\mathcal{C}$, and thus between observability and controllability properties.

In the linear case, however, we can be much more explicit. A linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\ y &= Cx, \qquad\quad y \in \mathbb{R}^m, \end{aligned} \tag{43}$$

is a port-controlled Hamiltonian system if there exists an $n \times n$ skew-symmetric matrix $J = -J^\top$ (defining a constant Poisson structure) and an $n \times n$ symmetric matrix $Q = Q^\top$ (defining the Hamiltonian $H_0(x) = \frac{1}{2} x^\top Q x$) such that

$$A = JQ, \qquad B^\top Q = C \tag{44}$$

It immediately follows that

$$\begin{aligned} &[C^\top \vdots A^\top C^\top \vdots \ldots \vdots (A^\top)^{n-1} C^\top] = \\ &Q[B \vdots - AB \vdots \ldots \vdots (-1)^{n-1} A^{n-1} B] \end{aligned} \tag{45}$$

and thus

Proposition 2. *Consider the linear port-controlled Hamiltonian system (43).* (i) *If (43) is observable, then* $\det Q \neq 0$ *and (43) is controllable.*
(ii) *Assume* $\det Q \neq 0$*, then (43) is observable iff (43) is controllable.*

Let us now consider the nonlinear port-controlled Hamiltonian system (37), and assume it has an equilibrium $x_0 \in M$, i.e. $X_{H_0}(x_0) = 0$, or equivalently $dH_0(x_0) = 0$. Denoting the linearisation of (37) around x_0 by the linear system (44) with

$$A = \frac{\partial X_{H_0}}{\partial x}(x_0), \qquad B = [g_1(x_0) \vdots \ldots \vdots g_m(x_0)]$$

$$C = \begin{bmatrix} \frac{\partial L_{g_1} H_0}{\partial x}(x_0) \\ \vdots \\ \frac{\partial L_{g_m} H_0}{\partial x}(x_0) \end{bmatrix} \tag{46}$$

we obtain by standard arguments (see e.g. [12], Exercise 3.4) that this linearisation satisfies (44) with $Q = \frac{\partial^2 H_0}{\partial x^2}(x_0)$. Hence by Proposition 3.1 observability of the linearisation implies that $\frac{\partial^2 H_0}{\partial x^2}(x_0)$ is non-singular, and if $\frac{\partial^2 H_0}{\partial x^2}(x_0)$ is non-singular then observability and controllability of the linearisation are equivalent. This leads one to speculate what the observability of a general port-controlled Hamiltonian system (37) implies about the (non-degeneracy of the) internal energy H_0, and subsequently what this means for controllability.

Note that in [13] a complete external characterization of a Hamiltonian input-output system (32) has been obtained, see also the work of Jakubczyk [14]. So far, we have only obtained an external characterization of *linear* port-controlled Hamiltonian systems:

Proposition 3.

(i). The transfermatrix $G(s) = C(Is - A)^{-1}B$ *of a linear port-controlled Hamiltonian system (43) satisfies* $G(s) = -G^\top(-s)$.

(ii). Let $G(s)$ *be a strictly proper rational* $m \times m$ *matrix satisfying* $G(s) = -G^\top(-s)$. *Then every minimal realization* $\dot{x} = Ax + Bu, \quad y = Cx$ *is a linear port-controlled Hamiltonian system, i.e.* (A, B, C) *satisfy (44) for some matrices* $J = -J^\top, \quad Q = Q^\top$. *Furthermore* $\det Q \neq 0$.

Proof. *(i)* can be checked immediately. For *(ii)* let us consider a minimal realization of $G(s)$, i.e. $G(s) = C(Is - A)^{-1}B$. Since $G(s) = -G^\top(-s) = B^\top(Is + A^\top)^{-1}C^\top$ it follows that also the triple $(-A^\top, C^\top, B^\top)$ is a minimal realization, and thus by the state space isomorphism theorem there exists a unique linear invertible mapping Q such that

$$-A^\top = QAQ^{-1}, \quad C^\top = QB, \quad B^\top = CQ^{-1} \tag{47}$$

Obviously, (47) is also satisfied by $Q^\top$ instead of Q, and thus by unicity $Q = Q^\top$. Finally, defining $J = AQ^{-1}$, it follows from the first equality in (47) that $J = -J^\top$. □

4 Conclusion

A class of physical input-output systems, called port-controlled Hamiltonian systems, was proposed based on the network representation of non-resistive physical systems in the bond graph formalism. This class encompasses the previously defined notions of Hamiltonian or Poisson input-output systems, although if in the latter case the interaction Hamiltonians are not commuting there is a subtle discrepancy. The relation with mechanical control systems with symmetries as studied e.g. in [18], [20] needs to be further investigated. In general we think that the port-controlled Hamiltonian systems constitute a class of systems which is important in applications.

The conjugacy of inputs and outputs of port-controlled Hamiltonian systems has important consequences for the relation between observability and controllability. Since the inputs and outputs satisfy the port concept of *energetic* transactions there is a close connection with the observability of the system with the energy function as output function. Furthermore it is demonstrated that an equivalence between observability and controllability hinges upon the *regularity* of the Hamiltonian (i.e. energy) function. Finally, in the linear case the realization problem of port-controlled Hamiltonian systems is solved.

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