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Portfolio Management with Stochastic Interest Rates and Inflation Ambiguity

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Abstract

We solve a stock-bond-cash portfolio choice problem for a risk- and ambiguity-averse investor in a setting where the inflation rate and interest rates are stochastic. The expected inflation rate is unobservable, but the investor may learn about it from realized inflation and observed stock and bond prices. The investor is aware that his model for the observed inflation is potentially misspecified, and he seeks an investment strategy that maximizes his expected utility from real terminal wealth and is also robust to inflation model misspecification. We solve the corresponding robust Hamilton-Jacobi-Bellman equation in closed form and derive and illustrate a number of interesting properties of the solution. For example, ambiguity aversion affects the optimal portfolio through the correlation of price level with the stock index, a bond, and the expected inflation rate. Furthermore, unlike other settings with model ambiguity, the optimal portfolio weights are not always decreasing in the degree of ambiguity aversion.

1 Introduction

Since the seminal work of Merton (1969, 1971), numerous studies have been devoted to the optimal portfolio choice of a risk-averse investor under various assumptions. The vast majority of these papers, including Merton's papers, make the unrealistic assumption that the probability distributions of all relevant random quantities are known by the investor. Following the ideas of Knight (1921), the experimental studies of Ellsberg (1961) and Bossaerts et al. (2010) show that individuals are not only averse to risk (known probability distribution), but also averse to ambiguity (unknown probability distribution). In this paper we solve the problem of a risk- and ambiguity-averse investor who can invest in a stock (index), a long-term nominal bond, and in short-term deposits (cash). Interest rates and the inflation rate vary stochastically. The investor does not observe the expected inflation rate and is uncertain about the correct process for the consumer price level. Using the robust control approach of Anderson, Hansen, and Sargent (2003), we derive the optimal investment strategy in closed form, compare it with important special cases, and illustrate the properties of the optimal portfolio by a numerical example.

Our paper extends the existing literature studying the impact of inflation on portfolio choice. Campbell and Viceira (2001) and Brennan and Xia (2002), among others, specify the price processes of tradable assets in nominal terms as well as the inflation process and derive and study optimal dynamic portfolios of investors with constant relative risk aversion. In these papers all relevant state variables are assumed observable and the probability distributions of all processes are assumed known. Bensoussan, Keppo, and Sethi (2007) assume the investor observes the consumer price index with noise and, thus, the inflation rate is not fully observed, but an estimate can be filtered from observed quantities. This estimate is then used for determining the real wealth and real consumption. Chou, Han, and Hung (2011) assume that the price level is fully observable but that the expected inflation rate is unobservable to investor. All these papers disregard model uncertainty. We allow for an unobservable expected inflation rate and uncertainty about the relevant consumer price index. Next, we motivate these two model features.

The inflation rate is the change in the price level of a basket of consumption goods. Although the price level is directly observable to the investor, it is reasonable to assume that the drift of the price level process – the expected inflation rate – is not directly observed from the prices of consumer goods or financial assets nor from publications of macroeconomic statistics. Using the Bayesian approach formalized by Liptser and Shiryaev (2001), the investor learns about the process for the expected inflation rate from observations of the price level, the stock price, and the interest rate if the expected inflation is correlated with these variables.

We also assume that the investor is uncertain about the correct process to use for the observed inflation process and wants to derive a portfolio strategy which is robust to a potential model misspecification. This is a reasonable model feature because the identification of a particular inflation process requires a substantial amount of data (see Anderson, Hansen, and Sargent (2003) for a discussion of this issue) which might not be available for the investor. First, the official consumer price index may reflect a different composition of consumption goods than that preferred by the investor and can therefore be inappropriate for the individual investor. Secondly, the composition of the basket of goods changes over time with weights varying, new goods entering and other goods leaving the index. Thirdly, the official consumer price index may not appropriately reflect changes in the quality of the different goods (see the discussion in Griliches (1961) and Prentice and Yin (2000)). We focus on uncertainty about the inflation process, but we allow uncertainty about the inflation process to spill over into uncertainty about the expected nominal return on the stock and the bond. Such an effect would be in line with the discussions of Uppal and Wang (2003) and Vardas and Xepapadeas (2012) who argue that when ambiguity is related to economy-wide factors, the preference for robustness is the same for all processes in the model.

The investor has a reference model for the observed inflation process but he is also aware of the fact that other models might be a better representation of reality. As a result, he wants to derive investment rules that are robust to the proposed type of inflation model misspecification and that perform reasonably well across a set of plausible models. The discrepancy between the reference model and alternative models is defined in terms of relative entropy which serves as a penalty in the optimization procedure. This penalty measures the investor's uncertainty about

the reference model. Following Anderson, Hansen, and Sargent (2003), the optimal portfolio is obtained in closed form after solving the robust Hamilton-Jacobi-Bellman equation associated with our dynamic decision problem.

In the optimal portfolio of our model, the ambiguity aversion parameter is multiplied by various combinations of the correlation coefficients between the stock, bond, inflation, and expected inflation rate. In particular, if the price level process is not correlated with the securities then the level of uncertainty about the inflation model misspecification does not influence the optimal portfolio. This stands in contrast with the results of Maenhout (2004), Flor and Larsen (2011), and Branger, Larsen, and Munk (2012), among others, where the ambiguity aversion parameter enters the optimal portfolio independently. In these papers the agent is uncertain about the models for tradable assets whereas in our model the ambiguity is about the inflation process which implies that the securities can only be used as a hedge against the inflation uncertainty if the former are correlated with the price level process.

We show that the uncertainty about the inflation process affects the investor's positions in both the stock and the bond. This differs from Flor and Larsen (2011) where ambiguity about the interest rate process influences only the optimal bond position because the bond is a perfect instrument for hedging against the interest rate risk. When the investor is ambiguous about the inflation process, his optimal positions in both assets are affected because neither the stock nor the bond can perfectly hedge against the inflation risk.

In our model with ambiguity about the inflation model, a more risk-averse investor does not necessarily have smaller speculative components in his optimal portfolio of the stock and a bond. Although in models with no ambiguity (see, for example, Sørensen (1999) and Munk and Sørensen (2004)) the speculative components decrease as the risk aversion increases, this is generally not so when model uncertainty is introduced (see, for example, Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Meyer and Ormiston (1985), Hadar and Seo (1990), Gollier (1995)). In our model the behavior of the speculative portfolios with respect to the risk aversion depends on numerous parameter values. However, in a special case when the price level process is positively correlated with the stock price and the stock price is negatively correlated with the bond, then the speculative components are indeed decreasing in risk aversion.

The optimal investment strategies with stochastic interest rates have been studied in many papers. Sørensen (1999) and Korn and Kraft (2001) provide a solution for an investor who can invest in the stock index and a bond in a setting with the Vasicek (1977) term structure. Campbell and Viceira (2001) and Brennan and Xia (2002) analyze the effect of inflation on the optimal investment strategy. Koijman, Nijman, and Werker (2011) study optimal consumption and portfolio problem taking into account annuity risk at retirement. Van Hemert (2005) considers mortgages as a part of a homeowner's financial portfolio. In Munk and Sørensen (2004) the solution is obtained for non-Markovian dynamics of the opportunity set. Munk and Sørensen (2010) solve the problem for the investor with stochastic labor income. All processes in these papers are assumed to be known and all parameters are observable.

By allowing both for learning about the expected inflation rate and for price level model uncertainty, our paper combines two strands of the portfolio choice

literature. Gennotte (1986), Brennan (1998), Lakner (1998), and Bjørk, Davis, and Landén (2010) assume that the expected rates of return on the risky assets are unobserved. As mentioned above, Bensoussan, Keppo, and Sethi (2007) and Chou, Han, and Hung (2011) investment problems with partial observability of inflation process parameters has been studied in other papers. On the other hand, several papers assume all parameters and variables are observable but incorporate model uncertainty into a portfolio choice problem. Maenhout (2004) adapts the general robust control framework of Anderson, Hansen, and Sargent (2003) to a dynamic portfolio choice problem with power utility. He considers the simple Merton setting with a single stock and a riskless asset with constant investment opportunities and assumes ambiguity about the expected rate of return on the stock. In an extension, Maenhout (2006) investigates the role of ambiguity aversion when the expected stock return varies over time following an Ornstein-Uhlenbeck process. Liu (2010) extends that analysis to Epstein-Zin preferences. Flor and Larsen (2011) solve the optimal investment problem when the investor is ambiguous about the models for the interest rate and the stock. These papers assume that all parameters and state variables are observable.

Finally, two recent papers study portfolio choice models involving both unobservability and ambiguity, as we do. Liu (2011) considers a model with a regime-switching expected stock return with the current regime being unobservable. Branger, Larsen, and Munk (2012) extend the model of Maenhout (2006) to the case where the expected stock return also has an unobservable component and the investor learns about this component based on observed stock returns and the observable component of the expected stock return. We focus on unobservability and ambiguity related to inflation instead of the expected stock return.

This paper if organized as follows. In Section 2 we formulate the portfolio choice problem. In Section 3 we provide the optimal solution, discuss it, and compare it with optimal solutions to other relevant models. In Section 4 we analyze the optimal portfolios in a numerical example based on estimates of all model parameters. In particular, to determine a reasonable range for the ambiguity aversion parameter, we compute the so-called detection-error probabilities. The proofs of some results are given in the Appendix.

2 Mathematical Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a right-continuous filtration $\{\mathcal{F}_s\}_{s\in[0,T]}$. All stochastic processes introduced below are defined on this probability space. We consider an investor who can trade in a stock (index), zero-coupon bonds, and a money market account (cash).

According to the Vasicek (1977) model, the nominal short-term interest rate follows an Ornstein-Uhlenbeck process

$$dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r dB_t^P \tag{2.1}$$

where $\kappa > 0$ is the degree of mean reversion, $\bar{r} > 0$ is the long-run mean of the interest rate, $\sigma_r > 0$ is the interest rate volatility, and B_t^P is a standard Brownian

motion. Let q_r be the market price of interest rate risk which is assumed to be constant. With this dynamics for the short-term interest rate, the price P_t of a nominal zero-coupon bond paying one unit of account at time $\bar{T}$ is given by

$$P_t = e^{-a(\bar{T}-t)-b_\kappa(\bar{T}-t)r_t}$$

where the functions a and b_κ are

$$b_\kappa(x) = \tfrac{1}{\kappa}(1 - e^{-\kappa x}), \tag{2.2}$$

$$a(x) = \Big(\bar{r} + \frac{\sigma_r q_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\Big)(x - b_\kappa(x)) + \frac{\sigma_r^2}{4\kappa} b_\kappa(x)^2.$$

From Ito's lemma, the dynamics of the price of such a bond is

$$dP_t = P_t\Big((r_t + q)dt + \sigma_P dB_t^P\Big),$$

where $q = q_r\sigma_P$ is the expected excess return on the bond and $\sigma_P = \sigma_r b_\kappa(\bar{T} - t)$ is the bond price volatility.[1] Note that because interest rates are driven by a one-factor model, an unconstrained investor would not benefit from trading in more than one bond, and the investor can obtain exactly the same utility no matter which bond he trades in.

The agent can also invest in a stock index with nominal price S_t modeled by

$$dS_t = S_t\Big((r_t + \alpha)dt + \sigma_S dB_t^S\Big) \tag{2.3}$$

where the positive constants α and σ_S are the expected excess return and volatility, respectively, and B_t^S is a standard Brownian motion.

Let X_t denote the nominal value of the investor's portfolio at time t. The evolution of the portfolio value is

$$\begin{aligned} dX_t &= r_t(X_t - \Theta_t^S - \Theta_t^P)dt + \Theta_t^S \frac{dS_t}{S_t} + \Theta_t^P \frac{dP_t}{P_t} \\ &= X_t\Big((r_t + \alpha\Pi_t^S + q\Pi_t^P)dt + \sigma_S \Pi_t^S dB_t^S + \sigma_P \Pi_t^P dB_t^P\Big) \end{aligned}$$

where Θ_t^S and Θ_t^P represent the amounts of wealth invested in the stock and the bond, respectively. Equivalently, Π_t^S and Π_t^P represent the fractions of wealth invested in the stock and the bond, respectively, so that $1 - \Pi_t^S - \Pi_t^P$ is the fraction of wealth invested in the bank account that provides a return given by the short-term interest rate. Thus, the control (strategy) is represented by (Π_t^S, Π_t^P).

Let Z_t be the price level of the consumption good or a basket of consumption goods. Define the price level process

$$dZ_t = \beta_t Z_t dt + \sigma_Z Z_t dB_t^Z \tag{2.4}$$

[1]For simplicity of notation we suppress the dependence of σ_P on $\bar{T} - t$.

with the unobserved drift parameter (the expected inflation rate) given by

$$d\beta_t = \lambda(\bar{\beta} - \beta_t)dt + \sigma_\beta dB_t^\beta \tag{2.5}$$

where $\lambda > 0$ is the degree of mean reversion, $\bar{\beta} > 0$ is the long-run mean of the expected inflation rate, σ_Z is the price level process volatility, σ_β is the volatility of the expected inflation rate, and $B_t^Z,\ B_t^\beta$ are standard Brownian motions. The Brownian motions $B_t^Z,\ B_t^S,\ B_t^P,\ B_t^\beta$ are assumed to be correlated with the correlation matrix

$$\begin{bmatrix} 1 & \rho_{ZS} & \rho_{ZP} & \rho_{Z\beta} \\ \rho_{ZS} & 1 & \rho_{SP} & \rho_{S\beta} \\ \rho_{ZP} & \rho_{SP} & 1 & \rho_{P\beta} \\ \rho_{Z\beta} & \rho_{S\beta} & \rho_{P\beta} & 1 \end{bmatrix}.$$

We assume that the correlation coefficients take values in the interval $(-1, 1)$. The process (2.4) and Ornstein-Uhlenbeck process (2.5) are quite common in modeling the price level and the expected inflation rate, respectively, see for example, Brennan and Xia (2002) and Bensoussan, Keppo, and Sethi (2009). However, in Brennan and Xia (2002) the expected inflation rate is assumed to be observable, whereas in Bensoussan, Keppo, and Sethi (2009) the expected inflation rate is constant, but the price level is assumed to be unobservable.

Since the Brownian motions $B_t^Z, B_t^S, B_t^P, B_t^\beta$ are assumed to be correlated, the investor can obtain an estimate $\hat{\beta}_t$ of the unobserved expected inflation rate β_t based on the observed processes S_t, Z_t, r_t using Bayesian learning. According to Liptser and Shiryaev (2002), the Equations (2.1), (2.3), (2.4), and (2.5) as observed by the investor are (see Appendix A)

$$dZ_t = \hat{\beta}_t Z_t dt + Z_t \sigma_Z d\hat{B}_t^Z, \tag{2.6}$$

$$dS_t = S_t\Big((r_t + \alpha)dt + \sigma_S(\rho_{ZS} d\hat{B}_t^Z + \sqrt{1 - \rho_{ZS}^2} d\hat{B}_t^S)\Big), \tag{2.7}$$

$$dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r\Big(R_3 d\hat{B}_t^Z + R_4 d\hat{B}_t^S + R_5 d\hat{B}_t^P\Big), \tag{2.8}$$

$$d\hat{\beta}_t = \lambda(\bar{\beta} - \hat{\beta}_t)dt + A_Z\sigma_Z d\hat{B}_t^Z + A_S\sigma_S d\hat{B}_t^S + A_P\sigma_P d\hat{B}_t^P, \tag{2.9}$$

where $(\hat{B}_t^Z, \hat{B}_t^S, \hat{B}_t^P)^\intercal$ is an $\mathcal{F}_t^{S,Z,r}$-adapted Brownian motion with the filtration $\mathcal{F}_t^{S,Z,r} = \sigma\{S_\tau, Z_\tau, r_\tau | \tau \le t\}$ and

$$A_P = \frac{m(R_1R_4 - R_2R_3) + \sigma_Z\sigma_\beta R_2R_5R_8}{\sigma_P\sigma_Z R_2R_5}, \quad A_S = \frac{-R_1 m + \sigma_Z\sigma_\beta R_2R_7}{\sigma_S\sigma_Z R_2},$$

$$A_Z = \frac{\sigma_Z\sigma_\beta R_6 + m}{\sigma_Z^2}, \quad R_1 = \rho_{ZS},\ R_2 = \sqrt{1 - \rho_{ZS}^2}, \quad R_3 = \rho_{ZP},$$

$$R_4 = \frac{\rho_{SP} - \rho_{ZS}\rho_{ZP}}{\sqrt{1 - \rho_{ZS}^2}}, \quad R_5 = \sqrt{1 - R_3^2 - R_4^2}, R_6 = \rho_{Z\beta}, \quad R_7 = \frac{\rho_{S\beta} - \rho_{ZS}\rho_{Z\beta}}{\sqrt{1 - \rho_{ZS}^2}},$$

$$R_8 = \frac{\rho_{P\beta} - R_3R_6 - R_4R_7}{R_5}, \quad R_9 = \sqrt{1 - R_6^2 - R_7^2 - R_8^2}.$$

Here m is the limit value (as $t \to \infty$) of the deterministic variance given by $m_t = E[(\beta_t - \hat{\beta}_t)^2 | \mathcal{F}_t^{S,Z,r}]$ and it can be shown that

$$m = \frac{-\bar{K}_2 + \sqrt{\bar{K}_2^2 - 4\bar{K}_1\bar{K}_3}}{2\bar{K}_1},$$

where

$$\bar{K}_1 = \frac{(R_2R_5)^2 + (R_1R_5)^2 + (R_1R_4 - R_2R_3)^2}{(\sigma_Z R_2 R_5)^2},$$

$$\bar{K}_2 = 2\lambda + \frac{2\sigma_Z\sigma_\beta\Big(\rho_{Z\beta}(R_2R_5)^2 - R_1R_2R_5^2R_7 + (R_1R_4 - R_2R_3)R_2R_5R_8\Big)}{(\sigma_Z R_2 R_5)^2},$$

$$\bar{K}_3 = \sigma_\beta^2(\rho_{Z\beta}^2 + R_7^2 + R_8^2) - \sigma_\beta^2.$$

We assume that learning was long enough and take the variance to be equal to m.[2] Equations (2.6)–(2.9) constitute the reference model of the investor.

As it was mentioned in the Introduction, our investor is uncertain about the probability distribution for the observed processes (2.6)–(2.9). In other words, he realizes that the reference model is only an approximation of reality, and he wants to consider a set of plausible, alternative models which we now specify. Let e_t be an $\mathcal{F}_t^{S,Z,r}$-progressively measurable process (valued in $\mathbb{R}$) and define the Radon-Nikodým derivative process

$$\begin{aligned}\xi_t^e &= E\Big[\frac{d\mathbb{P}^e}{d\mathbb{P}}\Big|\mathcal{F}_t^{S,Z,r}\Big] \\ &= \exp\Big(-\int_0^t\Big(\frac{(1+k_S^2+k_P^2)e_s^2}{2}ds - e_s(d\hat{B}_s^Z + k_S d\hat{B}_s^S + k_P d\hat{B}_s^P)\Big)\Big) \qquad (2.10)\end{aligned}$$

where k_S and k_P are constants. According to Girsanov's theorem, the process

$$\begin{pmatrix}\tilde{B}_t^Z\\ \tilde{B}_t^S\\ \tilde{B}_t^P\end{pmatrix} = \begin{pmatrix}\int_0^t e_s ds + \hat{B}_t^Z\\ k_S\int_0^t e_s ds + \hat{B}_t^S\\ k_P\int_0^t e_s ds + \hat{B}_t^P\end{pmatrix}$$

is a Brownian motion with respect to probability measure $\mathbb{P}^e$.

According to this model misspecification, we rewrite the equations for the wealth X_t, the price level process Z_t, the short-term interest rate r_t, and the estimate of expected inflation $\hat{\beta}_t$ in the form

$$\begin{aligned}dX_t &= X_t\Big(r_t + \alpha\Pi_t^S + q\Pi_t^P - (\sigma_S\Pi_t^S a_1 + \sigma_P\Pi_t^P a_2)e_t\Big)dt \\ &\quad + K_1 d\tilde{B}_t^Z + K_2 d\tilde{B}_t^S + K_3 d\tilde{B}_t^P, \qquad (2.11)\\ dZ_t &= Z_t\Big((\hat{\beta}_t - \sigma_Z e_t)dt + \sigma_Z d\tilde{B}_t^Z\Big), \qquad (2.12)\\ dr_t &= \Big(\kappa(\bar{r} - r_t) - a_3 e_t\Big)dt - \sigma_r R_3 d\tilde{B}_t^Z - \sigma_r R_4 d\tilde{B}_t^S - \sigma_r R_5 d\tilde{B}_t^P, \qquad (2.13)\\ d\hat{\beta}_t &= \Big(\lambda(\bar{\beta} - \hat{\beta}_t) - a_4 e_t\Big)dt + A_Z\sigma_Z d\tilde{B}_t^Z + A_S\sigma_S d\tilde{B}_t^S + A_P\sigma_P d\tilde{B}_t^P, \qquad (2.14)\end{aligned}$$

where for simplicity we introduced the following notation $a_1 = R_1 + k_S R_2$, $a_2 = R_3 + k_S R_4 + k_P R_5$, $a_3 = -\sigma_r a_2$, $a_4 = A_Z\sigma_Z + k_S A_S\sigma_S + k_P A_P\sigma_P$, $K_1 = \sigma_S\Pi_t^S X_t R_1 + \sigma_P\Pi_t^P X_t R_3$, $K_2 = \sigma_S\Pi_t^S X_t R_2 + \sigma_P\Pi_t^P X_t R_4$, and $K_3 = \sigma_P\Pi_t^P X_t R_5$. These equations

[2]The same assumption was made by Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009), and Branger, Larsen, and Munk (2012).

represent alternative models indexed by the process e_t. The investor is uncertain about which model from the set (2.11)–(2.14) is the true model and wants to derive robust investment rules that work reasonably well for all these models.

Since the processes in (2.6)–(2.9) are assumed to be correlated, the ambiguity about the inflation might translate into the inflation-specific ambiguity about the other processes. The constants k_S and k_P determine whether the price level uncertainty influences the stock price and interest rate processes. In particular, we have

$$dS_t = S_t\Big[\Big(r_t + \alpha - \sigma_S(R_1 + k_S R_2)e_t\Big)dt + \sigma_S(R_1 d\tilde{B}_t^Z + R_2 d\tilde{B}_t^S)\Big].$$

Thus, if $k_S = -\rho_{ZS}/\sqrt{1-\rho_{ZS}^2}$, then there is no ambiguity about the stock price ($a_1 = 0$). If in addition $k_P = \frac{R_1R_4 - R_2R_3}{R_2R_5}$, then there is also no uncertainty about the interest rate process ($a_2 = 0$). Any other values of k_S and k_P imply that uncertainty about the price level spills over into uncertainty about the expected nominal stock return and about the expected nominal bond return. This setting is similar to Uppal and Wang (2002) and Vardas and Xepapadeas (2012) where the cases with equal and different component perturbations to a Brownian motion are considered. Here, equal perturbations mean that ambiguity is related to economy-wide factors and, thus, the preference for robustness is the same for all processes.

We consider an agent with CRRA (constant relative risk aversion) utility, who wants to derive an investment strategy for the time interval $[0, T]$ in order to maximize the expected utility from real terminal wealth X_T/Z_T. Let us denote the state variables by $y \triangleq (x, z, r, \hat{\beta})$ and the optimal investment strategy by $\Pi \triangleq (\Pi_t^S, \Pi_t^P)$. Therefore, we define the reward functional realized when choosing an alternative model specified by e as

$$w^e(t, y, \Pi) = \frac{1}{1-\gamma} E_{t,y}^{\mathbb{P}^e}\Big[\Big(\frac{X_T}{Z_T}\Big)^{1-\gamma}\Big], \tag{2.15}$$

and the value function as

$$v(t, y) = \sup_{\Pi \in \mathcal{U}[t,T]} \inf_{e \in \mathcal{E}[t,T]} \Big(w^e(t, y, \Pi) + E_{t,y}^{\mathbb{P}^e}\Big[\int_t^T \frac{e_s^2}{2\Psi(s, Y_s)} ds\Big]\Big) \tag{2.16}$$

where the parameter $\gamma > 0, \gamma \neq 1$ is the constant relative risk aversion and

$$\int_t^T \frac{e_s^2}{2\Psi(s, X_s, Z_s, r_s, \hat{\beta}_s)} ds$$

is the penalty term for deviating from the reference model.[3] To obtain wealth-independent optimal portfolio weights, and also for analytical tractability, we follow Maenhout (2004) by assuming that $\Psi(t, y) = \frac{\theta}{(1-\gamma)v(t,y)}$, where $\theta > 0$ is called the ambiguity aversion parameter.[4] A large value of Ψ corresponds to a small penalty, which means that the investor is more uncertain about the model.

[3]To simplify the notation, we write Π instead of $\{(\Pi_s^S, \Pi_s^P)\}_{s\in[t,T]}$ and e instead of $\{e_s\}_{s\in[t,T]}$. The expectation operator with respect to the probability measure $\mathbb{P}^e$ is defined as $E_{t,y}^{\mathbb{P}^e}[\cdot] \triangleq E^{\mathbb{P}^e}[\cdot | X_t = x, Z_t = z, r_t = r, \hat{\beta}_t = \hat{\beta}]$.

[4]For a critique of this approach, see Pathak (2002).

We define the space $\mathcal{U}$ of admissible strategies $\{\Pi_s\}_{s\in[0,T]}$, taking values in $\mathbb{R}^2$, as strategies that satisfy the following conditions

1. $\Pi : [0,T]\times\Omega \to \mathbb{R}^2$ is an $\mathcal{F}_t^{S,Z,r}$-progressively measurable process;
2. Under Π, for any $x \in (0,\infty)$, the wealth equation (2.11) admits a unique strong solution;
3. The integrability conditions necessary for the expectation operator in (2.15) to be well defined are satisfied;
4. $X_t \geq 0, a.s.,\ t \in [0,T]$.

The space $\mathcal{E}[0,T]$ is defined to be the space of $\mathcal{F}_t^{S,Z,r}$-progressively measurable processes e_t such that the process (2.10) is a Radon-Nikodým derivative.

3 Solution

The problem (2.16) is difficult to solve directly. We derive and solve a corresponding highly non-linear second-order partial differential equation that the value function $v(t,y)$ should satisfy, the so-called robust Hamilton-Jacobi-Bellman (HJB) equation, see Anderson, Hansen, and Sargent (2003). Let $\pi = (\pi^S, \pi^P)$ be the vector of fractions of wealth invested at time $t \in [0,T]$ in the stock (π^S) and the bond (π^P), then the corresponding robust HJB equation is

$$\begin{aligned}
\sup_{\pi\in\mathbb{R}^2}\inf_{e\in\mathbb{R}} \Big\{ & v_t + z\big(\hat{\beta} - \sigma_Z e\big)v_z + x\big(r + \alpha\pi^S + q\pi^P - [\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2]e\big)v_x \\
& + \big(\kappa(\bar{r} - r) - a_3 e\big)v_r + \big(\lambda(\bar{\beta} - \hat{\beta}) - a_4 e\big)v_{\hat{\beta}} + \tfrac{1}{2}(z\sigma_Z)^2 v_{zz} \\
& + \sigma_Z x z\big(\sigma_S\pi^S\rho_{ZS} + \sigma_P\pi^P\rho_{ZP}\big)v_{zx} - \sigma_Z\sigma_r z\rho_{ZP}v_{zr} + \sigma_Z^2 A_Z z v_{z\hat{\beta}} \\
& + \tfrac{1}{2}x^2\big((\sigma_S\pi^S)^2 + (\sigma_P\pi^P)^2 + 2\sigma_S\sigma_P\pi^S\pi^P\rho_{SP}\big)v_{xx} \\
& - \sigma_r x\big(\sigma_P\pi^P + \sigma_S\pi^S\rho_{SP}\big)v_{xr} + \sigma_\beta x\big(\sigma_S\pi^S\rho_{S\beta} + \sigma_P\pi^P\rho_{P\beta}\big)v_{x\hat{\beta}} + \tfrac{1}{2}\sigma_r^2 v_{rr} \\
& - \sigma_r\sigma_\beta\rho_{P\beta}v_{r\hat{\beta}} + \tfrac{1}{2}\big((A_Z\sigma_Z)^2 + (A_S\sigma_S)^2 + (A_P\sigma_P)^2\big)v_{\hat{\beta}\hat{\beta}} + \frac{e^2}{2\Psi}\Big\} = 0.
\end{aligned}$$

We assume that the value function is sufficiently smooth and that the HJB equation admits a classical solution.

Proposition 3.1. *The solution to problem* (2.16) *is of the form*

$$v(t,x,z,r,\hat{\beta}) = \frac{1}{1-\gamma}\Big(\frac{x}{z}\Big)^{1-\gamma} h(t,r,\hat{\beta}).$$

The function $h(t,r,\hat{\beta})$ is given by

$$h(t,r,\hat{\beta}) = \exp\big(-(1-\gamma)b_\lambda(T-t)\hat{\beta} + (1-\gamma)b_\kappa(T-t)r + c(t)\big),$$

where b_κ is defined in (2.2) *(b_λ is defined similarly), and the function $c(t)$ solves the ordinary differential equation* (B.5) *in Appendix B. The worst-case shock is*

$$e^* = \theta\big(\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2 - \sigma_Z + a_3 b_\kappa(T-t) - a_4 b_\lambda(T-t)\big) \tag{3.1}$$

and the optimal investments in the stock and the bond, respectively are

$$\pi_1^S = \pi_{spec}^S + \pi_{infl}^S + \pi_{rate}^S b_\kappa(T-t) + \pi_{unobs}^S b_\lambda(T-t),$$
$$\pi_1^P = \pi_{spec}^P + \pi_{infl}^P + \pi_{rate}^P b_\kappa(T-t) + \pi_{unobs}^P b_\lambda(T-t),$$

where

$$\begin{aligned}
\pi_{spec}^S &= \frac{1}{\sigma_S K}\Big(\alpha\frac{\gamma+\theta a_2^2}{\gamma\sigma_S} - q\frac{\gamma\rho_{SP}+\theta a_1 a_2}{\gamma\sigma_P}\Big),\\
\pi_{infl}^S &= \frac{\sigma_Z(\theta a_1 - (1-\gamma)\rho_{ZS})(\gamma+\theta a_2^2) - \sigma_Z(\theta a_2 - (1-\gamma)\rho_{ZP})(\gamma\rho_{SP}+\theta a_1 a_2)}{\gamma\sigma_S K},\\
\pi_{rate}^S &= \frac{\theta\sigma_r a_2(a_1 - a_2\rho_{SP})}{\gamma\sigma_S K},\\
\pi_{unobs}^S &= \frac{1}{\gamma\sigma_S K}\Big[\theta\gamma a_4(a_1 - a_2\rho_{SP})\\
&\qquad - \sigma_\beta(1-\gamma)\Big((\gamma+\theta a_2^2)\rho_{S\beta} - (\gamma\rho_{SP}+\theta a_1 a_2)\rho_{P\beta}\Big)\Big],\\
\pi_{spec}^P &= \frac{1}{\sigma_P K}\Big(q\frac{\gamma+\theta a_1^2}{\gamma\sigma_P} - \alpha\frac{\gamma\rho_{SP}+\theta a_1 a_2}{\gamma\sigma_S}\Big),\\
\pi_{infl}^P &= \frac{\sigma_Z(\theta a_2 - (1-\gamma)\rho_{ZP})(\gamma+\theta a_1^2) - \sigma_Z(\theta a_1 - (1-\gamma)\rho_{ZS})(\gamma\rho_{SP}+\theta a_1 a_2)}{\gamma\sigma_P K},\\
\pi_{rate}^P &= \frac{\theta\sigma_r a_2(a_2 - a_1\rho_{SP})}{\gamma\sigma_P K} + \frac{\sigma_r(\gamma-1)}{\gamma\sigma_P},\\
\pi_{unobs}^P &= \frac{1}{\gamma\sigma_P K}\Big[\gamma\theta a_4(a_2 - a_1\rho_{SP})\\
&\qquad - \sigma_\beta(1-\gamma)\Big((\gamma+\theta a_1^2)\rho_{P\beta} - (\gamma\rho_{SP}+\theta a_1 a_2)\rho_{S\beta}\Big)\Big],
\end{aligned}$$

and

$$K = \gamma(1-\rho_{SP}^2) + \theta(a_1^2 + a_2^2 - 2a_1 a_2\rho_{SP}).$$

Proof. See Appendix B. □

We analyze the portfolio given in Proposition 3.1 for the case when the investor's uncertainty about the price level also means that he is ambiguous about the stock price and the interest rate process, namely, we assume $k_S = k_P = 0$ and, thus, $a_1 = \rho_{ZS}$ and $a_2 = \rho_{ZP}$. A similar analysis for the case when the agent is uncertain about the inflation only, which means that $a_1 = a_2 = 0$, follows easily.

The optimal wealth allocation to the available securities consists of four components. First, we discuss the speculative components π_{spec}^S and π_{spec}^P which involve weighted combinations of expected excess returns on the stock (α) and the bond (q). Similarly to other models with stochastic interest rates (Sørensen (1999), Korn and Kraft (2001), Flor and Larsen (2011) among others), we also have that if the expected excess return on the stock increases, then the stock becomes more attractive which corresponds to the increase in π_{spec}^S. On the other hand, the increase in the expected excess return on the bond makes the bond more attractive for the investor and the value of π_{spec}^P becomes larger. Another difference between the speculative

components is that π^S_{spec} is constant (because $q = q_r\sigma_P$, see Section 2) whereas π^P_{spec} is time-dependent ($\sigma_P = \sigma_r b(\bar{T} - t)$).

The terms π^S_{infl} and π^P_{infl} represent the hedge against the inflation risk. The investor includes this hedge in the portfolio to protect the real value of his wealth. Interestingly, even if the stock (bond) is not correlated with the inflation, it still can be used as the hedge if it is correlated with the bond (stock) which in turn is correlated with the inflation. On the other hand, these terms vanish if the available securities cannot be used to hedge against the inflation risk (for the stock $\rho_{ZS} = \rho_{ZP} = 0$ or $\rho_{ZS} = \rho_{SP} = 0$, and for the bond $\rho_{ZP} = \rho_{ZS} = 0$ or $\rho_{ZP} = \rho_{SP} = 0$). The terms also vanish if the inflation is locally deterministic ($\sigma_Z = 0$) and if $\rho_{ZS} - \rho_{ZP}\rho_{SP} = 0$ (for the stock) and $\rho_{ZP} - \rho_{ZS}\rho_{SP} = 0$ (for the bond).[5] This property of the optimal portfolio is similar to Bensoussan, Keppo and Sethi (2009) where the optimal portfolio includes the hedge against inflation risk if the stock price is correlated with the inflation.

If the bond price is correlated with the inflation (which is usually the case), the uncertainty about the latter introduces an extra term π^S_{rate} in π^S and an additional term in the component π^P_{rate}. These terms vanish if there is no ambiguity ($\theta = 0$), or the interest rate is locally deterministic ($\sigma_r = 0$), or $\rho_{ZP} - \rho_{ZS}\rho_{SP} = 0$. This is in contrast with Flor and Larsen (2011) where stock price and interest rate (not inflation) model ambiguity does not introduce additional terms to the stock investment. It should also be pointed out that the influence of these components on the optimal portfolio decreases to zero when the investment horizon $T - t$ approaches zero.

The terms π^S_{unobs} and π^P_{unobs} arise from unobservability of the stochastic expected inflation rate. These components appear because the expected inflation rate is assumed to be stochastic and unobservable. The terms disappear if the expected inflation rate is deterministic. The presence of terms that hedge against changes in unobserved parameters is common for the portfolio choice problems (see for example Lakner (1998), Bjørk, Davis, and Landen (2010), Branger, Larsen, and Munk (2012)).

In contrast to Maenhout (2006), where the ambiguity aversion parameter is simply added to the risk aversion parameter, the ambiguity aversion parameter θ in our model is multiplied by various combinations of the correlation coefficients ρ_{ZS}, ρ_{ZP}, and ρ_{SP}. In particular, if the inflation is uncorrelated with the risky assets ($\rho_{ZS} = \rho_{ZP} = 0$), then the model uncertainty does not influence the optimal portfolio because the securities cannot be used in hedging against the inflation model misspecification. The same explanation holds for the case when $a_1 = a_2 = 0$, which means that the price level ambiguity does not translate into the uncertainty about the stock price and the interest rate.

Next, we compare the optimal portfolio given in Proposition 3.1 with solutions to similar investment problems. To make the paper self-contained we briefly describe each model and provide the corresponding solutions. The following portfolios are optimal for the investor who wants to maximize the expected utility of:

- terminal wealth (Sørensen 1999, Korn and Kraft 2001). All variables (stock, bond, and interest rate) are assumed observable with known dynamics. The

[5] The hedge against the inflation risk is also zero if $\theta = 1-\gamma$, but since empirical studies support $\gamma > 1$ and θ has to be positive, this is unlikely to be the case.

model is implicitly stated in real terms as inflation is not modeled. The optimal portfolio is

$$\pi_2^S = \frac{\alpha\sigma_P - q\sigma_S\rho_{SP}}{\gamma\sigma_S^2\sigma_P(1-\rho_{SP}^2)},$$
$$\pi_2^P = \underbrace{\frac{q\sigma_S - \alpha\sigma_P\rho_{SP}}{\gamma\sigma_P^2\sigma_S(1-\rho_{SP}^2)}}_{speculative} + \underbrace{\frac{\gamma-1}{\gamma}\frac{b(\bar{T}-t)}{b(T-t)}}_{hedge}.$$

The optimal stock investment π_2^S is represented by the speculative component only. On the other hand, the proportion π_2^P of wealth invested in the bond consists of both a speculative and an interest rate hedge component.

- terminal wealth with stock price model ambiguity (Flor and Larsen 2011). All variables are observable, inflation is not modeled. The investor in uncertain about the drift of the stock price with associated ambiguity aversion parameter θ_S. The optimal portfolio weights in the stock and the bond, respectively, are

$$\pi_3^S = \frac{\alpha\sigma_P - q\sigma_S\rho_{SP}}{(\gamma+\theta_S)\sigma_S^2\sigma_P(1-\rho_{SP}^2)},$$
$$\pi_3^P = \underbrace{\frac{q\sigma_S\Big(\gamma\rho_{SP}^2 + (\gamma+\theta_S)(1-\rho_{SP}^2)\Big) - \alpha\gamma\sigma_P\rho_{SP}}{\gamma(\gamma+\theta_S)\sigma_P^2\sigma_S(1-\rho_{SP}^2)}}_{speculative} + \underbrace{\frac{\gamma-1}{\gamma}\frac{b(\bar{T}-t)}{b(T-t)}}_{hedge}.$$

The introduced uncertainty about the stock price process alters the speculative components of π_3^S and π_3^P. It also follows that the component of π_3^P that hedges the interest rate risk does not change when the uncertainty is introduced.

- terminal wealth with bond price model ambiguity (Flor and Larsen 2011). All variables are observable, inflation is not modeled. The investor in uncertain about the drift of the bond price with associated ambiguity aversion parameter θ_P. Then the optimal investment strategy is

$$\pi_4^S = \frac{\alpha\sigma_P - q\sigma_S\rho_{SP}}{\gamma\sigma_S^2\sigma_P(1-\rho_{SP}^2)},$$
$$\pi_4^P = \underbrace{\frac{q\sigma_S\Big((\gamma+\theta_P)\rho_{SP}^2 + \gamma(1-\rho_{SP}^2)\Big) - \alpha(\gamma+\theta_P)\sigma_P\rho_{SP}}{\gamma(\gamma+\theta_P)\sigma_P^2\sigma_S(1-\rho_{SP}^2)}}_{speculative}$$
$$+ \underbrace{\frac{\gamma-1}{\gamma+\theta_P}\frac{b(\bar{T}-t)}{b(T-t)}}_{hedge} + \frac{\theta_P}{\gamma+\theta_P}.$$

In contrast with the previous model (stock price process ambiguity), the uncertainty about the bond price process influences only the components of π_4^P, the optimal wealth allocation in the bond. The optimal investment in the stock π_4^S is the same as π_2^S. Comparison of π_3^P and π_4^P shows that the hedge component of π_4^P hedges both the interest rate risk and the model uncertainty; see Flor and Larsen (2011) for a discussion of this issue.

- real terminal wealth with inflation model ambiguity. The investor is uncertain about the price level dynamics, but can observe the expected inflation rate. This is our model except that the expected inflation rate in our model is assumed unobservable. The solution is the same as in Proposition 3.1 but with $m = 0$, which in turn means that only components π^S_{unobs} and π^P_{unobs} change.[6] This also implies that these components do not depend on the price level process volatility σ_Z.
- real terminal wealth with no ambiguity, but an unobserved, stochastic expected inflation rate). This is our model without ambiguity. The solution is the same as in Proposition 3.1 but with $\theta = 0$.

A comparison of the models shows that different sources of ambiguity influence different components in the optimal portfolio. Since only bonds are used in hedging the interest rate risk, the ambiguity about the stock price process does not influence the hedge (compare π^P_2 and π^P_3). In our model, if the tradable assets are not correlated with the inflation, they cannot be used in hedging against the inflation and, therefore, the uncertainty about the price level process has no effect on the optimal portfolio. On the other hand, the ambiguity about the bond price process adds an extra term to the bond portfolio (not to the stock portfolio) and this term represents the hedge against the model uncertainty (see π^P_4). If the uncertainty is about the price level process, then extra terms appear in both the stock portfolio and the bond portfolio (see π^S_1 and π^P_1).

As it is usually the case, the speculative component of the optimal portfolio decreases as the investor's risk aversion increases (see for example the models in Sørensen (1999), Munk and Sørensen (2004), Flor and Larsen (2011) among others). On the other hand, pessimistic deteriorations in beliefs do not necessarily decrease the demand for the risky assets.[7] Since in our model the investor chooses the financial strategy that is optimal under the worst-case probability distribution for the inflation process, the speculative demand for the risky assets does not necessarily decrease when the investor's risk aversion increases. In particular, if the price level process is misspecified, then $\frac{d\pi^S_{spec}}{d\gamma} < 0$ and $\frac{d\pi^P_{spec}}{d\gamma} < 0$ are equivalent to the following two values being positive

$$\gamma^2(1-\rho^2_{SP})\Big(\frac{\alpha}{\sigma_S} - \frac{q}{\sigma_P}\rho_{SP}\Big) + \theta\rho_{ZP}\Big(\gamma(1-\rho^2_{SP}) + K\Big)\Big(\frac{\alpha}{\sigma_S}\rho_{ZP} - \frac{q}{\sigma_P}\rho_{ZS}\Big),$$
$$\gamma^2(1-\rho^2_{SP})\Big(\frac{q}{\sigma_P} - \frac{\alpha}{\sigma_S}\rho_{SP}\Big) + \theta\rho_{ZS}\Big(\gamma(1-\rho^2_{SP}) + K\Big)\Big(\frac{q}{\sigma_P}\rho_{ZP} - \frac{\alpha}{\sigma_S}\rho_{ZS}\Big),$$

respectively, where K is defined in Proposition 3.1. Therefore, the behavior of the speculative portfolios π^S_{spec} and π^P_{spec}, as functions of the ambiguity aversion parameter γ, depend on parameter values.

[6]To obtain the optimal portfolio, one should append fourth column $(0, 0, 0, \sigma_\beta R_9)^\mathsf{T}$ to Λ, use $(\sigma_\beta R_6, \sigma_\beta R_7, \sigma_\beta R_8, \sigma_\beta R_9)$ as the fourth row, and consider Brownian motion $(W^Z_t, W^S_t, W^P_t, W^\beta_t)^\mathsf{T}$ instead of $(\hat{B}^Z_t, \hat{B}^S_t, \hat{B}^P_t)^\mathsf{T}$. In the HJB equation, this change is equivalent to setting $m = 0$ and using $\sigma_{44} = \sigma^2_\beta$. As far as the optimal portfolio is concerned, this change is the same as taking the variance m of $\hat{\beta}_t$ to be zero.

[7]See Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Meyer and Ormiston (1985), Hadar and Seo (1990), Gollier (1995), and the references in these papers.

In our model, assuming that the Sharpe ratio of the stock ($\frac{\alpha}{\sigma_S}$) is greater than that of the bond ($\frac{q}{\sigma_P}$), the speculative portfolio π^S_{spec} decreases when γ increases if the inflation is positively correlated with the stock price or, more generally, if $\frac{\alpha/\sigma_S}{q/\sigma_P} > \frac{\rho_{ZS}}{\rho_{ZP}}$. Similarly, conditions that ensure that the speculative demand in the bond is a decreasing function of the ambiguity aversion parameter γ can be deduced.

4 Numerical Example

Since the optimal robust portfolio and the corresponding worst-case model depend on the preference parameter θ, some tools of its estimation are necessary.[8] We assume that the investor has measurements of $S_t, Z_t, \hat{\beta}_t$ over some finite time interval of length N. As suggested by Anderson, Hansen, and Sargent (2003), the parameter θ should be chosen in such a way that the approximating model and the worst-case model are sufficiently similar, which makes it difficult for the investor to use a likelihood ratio test in choosing either model based on the time series of length N.

4.1 Detection-Error Probabilities

In this section we follow the procedure suggested by Maenhout (2006), namely, we apply Fourier inversion to find the detection-error probability $\varepsilon_N(\theta)$ which is then used to determine how similar the reference and the worst-case models are. Anderson, Hansen, and Sargent (2003) suggest using θ such that $\varepsilon_N(\theta)$ is not less than 0.1. This choice will make it difficult for the robust investor to distinguish the two models statistically.

The worst-case model considered by the robust investor is given by (2.11)–(2.14) with $e_t = e^*_t$, where e^*_t is defined in (3.1). Define the Radon-Nikodým derivatives $\Xi_{1,t} \triangleq E^{\mathbb{P}}\left[\frac{d\mathbb{P}^{e^*}}{d\mathbb{P}}\middle|\mathcal{F}^{S,Z,r}_t\right]$ and $\Xi_{2,t} \triangleq E^{\mathbb{P}^{e^*}}\left[\frac{d\mathbb{P}}{d\mathbb{P}^{e^*}}\middle|\mathcal{F}^{S,Z,r}_t\right]$ and consider the logarithm of these derivatives,

$$\begin{aligned}
\xi_{1,t} &\triangleq \ln \Xi_{1,t} \\
&= -\int_0^t \left(e^*_s d\hat{B}^Z_s + k_S e^*_s d\hat{B}^S_s + k_P e^*_s d\hat{B}^P_s\right) - \frac{1+k_S^2+k_P^2}{2}\int_0^t (e^*_s)^2 ds, \\
\xi_{2,t} &\triangleq \ln \Xi_{2,t} \\
&= \int_0^t \left(e^*_s d\hat{B}^Z_s + k_S e^*_s d\hat{B}^S_s + k_P e^*_s d\hat{B}^P_s\right) + \frac{1+k_S^2+k_P^2}{2}\int_0^t (e^*_s)^2 ds.
\end{aligned}$$

Based on the sample with size N, the decision maker will discard the reference model mistakenly for the worst-case model if $\xi_{1,N} > 0$. On the other hand, if the worst-case model is true, then it will be rejected erroneously if $\xi_{2,N} > 0$ (or $\xi_{1,N} < 0$). According to this, we define the detection error probability

$$\varepsilon_N(\theta) = \tfrac{1}{2}\Pr(\xi_{1,N} > 0 \mid \mathbb{P}, \mathcal{F}_0) + \tfrac{1}{2}\Pr(\xi_{1,N} < 0 \mid \mathbb{P}^{e^*}, \mathcal{F}_0).$$

[8] Note that the ambiguity aversion parameter depends on the precise model set-up and source of ambiguity and therefore has to be estimated on a case-by-case basis.

It can then be shown that (see Appendix C)

$$\varepsilon_N(\theta) = \tfrac{1}{2} - \tfrac{1}{2}\,\mathrm{erf}\left(\frac{\sqrt{\tilde{K}}}{2}\right),$$

where $\tilde{K} = \dfrac{1}{2}\displaystyle\int_0^N (e_s^*)^2 ds$ and $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2} dt$. Note that we obtained a closed-form expression for the detection-error probability, in contrast to Maenhout (2006) and Branger, Larsen, and Munk (2012) who rely on numerical techniques of solving differential equations.

4.2 Model Parameters

For the numerical analysis of our model, we apply the parameters estimated by Brennan and Xia (2002) from a time series for 25 years. For concreteness, we assume that the bond the investor trades in at any date is a zero-coupon bond maturing 10 years later. This implies a constant bond price volatility $\sigma_P = \sigma_r b_\kappa(10)$. The assumed parameter values are shown in Table 1.[9]

Table 1: Parameter values in our numerical example.

σ_S	σ_Z	σ_β	σ_r	σ_P	λ	$\bar{\beta}$	q_r
0.158	0.013	0.014	−0.019	−0.143	0.027	0.054	0.209

κ	α	ρ_{SZ}	$\rho_{S\beta}$	ρ_{ZP}	ρ_{SP}	$\rho_{P\beta}$	$\rho_{Z\beta}$
0.060	0.054	±0.300	−0.024	−0.300	0.106	−0.695	±0.300

In Table 2 we present the detection-error probabilities for different values of the risk aversion parameter γ and the ambiguity aversion parameter θ for $N = 25$ years and $T = 10$ years. It can be shown that $\frac{\partial \varepsilon_N(\theta)}{\partial N} < 0$ regardless of the parameter values so that the detection-error probability decreases when the data sample increases. Furthermore, $\lim\limits_{N\to\infty} \varepsilon_N(\theta) = 0$. In the following example we choose $\gamma = 4$ and $\theta = 5$. With this choice of θ the detection-error probability is greater than 0.1.

[9]Since Brennan and Xia (2002) estimate the parameters for real interest rates, we accordingly adjust their parameters to be applicable in our model. Although ρ_{SZ} and $\rho_{Z\beta}$ were not estimated by Brennan and Xia (2002), we perform the analysis for their values equal to 0.3 and −0.3. These values of ρ_{SZ} were used in Bensoussan, Keppo, and Sethi (2009). Estimations of Fama and Schwert (1977), Gultekin (1983), Ferson and Harvey (1991), and Moerman and van Dijk (2010) also show that these correlation coefficients can be quite different. Similarly, we take ρ_{ZP} to be equal to -0.3 because bond prices are negatively correlated with inflation.

Table 2: Detection-error probabilities $\varepsilon_N(\theta)$ for different values of γ and θ for $N = 25$ years and $T = 10$ years.

θ	1	2	3	4	5	6	7	8
$\gamma = 2$	0.3915	0.3069	0.2425	0.1937	0.1567	0.1285	0.1067	0.0897
$\gamma = 4$	0.4101	0.3322	0.2669	0.2133	0.1701	0.1356	0.1083	0.0867
$\gamma = 6$	0.4165	0.3415	0.2763	0.2212	0.1755	0.1383	0.1085	0.0847
$\gamma = 8$	0.4198	0.3464	0.2813	0.2254	0.1783	0.1397	0.1084	0.0835

4.3 Optimal Portfolios

Next, we analyze the optimal portfolios π_1^S, π_1^P and their components. To better understand the influence of the unobservability of the expected inflation rate and ambiguity about the inflation process, the optimal portfolios are compared with the following special cases provided in Section 3:

- the expected inflation rate is observed;
- there is no ambiguity about the price level process.

We also discuss the influence of the ambiguity aversion parameter θ and the risk aversion parameter γ on the optimal portfolio.

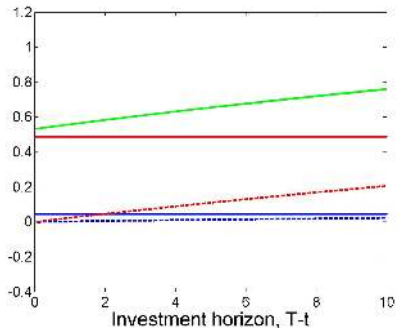
Figure 1 illustrates the optimal portfolios π_1^S and π_1^P with the corresponding components given in Proposition 3.1. The figure shows that if the stock price is positively correlated with the inflation ($\rho_{ZS} = 0.3$), the investor should decrease his optimal stock holdings as his investment horizon $T - t$ decreases which is in line with typical investment advice. On the other hand, if $\rho_{ZS} = -0.3$, the stock becomes more attractive for the investor as $T - t$ decreases. The most influential time-varying component in his stock portfolio is $\pi_{unobs}^S b_\lambda(T - t)$ that adjusts the optimal portfolio due to unobservability of the stochastic expected inflation rate. It is worth pointing out that the speculative component is the largest in the portfolio.

The optimal bond position is an increasing function of the investment horizon $T - t$ both when $\rho_{ZS} = 0.3$ and when $\rho_{ZS} = -0.3$. The components $\pi_{rate}^P b_\kappa(T - t)$ and $\pi_{unobs}^P b_\lambda(T - t)$ of the portfolio significantly adjust the optimal wealth allocation in the bond. However, this influence weakens over time because, as it was pointed out in Section 3, these components decrease to zero as the remaining investment horizon shortens. As a result, the optimal bond position changes from long to short as the investment horizon becomes smaller.

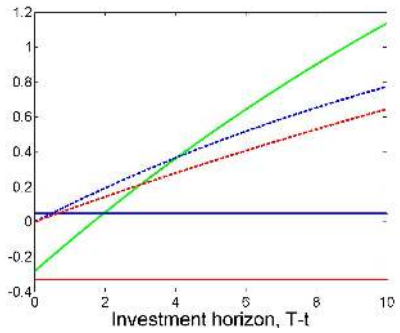
Comparing the optimal stock and bond portfolios we see that the time-varying components have more effect on the bond holdings than on the stock holdings. Interestingly, the speculative components for the stock and the bond portfolios are of opposite sign regardless of the correlation between the stock and the inflation.

Figure 2 shows the optimal portfolios π_1^S and π_1^P when the expected inflation rate is observed by the investor. As it was pointed out in Section 3, only the components π_{unobs}^S and π_{unobs}^P are affected by the change in the assumption. Compared to the unobservable case, the investor should invest less (more) in the stock if $\rho_{ZS} = 0.3$ ($\rho_{ZS} = -0.3$). At the same time, the optimal investment in the bond is smaller when

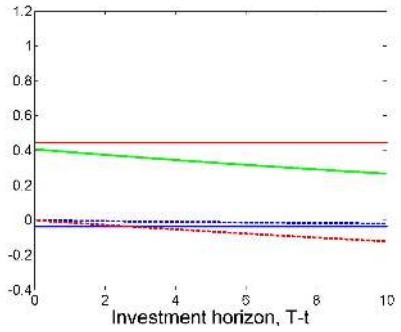
(a) Optimal stock portfolio and its components ($\rho_{SZ} = 0.3$)



(b) Optimal bond portfolio and its components ($\rho_{SZ} = 0.3$)



(c) Optimal stock portfolio and its components ($\rho_{SZ} = -0.3$)



(d) Optimal bond portfolio and its components ($\rho_{SZ} = -0.3$)

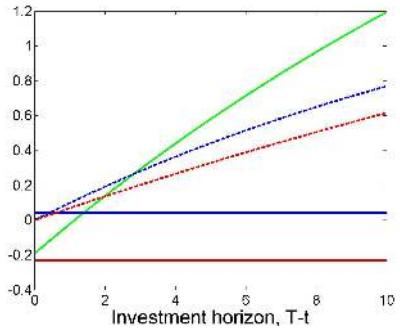
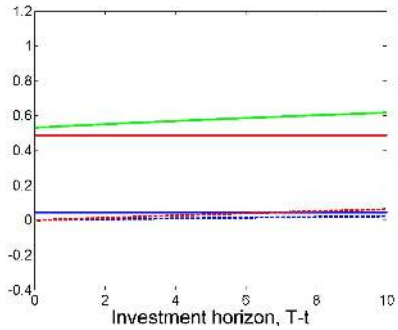
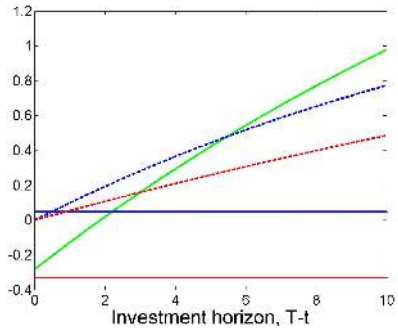


Figure 1: Optimal wealth allocation π_1^S in the stock (green line on the left plots) with its components and optimal wealth allocation π_1^P in the bond (green line on the right plots) with its components. The top plots are for $\rho_{SZ} = 0.3$ and the bottom plots are for $\rho_{SZ} = -0.3$. Red line is the speculative component. Blue line is the hedge against the price level process Z_t. Dashed blue line represents the model ambiguity adjustment. Dashed red line represents the component that arises from unobservable stochastic expected inflation rate β_t.

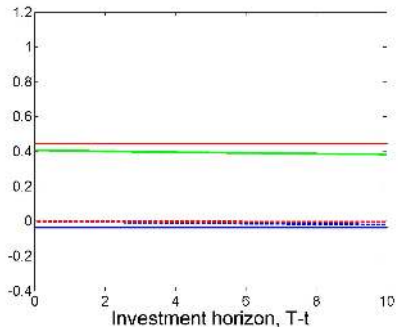
(a) Optimal stock portfolio and its components ($\rho_{SZ} = 0.3$)



(b) Optimal bond portfolio and its components ($\rho_{SZ} = 0.3$)



(c) Optimal stock portfolio and its components ($\rho_{SZ} = -0.3$)



(d) Optimal bond portfolio and its components ($\rho_{SZ} = -0.3$)

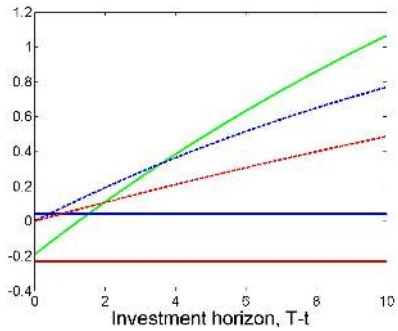


Figure 2: Optimal wealth allocation π_1^S in the stock (green line on the left plots) with its components and optimal wealth allocation π_1^P in the bond (green line on the right plots) with its components when the expected inflation rate is observable. The top plots are for $\rho_{SZ} = 0.3$ and the bottom plots are for $\rho_{SZ} = -0.3$. Red line is the speculative component. Blue line is the hedge against the price level process Z_t. Dashed blue line represents the model ambiguity adjustment. Dashed red line represents the component that arises from stochastic expected inflation rate β_t.

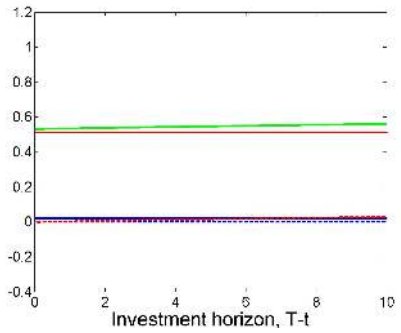
the expected inflation rate is observed. The rest of the analysis of Figure 2 is similar to that done for Figure 1.

The optimal portfolios for the investor with no ambiguity about the price level process are shown in Figure 3. The behavior of the optimal portfolios and the corresponding components in Figure 3 is similar to that in Figure 1 and Figure 2 so that the above discussion applies. However, since the component π_{rate}^S in the optimal stock portfolio becomes zero, this portfolio is heavily dominated by the speculative component. The corresponding component π_{rate}^P in the bond portfolio is not zero because it includes the hedge against the interest rate risk that does not vanish when the investor is ambiguous about the inflation.

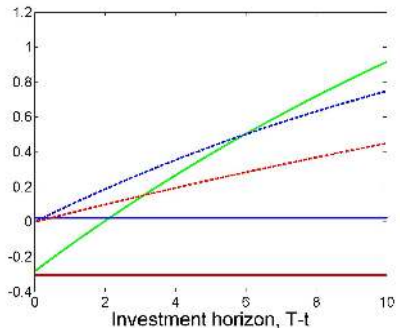
For the ease of exposition Figure 4 shows the optimal portfolios for an investor who is

- ambiguous about the price level and does not observe the expected inflation rate (Figure 1);

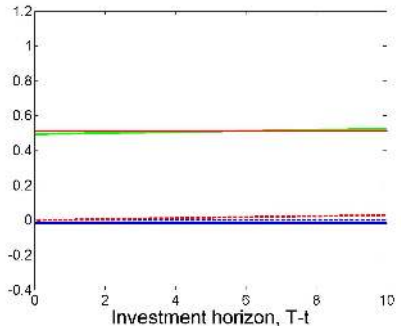
(a) Optimal stock portfolio and its components ($\rho_{SZ} = 0.3$)



(b) Optimal bond portfolio and its components ($\rho_{SZ} = 0.3$)



(c) Optimal stock portfolio and its components ($\rho_{SZ} = -0.3$)



(d) Optimal bond portfolio and its components ($\rho_{SZ} = -0.3$)

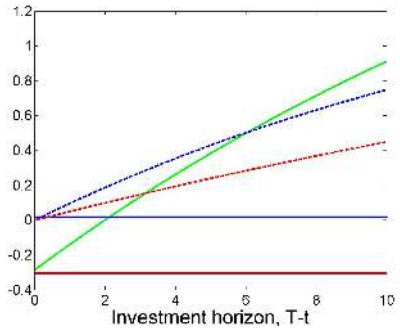


Figure 3: Optimal wealth allocation π_1^S in the stock (green line on the left plots) with its components and optimal wealth allocation π_1^P in the bond (green line on the right plots) with its components when there is no ambiguity about the inflation process. The top plots are for $\rho_{SZ} = 0.3$ and the bottom plots are for $\rho_{SZ} = -0.3$. Red line is the speculative component. Blue line is the hedge against the price level process Z_t. Dashed blue line represents the model ambiguity adjustment. Dashed red line represents the component that arises from unobservable stochastic expected inflation rate β_t.

- ambiguous about the price level and observes the expected inflation rate (Figure 2);
- not ambiguous about the price level and does not observe the expected inflation rate (Figure 3).

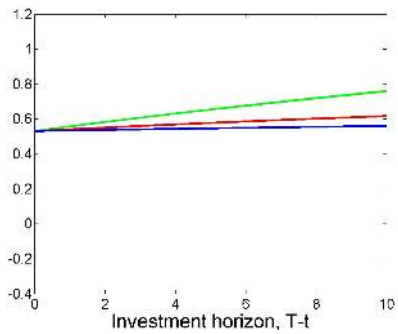
It is clear from the figure that an ambiguity-averse investor who does not observe the expected inflation rate invests more in the bond compared to an investor who either observes the expected inflation rate or is not ambiguous about the price level. In this setting the optimal bond investment is the smallest when the investor is not ambiguous about the price level. Interestingly, this is true for both $\rho_{SZ} = 0.3$ and $\rho_{SZ} = -0.3$. On the other hand, a change in the correlation ρ_{SZ} also changes the attitude of an ambiguous investor toward the stock investment, making him invest less (more) in the stock over time when $\rho_{SZ} = 0.3$ ($\rho_{SZ} = -0.3$). However, if the investor is not ambiguous about the price level process, then he decreases his stock holdings with time for both values of the correlation.

Figure 5 illustrates how the optimal portfolios π_1^S and π_1^P depend on the ambiguity aversion parameter θ. As one can see from the figure, the more the investor is ambiguity-averse, the more he invests in the bond. On the other hand, higher values of the ambiguity aversion parameter lead to higher (lower) values of the stock investment if $\rho_{ZS} = 0.3$ ($\rho_{ZS} = -0.3$).

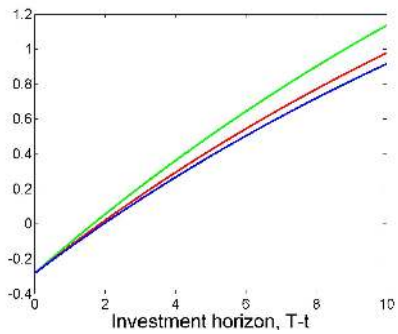
Note that some of the components of the optimal portfolios are increasing in θ and other components are decreasing in θ. This is in contrast to Maenhout (2006) where the ambiguity aversion parameter is simply added to the risk aversion parameter and, thus, the optimal portfolio is decreasing in θ. On the other hand, our findings are similar to the model of Flor and Larsen (2011) in which the speculative component of the bond portfolio decreases in θ if the investor is ambiguous about the bond price dynamics only and increases in θ if the investor is uncertain about the stock price process only.

Figure 6 shows the optimal portfolios π_1^S and π_1^P as functions of the risk aversion parameter γ. As discussed in Section 3, the impact of γ on the speculative components of the optimal portfolios depend on parameter values. It follows from the figure that the same applies for the (total) optimal portfolios. The more risk-averse investor invests less in the stock and more in the bond if the correlation between the stock price and inflation is $\rho_{ZS} = 0.3$. On the other hand, if $\rho_{ZS} = -0.9$, then the more risk-averse investor invests more in the stock.

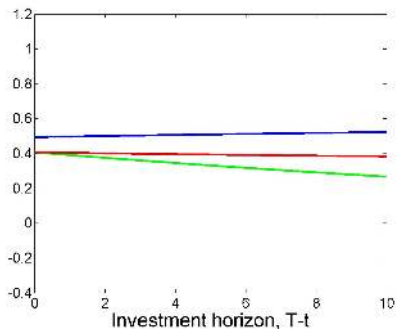
(a) Optimal stock portfolios ($\rho_{SZ} = 0.3$)



(b) Optimal bond portfolios ($\rho_{SZ} = 0.3$)



(c) Optimal stock portfolios ($\rho_{SZ} = -0.3$)



(d) Optimal bond portfolios ($\rho_{SZ} = -0.3$)

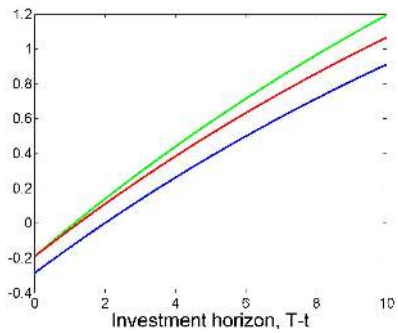
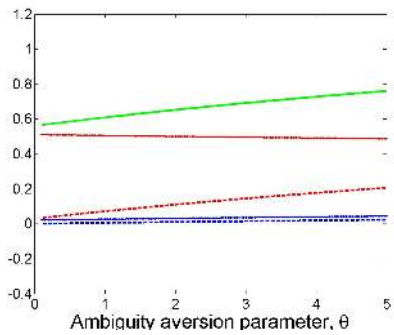
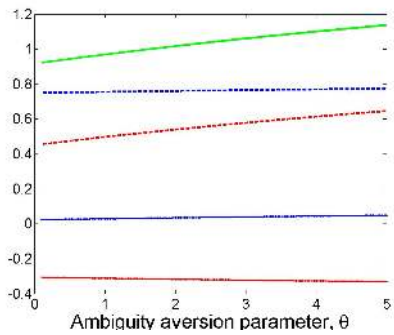


Figure 4: Optimal wealth allocation π_1^S in the stock (left plots) and optimal wealth allocation π_1^P in the bond (right plots) under different assumptions on the expected inflation rate and ambiguity. The top plots are for $\rho_{SZ} = 0.3$ and the bottom plots are for $\rho_{SZ} = -0.3$. Green line is the optimal portfolio when the expected inflation rate is unobserved and there is ambiguity about the price level process. Red line is the optimal portfolio when the expected inflation rate is observable and there is ambiguity about the price level process. Blue line is the optimal portfolio when the expected inflation rate is unobservable and there is no ambiguity about the price level process.

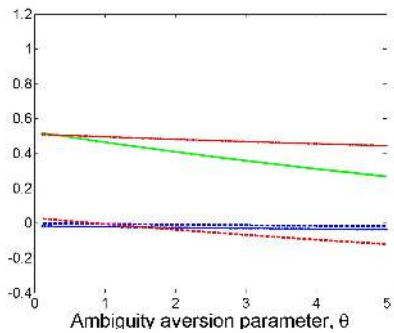
(a) Optimal stock portfolio and its components ($\rho_{SZ} = 0.3$)



(b) Optimal bond portfolio and its components ($\rho_{SZ} = 0.3$)



(c) Optimal stock portfolio and its components($\rho_{SZ} = -0.3$)



(d) Optimal bond portfolio and its components ($\rho_{SZ} = -0.3$)

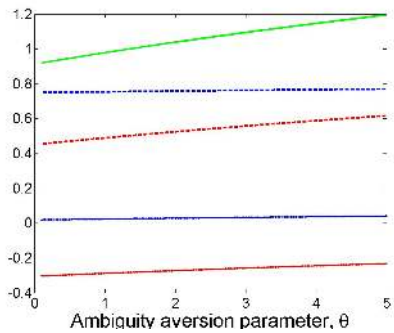
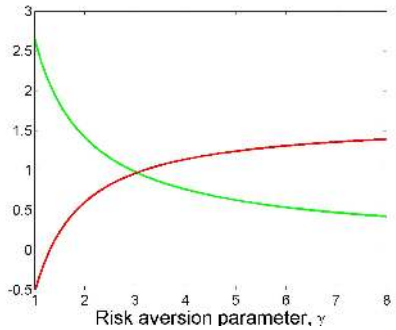


Figure 5: Optimal wealth allocation π_1^S in the stock (green line on the left plots) with its components and optimal wealth allocation π_1^P in the bond (green line on the right plots) with its components as functions of the ambiguity aversion parameter θ. The value of the investment horizon $T - t$ is 10 years. The top plots are for $\rho_{SZ} = 0.3$ and the bottom plots are for $\rho_{SZ} = -0.3$. Red line is the speculative component. Blue line is the hedge against the price level process Z_t. Dashed blue line represents the model ambiguity adjustment. Dashed red line represents the component that arises from unobservable stochastic expected inflation rate β_t.

(a) Optimal stock and optimal bond portfolios ($\rho_{SZ} = 0.3$)



(b) Optimal stock and optimal bond portfolios ($\rho_{SZ} = -0.9$)

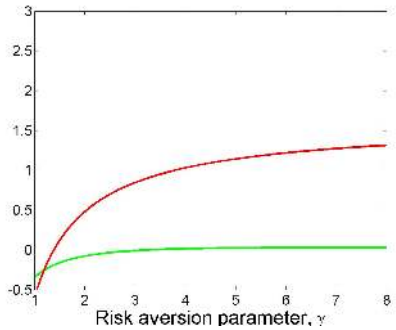


Figure 6: Optimal wealth allocation π_1^S in the stock (green line) and optimal wealth allocation π_1^P in the bond (red line) as functions of the risk aversion parameter γ. The value of the investment horizon $T - t$ is 10 years. The left plot is for $\rho_{SZ} = 0.3$ and the right plot is for $\rho_{SZ} = -0.9$.

5 Conclusion

In this paper we solve the problem of optimal portfolio choice under the assumptions that the investor is ambiguous about the price level process and that the expected inflation rate is unobservable in a setting with stochastic interest rates. The optimal wealth allocation in the stock index and a zero-coupon bond is obtained in closed form. We show that the influence of the ambiguity aversion parameter on the optimal portfolio depends on the correlation between the state variables. The uncertainty about the price level process influences the optimal positions in both the stock index and the bond. We also show that when there is ambiguity about the model for the inflation process, the more risk-averse investor does not necessarily invest less in the speculative portfolios. The optimal portfolio is illustrated by a numerical example.

A Optimal Filtering

To keep the same notation as in Liptser and Shiryaev (2001), we rewrite the Equations (2.3), (2.4), and (2.5) in the following form

$$\begin{bmatrix} \frac{dZ_t}{Z_t} \\ \frac{dS_t}{S_t} \\ dr_t \end{bmatrix} = \Bigg[\underbrace{\begin{pmatrix} 0 \\ r_t + \alpha \\ \kappa(\bar{r} - r_t) \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{A_1} \beta_t \Bigg] dt$$
$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{B_1} dW_t^\beta + \underbrace{\begin{bmatrix} \sigma_Z & 0 & 0 \\ \sigma_S R_1 & \sigma_S R_2 & 0 \\ -\sigma_r R_3 & -\sigma_r R_4 & -\sigma_r R_5 \end{bmatrix}}_{B_2} \begin{bmatrix} dW_t^Z \\ dW_t^S \\ dW_t^P \end{bmatrix}$$

and

$$d\beta_t = (\underbrace{\lambda\bar{\beta}}_{a_0} + \underbrace{(-\lambda)}_{a_1}\beta_t)dt + \underbrace{\sigma_\beta R_9}_{b_1} dW_t^\beta + \underbrace{[\sigma_\beta R_6 \quad \sigma_\beta R_7 \quad \sigma_\beta R_8]}_{b_2} \begin{bmatrix} dW_t^Z \\ dW_t^S \\ dW_t^P \end{bmatrix},$$

where $(W_t^Z, W_t^S, W_t^P, W_t^\beta)^\mathsf{T}$ is a standard Brownian motion relative to the filtration $\mathcal{F}_t$ and the coefficients $R_i,\ i = 1, ..., 9$ are defined in such a way that

$$\begin{bmatrix} dB_t^Z \\ dB_t^S \\ dB_t^P \\ dB_t^\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ R_1 & R_2 & 0 & 0 \\ R_3 & R_4 & R_5 & 0 \\ R_6 & R_7 & R_8 & R_9 \end{bmatrix} \begin{bmatrix} dW_t^Z \\ dW_t^S \\ dW_t^P \\ dW_t^\beta \end{bmatrix}$$

and

$$R_1 = \rho_{ZS}, \quad R_2 = \sqrt{1-\rho_{ZS}^2}, \quad R_3 = \rho_{ZP}, \quad R_4 = \frac{\rho_{SP} - \rho_{ZS}\rho_{ZP}}{\sqrt{1-\rho_{ZS}^2}},$$
$$R_5 = \sqrt{1 - R_3^2 - R_4^2}, \quad R_6 = \rho_{Z\beta}, \quad R_7 = \frac{\rho_{S\beta} - \rho_{ZS}\rho_{Z\beta}}{\sqrt{1-\rho_{ZS}^2}},$$
$$R_8 = \frac{\rho_{P\beta} - R_3R_6 - R_4R_7}{R_5}, \quad R_9 = \sqrt{1 - R_6^2 - R_7^2 - R_8^2}.$$

Assuming that for $a \in \mathbb{R}$ the conditional distribution $P(\beta_0 \leq a | Z_0, S_0, r_0)$ is Gaussian $\mathbb{P}$-a.s. with mean $\hat{\beta}_0 = E[\beta_0 | Z_0, S_0, r_0]$, and variance $m_0 = E[(\beta_0 - \hat{\beta}_0)^2 | Z_0, S_0, r_0]$ (equivalently, the distribution of β_0 is *conditionally* Gaussian), we have from Theorem 12.6 in Liptser and Shiryaev (2001) that the conditional distribution $P(\beta_t \leq a | \mathcal{F}_t^{S,Z,r})$ is also Gaussian $\mathbb{P}$-a.s.[10]

Therefore, applying Theorem 12.7 in Liptser and Shiryaev (2001) we have that the observed expected inflation rate $\hat{\beta}_t = E[\beta_t | \mathcal{F}_t^{S,Z,r}]$ satisfies

$$\begin{aligned} d\hat{\beta}_t = {} & (\underbrace{\lambda\bar{\beta}}_{a_0} + \underbrace{(-\lambda)}_{a_1}\hat{\beta}_t)dt \\ & + \Big\{ \underbrace{\Big[\sigma_Z\sigma_\beta\rho_{Z\beta},\ \sigma_S\sigma_\beta\rho_{S\beta},\ -\sigma_r\sigma_\beta\rho_{P\beta}\Big]}_{b\circ B} + m_t \underbrace{[1,\ 0,\ 0]}_{A_1^\mathsf{T}} \Big\} \\ & \times \underbrace{\begin{bmatrix} \sigma_Z & \sigma_S R_1 & -\sigma_r R_3 \\ 0 & \sigma_S R_2 & -\sigma_r R_4 \\ 0 & 0 & -\sigma_r R_5 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_Z & 0 & 0 \\ \sigma_S R_1 & \sigma_S R_2 & 0 \\ -\sigma_r R_3 & -\sigma_r R_4 & -\sigma_r R_5 \end{bmatrix}^{-1}}_{(B\circ B)^{-1}} \\ & \times \left\{ \begin{bmatrix} \frac{dZ_t}{Z_t} \\ \frac{dS_t}{S_t} \\ dr_t \end{bmatrix} - \Bigg[\underbrace{\begin{pmatrix} 0 \\ r_t + \alpha \\ \kappa(\bar{r} - r_t) \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{A_1} \hat{\beta}_t \Bigg] dt \right\}, \end{aligned}$$

[10]It is easy to check that the assumptions of the theorem are satisfied because entries in all matrices are constant.

where $m_t = E[(\beta_t - \hat{\beta}_t)^2 | \mathcal{F}_t^{S,Z,r}]$ is deterministic and

$$b \circ B = b_1 B_1^\mathsf{T} + b_2 B_2^\mathsf{T} = [0,\ 0,\ 0] + [\sigma_\beta R_6,\ \sigma_\beta R_7,\ \sigma_\beta R_8] \begin{bmatrix} \sigma_Z & \sigma_S R_1 & -\sigma_r R_3 \\ 0 & \sigma_S R_2 & -\sigma_r R_4 \\ 0 & 0 & -\sigma_r R_5 \end{bmatrix}$$
$$= \Big[\sigma_Z \sigma_\beta \rho_{Z\beta},\ \sigma_S \sigma_\beta \rho_{S\beta},\ -\sigma_r \sigma_\beta \rho_{P\beta}\Big],$$
$$(B \circ B)^{-1} = (B_1 B_1^\mathsf{T} + B_2 B_2^\mathsf{T})^{-1} = (B_2 B_2^\mathsf{T})^{-1} = (B_2^\mathsf{T})^{-1} (B_2)^{-1}$$

where

$$(B_2^\mathsf{T})^{-1} = \begin{bmatrix} \frac{1}{\sigma_Z} & -\frac{R_1}{\sigma_Z R_2} & \frac{R_1 R_4 - R_2 R_3}{\sigma_Z R_2 R_5} \\ 0 & \frac{1}{\sigma_S R_2} & -\frac{R_4}{\sigma_S R_2 R_5} \\ 0 & 0 & -\frac{1}{\sigma_r R_5} \end{bmatrix}.$$

We first evaluate $b \circ B + m_t A_1^\mathsf{T}$ which yields

$$b \circ B + m_t A_1^\mathsf{T} = \Big[\sigma_Z \sigma_\beta \rho_{Z\beta} + m_t,\ \sigma_S \sigma_\beta \rho_{S\beta},\ -\sigma_r \sigma_\beta \rho_{P\beta}\Big]$$

Now we multiply vector $b \circ B + m_t A_1^\mathsf{T}$ by matrix $(B_2^\mathsf{T})^{-1}$ to obtain

$$\begin{aligned} &\Big(b \circ B + m_t A_1^\mathsf{T}\Big)(B_2^\mathsf{T})^{-1} \\ &\quad = \Big[\sigma_\beta \rho_{Z\beta} + \frac{m_t}{\sigma_Z},\ \frac{-m_t R_1 + \sigma_Z \sigma_\beta R_2 R_7}{\sigma_Z R_2},\ \frac{m_t (R_1 R_4 - R_2 R_3) + \sigma_Z \sigma_\beta R_2 R_5 R_8}{\sigma_Z R_2 R_5}\Big] \\ &\quad = [\sigma_Z A_Z,\ \sigma_S A_S,\ \sigma_P A_P], \end{aligned}$$

where

$$\begin{aligned} A_P &= \frac{m_t (R_1 R_4 - R_2 R_3) + \sigma_Z \sigma_\beta R_2 R_5 R_8}{\sigma_P \sigma_Z R_2 R_5}, A_S = \frac{-R_1 m_t + \sigma_Z \sigma_\beta R_2 R_7}{\sigma_S \sigma_Z R_2}, \\ A_Z &= \frac{\sigma_Z \sigma_\beta \rho_{Z\beta} + m_t}{\sigma_Z^2}. \end{aligned}$$

The following vector defines a Brownian motion (see Liptser and Shiryaev (2001), Vol.2, p.35) relative to filtration $\mathcal{F}_t^{S,Z,r}$

$$\begin{aligned} &\begin{bmatrix} \sigma_Z & 0 & 0 \\ \sigma_S R_1 & \sigma_S R_2 & 0 \\ -\sigma_r R_3 & -\sigma_r R_4 & -\sigma_r R_5 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \frac{dZ_t}{Z_t} \\ \frac{dS_t}{S_t} \\ dr_t \end{bmatrix} - \left[\begin{pmatrix} 0 \\ r_t + \alpha \\ \kappa(\bar{r} - r_t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \hat{\beta}_t \right] dt \right\} \\ &= \begin{bmatrix} \frac{1}{\sigma_Z} & 0 & 0 \\ -\frac{R_1}{\sigma_Z R_2} & \frac{1}{\sigma_S R_2} & 0 \\ \frac{R_1 R_4 - R_2 R_3}{\sigma_Z R_2 R_5} & -\frac{R_4}{\sigma_S R_2 R_5} & -\frac{1}{\sigma_r R_5} \end{bmatrix} \begin{bmatrix} \frac{dZ_t}{Z_t} - \hat{\beta}_t dt \\ \sigma_S dB_t^S \\ -\sigma_r dB_t^P \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sigma_Z}\Big(\frac{dZ_t}{Z_t} - \hat{\beta}_t dt\Big) \\ -\frac{R_1}{\sigma_Z R_2}\Big(\frac{dZ_t}{Z_t} - \hat{\beta}_t dt\Big) + \frac{1}{R_2} dB_t^S \\ \frac{R_1 R_4 - R_2 R_3}{\sigma_Z R_2 R_5}\Big(\frac{dZ_t}{Z_t} - \hat{\beta}_t dt\Big) - \frac{R_4}{R_2 R_5} dB_t^S + \frac{1}{R_5} dB_t^P \end{bmatrix} = \begin{bmatrix} d\hat{B}_t^Z \\ d\hat{B}_t^S \\ d\hat{B}_t^P \end{bmatrix} \end{aligned} \tag{A.1}$$

where $(\hat{B}_t^Z, \hat{B}_t^S, \hat{B}_t^P)^\mathsf{T}$ is the Brownian motion. From this equality we easily obtain that $dB_t^S = R_2 d\hat{B}_t^S + R_1 d\hat{B}_t^Z$, $dB_t^P = R_5 d\hat{B}_t^P + R_4 d\hat{B}_t^S + R_3 d\hat{B}_t^Z$.

Thus, we have

$$d\hat{\beta}_t = \lambda(\bar{\beta} - \hat{\beta}_t)dt + A_Z\sigma_Z d\hat{B}_t^Z + A_S\sigma_S d\hat{B}_t^S + A_P\sigma_P d\hat{B}_t^P.$$

Therefore, the filtered equations can be written in terms of the Brownian motion (A.1) as

$$\begin{aligned}
dZ_t &= Z_t\Big(\hat{\beta}_t dt + \sigma_Z d\hat{B}_t^Z\Big),\\
dS_t &= S_t\Big((r_t + \alpha)dt + \sigma_S(R_1 d\hat{B}_t^Z + R_2 d\hat{B}_t^S)\Big),\\
dr_t &= \kappa(\bar{r} - r_t)dt - \sigma_r\Big(R_5 d\hat{B}_t^P + R_4 d\hat{B}_t^S + R_3 d\hat{B}_t^Z\Big).
\end{aligned}$$

B Robust HJB Equation

We rewrite equations for wealth X_t, price level process Z_t, short-term interest rate r_t, and drift $\hat{\beta}_t$ in matrix form

$$\begin{bmatrix} dZ_t \\ dX_t \\ dr_t \\ d\hat{\beta}_t \end{bmatrix} = \begin{bmatrix} Z_t\hat{\beta}_t \\ X_t(r_t + \alpha\Pi_t^S + q\Pi_t^P) \\ \kappa(\bar{r} - r_t) \\ \lambda(\bar{\beta} - \hat{\beta}_t) \end{bmatrix} dt + \begin{bmatrix} Z_t\sigma_Z & 0 & 0 \\ K_1 & K_2 & K_3 \\ -\sigma_r R_3 & -\sigma_r R_4 & -\sigma_r R_5 \\ A_Z\sigma_Z & A_S\sigma_S & A_P\sigma_P \end{bmatrix} \begin{bmatrix} d\hat{B}_t^Z \\ d\hat{B}_t^S \\ d\hat{B}_t^P \end{bmatrix}$$

We introduce perturbations to this system by adding a drift $\int_0^t e_s ds(1, k_S, k_P)^\mathsf{T}$ to the Brownian motion $(\hat{B}_t^Z, \hat{B}_t^S, \hat{B}_t^P)^\mathsf{T}$. The resulting vector $(\tilde{B}_t^Z, \tilde{B}_t^S, \tilde{B}_t^P)^\mathsf{T}$ is a Brownian motion under probability measure $\mathbb{P}^e$. The perturbed system of equations is

$$\begin{bmatrix} dZ_t \\ dX_t \\ dr_t \\ d\hat{\beta}_t \end{bmatrix} = \underbrace{\begin{bmatrix} Z_t(\hat{\beta}_t - \sigma_Z e_t) \\ X_t\Big(r_t + \alpha\Pi_t^S + q\Pi_t^P - \frac{K_1 + k_S K_2 + k_P K_3}{X_t} e_t\Big) \\ \kappa(\bar{r} - r_t) + \sigma_r(R_3 + k_S R_4 + k_P R_5)e_t \\ \lambda(\bar{\beta} - \hat{\beta}_t) - (A_Z\sigma_Z + k_S A_S\sigma_S + k_P A_P\sigma_P)e_t \end{bmatrix}}_{M} dt$$
$$+ \underbrace{\begin{bmatrix} Z_t\sigma_Z & 0 & 0 \\ K_1 & K_2 & K_3 \\ -\sigma_r R_3 & -\sigma_r R_4 & -\sigma_r R_5 \\ A_Z\sigma_Z & A_S\sigma_S & A_P\sigma_P \end{bmatrix}}_{\Lambda} \begin{bmatrix} d\tilde{B}_t^Z \\ d\tilde{B}_t^S \\ d\tilde{B}_t^P \end{bmatrix}.$$

According to Anderson, Hansen, and Sargent (2003), we evaluate the symmetric matrix

$$\Sigma = \Lambda\Lambda^\mathsf{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix}$$

where we defined $\sigma_{11} = (Z_t\sigma_Z)^2$, $\sigma_{22} = X_t^2\big((\sigma_S\Pi_t^S)^2 + (\sigma_P\Pi_t^P)^2 + 2\sigma_S\sigma_P\Pi_t^S\Pi_t^P\rho_{SP}\big)$, $\sigma_{33} = \sigma_r^2$, $\sigma_{44} = (\sigma_Z A_Z)^2 + (\sigma_S A_S)^2 + (\sigma_P A_P)^2$, $\sigma_{12} = \sigma_Z X_t Z_t\big(\sigma_S\Pi_t^S R_1 + \sigma_P\Pi_t^P R_3\big)$, $\sigma_{13} = -\sigma_Z\sigma_r Z_t R_3$, $\sigma_{14} = \sigma_Z^2 A_Z Z_t$, $\sigma_{34} = -\sigma_r\sigma_\beta\rho_{P\beta}$, $\sigma_{23} = -\sigma_r X_t\big(\sigma_P\Pi_t^P + \sigma_S\Pi_t^S\rho_{SP}\big)$, $\sigma_{24} = \sigma_\beta X_t\big(\sigma_S\Pi_t^S\rho_{S\beta} + \sigma_P\Pi_t^P\rho_{P\beta}\big)$.

We denote the Hessian and the gradient of the value function v with respect to state variables z, x, r and $\hat{\beta}$, respectively as

$$\frac{\partial^2 v}{\partial x_i \partial x_j} \triangleq \begin{bmatrix} v_{zz} & v_{zx} & v_{zr} & v_{z\hat{\beta}} \\ v_{zx} & v_{xx} & v_{xr} & v_{x\hat{\beta}} \\ v_{zr} & v_{xr} & v_{rr} & v_{r\hat{\beta}} \\ v_{z\hat{\beta}} & v_{x\hat{\beta}} & v_{r\hat{\beta}} & v_{\hat{\beta}\hat{\beta}} \end{bmatrix}, \quad \Big(\frac{\partial v}{\partial x_i}\Big) \triangleq \begin{bmatrix} v_z \\ v_x \\ v_r \\ v_{\hat{\beta}} \end{bmatrix}.$$

Let $\pi = (\pi^S, \pi^P)$ be the vector of fractions of wealth invested at time $t \in [0, T]$ in the stock (π^S) and the bond (π^P), then according to Anderson, Hansen, and Sargent (2003), the robust HJB equation is

$$v_t + \sup_{\pi\in\mathbb{R}^2}\inf_{e\in\mathbb{R}}\Big(M^\intercal\Big(\frac{\partial v}{\partial x_i}\Big) + \tfrac{1}{2}\,\mathrm{trace}\,\Big(\Sigma\frac{\partial^2 v}{\partial x_i\partial x_j}\Big) + \frac{e^2}{2\Psi}\Big) = 0.$$

In particular,

$$\begin{aligned}
\sup_{\pi\in\mathbb{R}^2}\inf_{e\in\mathbb{R}}\Big\{&v_t + z\big(\hat{\beta} - \sigma_Z e\big)v_z + x\big(r + \alpha\pi^S + q\pi^P - [\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2]e\big)v_x \\
&+ \big(\kappa(\bar{r} - r) - a_3 e\big)v_r + \big(\lambda(\bar{\beta} - \hat{\beta}) - a_4 e\big)v_{\hat{\beta}} + \tfrac{1}{2}(z\sigma_Z)^2 v_{zz} \\
&+ \sigma_Z xz\big(\sigma_S\pi^S n_1 + \sigma_P\pi^P n_2\big)v_{zx} - \sigma_Z\sigma_r z n_2 v_{zr} + \sigma_Z^2 A_Z z v_{z\hat{\beta}} \\
&+ \tfrac{1}{2}x^2\big((\sigma_S\pi^S)^2 + (\sigma_P\pi^P)^2 + 2\sigma_S\sigma_P\pi^S\pi^P n_4\big)v_{xx} \\
&- \sigma_r x\big(\sigma_P\pi^P + \sigma_S\pi^S n_4\big)v_{xr} + \sigma_\beta x\big(\sigma_S\pi^S n_5 + \sigma_P\pi^P n_6\big)v_{x\hat{\beta}} + \tfrac{1}{2}\sigma_r^2 v_{rr} \\
&- \sigma_r\sigma_\beta n_6 v_{r\hat{\beta}} + \tfrac{1}{2}\big((A_Z\sigma_Z)^2 + (A_S\sigma_S)^2 + (A_P\sigma_P)^2\big)v_{\hat{\beta}\hat{\beta}} + \frac{e^2}{2\Psi}\Big\} = 0,
\end{aligned} \tag{B.1}$$

where $n_1 = \rho_{ZS}$, $n_2 = \rho_{ZP}$, $n_3 = \rho_{Z\beta}$, $n_4 = \rho_{SP}$, $n_5 = \rho_{S\beta}$, and $n_6 = \rho_{P\beta}$.

To find the infimum over e, we take the derivative with respect to e and set it equal to zero.

$$\frac{d}{de}\Big(-z\sigma_Z e v_z - x(\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2)e v_x - a_3 e v_r - a_4 e v_{\hat{\beta}} + \frac{e^2}{2\Psi}\Big) = 0.$$

The value e^* that gives the infimum is

$$e^* = \Psi\big(z\sigma_Z v_z + x(\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2)v_x + a_3 v_r + a_4 v_{\hat{\beta}}\big).$$

To simplify the notation we use $\phi = 1 - \gamma$. Let us look for a solution in the form $v(t, z, x, r, \hat{\beta}) = \frac{1}{\phi}\big(\frac{x}{z}\big)^\phi h(t, r, \hat{\beta})$ and assume that $\Psi = \frac{\theta}{h}\big(\frac{x}{z}\big)^{-\phi}$. Plugging these functions into e^* we obtain

$$e^* = \theta\Big(\sigma_S\pi^S a_1 + \sigma_P\pi^P a_2 - \sigma_Z + \frac{a_3}{\phi}\frac{h_r}{h} + \frac{a_4}{\phi}\frac{h_{\hat{\beta}}}{h}\Big).$$

Plugging $v(t,z,x,r,\hat{\beta})$, e^* into (B.1) and dividing by $\frac{1}{\phi}\big(\frac{x}{z}\big)^{\phi}$ we have

$$\begin{aligned}
\sup_{\pi\in\mathbb{R}^2}\Big\{&h_t-\phi\hat{\beta}h+\phi\big(r+\alpha\pi^S+q\pi^P\big)h+\kappa\big(\bar{r}-r\big)h_r+\lambda\big(\bar{\beta}-\hat{\beta}\big)h_{\hat{\beta}}\\
&+\frac{\phi(\phi+1)\sigma_Z^2}{2}h-\phi^2\sigma_Z\big(\sigma_S\pi^Sn_1+\sigma_P\pi^Pn_2\big)h+\phi\sigma_Z\sigma_rn_2h_r\\
&-\phi\sigma_Z^2A_Zh_{\hat{\beta}}+\frac{\phi(\phi-1)}{2}\big((\sigma_S\pi^S)^2+(\sigma_P\pi^P)^2+2\sigma_S\sigma_P\pi^S\pi^Pn_4\big)h\\
&-\phi\sigma_r\big(\sigma_P\pi^P+\sigma_S\pi^Sn_4\big)h_r+\phi\sigma_\beta\big(\sigma_S\pi^Sn_5+\sigma_P\pi^Pn_6\big)h_{\hat{\beta}}+\frac{\sigma_r^2}{2}h_{rr}\\
&-\sigma_r\sigma_\beta n_6h_{r\hat{\beta}}+\tfrac{1}{2}\big((A_Z\sigma_Z)^2+(A_S\sigma_S)^2+(A_P\sigma_P)^2\big)h_{\hat{\beta}\hat{\beta}}\\
&-\frac{\phi\theta}{2}h\Big(\sigma_S\pi^Sa_1+\sigma_P\pi^Pa_2-\sigma_Z+\frac{a_3}{\phi}\frac{h_r}{h}+\frac{a_4}{\phi}\frac{h_{\hat{\beta}}}{h}\Big)^2\Big\}=0.
\end{aligned}\tag{B.2}$$

The values of π^S and π^P that give the supremum in (B.2) are

$$\begin{aligned}
\pi^S&=A+B\tfrac{h_r}{h}+C\tfrac{h_{\hat{\beta}}}{h},\\
\pi^P&=D+E\tfrac{h_r}{h}+F\tfrac{h_{\hat{\beta}}}{h},
\end{aligned}\tag{B.3}$$

where

$$\begin{aligned}
A&=\frac{\alpha\sigma_P(1-\phi+\theta a_2^2)+q\sigma_S\big((\phi-1)n_4-\theta a_1a_2\big)}{(\phi-1)\sigma_S^2\sigma_P\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)}\\
&\quad+\frac{\sigma_Z\sigma_S\sigma_P\big((\phi n_1-\theta a_1)(\phi-1-\theta a_2^2)-(\phi n_2-\theta a_2)((\phi-1)n_4-\theta a_1a_2)\big)}{(\phi-1)\sigma_S^2\sigma_P\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)},\\
B&=\frac{\theta\big((\phi-1)a_1a_3-\phi\sigma_ra_2^2n_4-(\phi-1)a_2a_3n_4+\phi\sigma_ra_1a_2\big)}{\phi(\phi-1)\sigma_S\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)},\\
C&=\frac{\theta(\phi-1)a_4(a_1-a_2n_4)+\phi\sigma_\beta\big((1-\phi+\theta a_2^2)n_5-((1-\phi)n_4+\theta a_1a_2)n_6\big)}{\phi(\phi-1)\sigma_S\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)},\\
D&=\frac{q\sigma_S(1-\phi+\theta a_1^2)+\alpha\sigma_P\big((\phi-1)n_4-\theta a_1a_2\big)}{(\phi-1)\sigma_P^2\sigma_S\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)}\\
&\quad+\frac{\sigma_Z\sigma_P\sigma_S\big((\phi n_2-\theta a_2)(\phi-1-\theta a_1^2)-(\phi n_1-\theta a_1)((\phi-1)n_4-\theta a_1a_2)\big)}{(\phi-1)\sigma_P^2\sigma_S\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)},\\
E&=\frac{\theta(\phi-1)(a_2a_3-a_1a_3n_4)+\phi(\phi-1)\sigma_r(1-n_4^2)-\phi\theta\sigma_r(a_1^2-a_1a_2n_4)}{\phi(\phi-1)\sigma_P\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)},\\
F&=\frac{\theta(\phi-1)a_4(a_2-a_1n_4)+\phi\sigma_\beta\big((1-\phi)(n_6-n_4n_5)+\theta(a_1^2n_6-a_1a_2n_5)\big)}{\phi(\phi-1)\sigma_P\big((\phi-1)(1-n_4^2)-\theta(a_1^2+a_2^2-2a_1a_2n_4)\big)}.
\end{aligned}$$

Substituting the values of π^S,π^P given in (B.3) into the Equation (B.2) and dividing by h, we obtain

$$\begin{aligned}
&h_t+(\phi r-\phi\hat{\beta}+C_1)h+(-\kappa r+C_2)h_r+(-\lambda\hat{\beta}+C_3)h_{\hat{\beta}}\\
&\quad+C_4h_{rr}+C_5h_{r\hat{\beta}}+C_6h_{\hat{\beta}\hat{\beta}}+C_7\frac{h_r^2}{h}+C_8\frac{h_{\hat{\beta}}^2}{h}+C_9\frac{h_rh_{\hat{\beta}}}{h}=0,
\end{aligned}\tag{B.4}$$

where

$$
\begin{aligned}
C_1 = & \frac{\phi\sigma_Z^2}{2}\big(1+\phi-\theta\big) + \phi\big(\alpha A + qD\big) \\
& + \frac{\phi(\phi-1)}{2}\big(\sigma_S^2 A^2 + \sigma_P^2 D^2 + 2\sigma_S\sigma_P n_4 AD\big) - \phi^2\sigma_Z\big(\sigma_S n_1 A + \sigma_P n_2 D\big) \\
& - \frac{\phi\theta}{2}\big(\sigma_S a_1 A + \sigma_P a_2 D\big)^2 + \phi\theta\big(\sigma_S a_1 A + \sigma_P a_2 D\big)\sigma_Z, \\
C_2 = & \kappa\bar{r} + \sigma_Z\sigma_r n_2\big(\phi-\theta\big) + \phi\big(\alpha B + qE\big) \\
& - \phi\theta\big(\sigma_S a_1 A + \sigma_P a_2 D\big)\big(\sigma_S a_1 B + \sigma_P a_2 E\big) \\
& + \phi(\phi-1)\big(\sigma_S^2 AB + \sigma_P^2 DE + \sigma_S\sigma_P n_4(AE+BD)\big) - \phi\sigma_r\big(\sigma_P D + \sigma_S n_4 A\big) \\
& - \phi^2\sigma_Z\big(\sigma_S n_1 B + \sigma_P n_2 E\big) \\
& - \phi\theta\Big(\frac{a_3}{\phi}\big(\sigma_S a_1 A + \sigma_P a_2 D\big) - \sigma_Z\big(\sigma_S a_1 B + \sigma_P a_2 E\big)\Big), \\
C_3 = & \lambda\bar{\beta} - \big(\sigma_Z\sigma_\beta n_3 + m\big)\big(\phi-\theta\big) + \phi\big(\alpha C + qF\big) - \phi^2\sigma_Z\big(\sigma_S n_1 C + \sigma_P n_2 F\big) \\
& + \phi(\phi-1)\big(\sigma_S^2 AC + \sigma_P^2 DF + \sigma_S\sigma_P n_4(AF+CD)\big) \\
& + \phi\sigma_\beta\big(\sigma_S n_5 A + \sigma_P n_6 D\big) - \phi\theta\big(\sigma_S a_1 A + \sigma_P a_2 D\big)\big(\sigma_S a_1 C + \sigma_P a_2 F\big) \\
& - \phi\theta\Big(\frac{a_4}{\phi}\big(\sigma_S a_1 A + \sigma_P a_2 D\big) - \sigma_Z\big(\sigma_S a_1 C + \sigma_P a_2 F\big)\Big) \\
C_4 = & \frac{\sigma_r^2}{2}, \\
C_5 = & -\sigma_r\sigma_\beta n_6, \\
C_6 = & \tfrac{1}{2}\big((A_Z\sigma_Z)^2 + (A_S\sigma_S)^2 + (A_P\sigma_P)^2\big), \\
C_7 = & -\frac{\phi\theta}{2}\Big(\frac{\sigma_r n_2}{\phi}\Big)^2 + \frac{\phi(\phi-1)}{2}\big(\sigma_S^2 B^2 + \sigma_P^2 E^2 + 2\sigma_S\sigma_P n_4 BE\big) \\
& - \phi\sigma_r\big(\sigma_S n_4 B + \sigma_P E\big) - \frac{\phi\theta}{2}\big(\sigma_S a_1 B + \sigma_P a_2 E\big)^2 - \theta a_3\big(\sigma_S a_1 B + \sigma_P a_2 E\big), \\
C_8 = & -\frac{\phi\theta}{2}\Big(\frac{\sigma_Z\sigma_\beta n_3 + m}{\phi\sigma_Z}\Big)^2 + \frac{\phi(\phi-1)}{2}\big(\sigma_S^2 C^2 + \sigma_P^2 F^2 + 2\sigma_S\sigma_P n_4 CF\big) \\
& + \phi\sigma_\beta\big(\sigma_S n_5 C + \sigma_P n_6 F\big) - \frac{\phi\theta}{2}\big(\sigma_S a_1 C + \sigma_P a_2 F\big)^2 - \theta a_4\big(\sigma_S a_1 C + \sigma_P a_2 F\big), \\
C_9 = & \phi(\phi-1)\big(\sigma_S^2 BC + \sigma_P^2 EF + \sigma_S\sigma_P n_4(BF+CE)\big) - \phi\sigma_r\big(\sigma_P F + \sigma_S n_4 C\big) \\
& + \phi\sigma_\beta\big(\sigma_S n_5 B + \sigma_P n_6 E\big) - \phi\theta\big(\sigma_S a_1 B + \sigma_P a_2 E\big)\big(\sigma_S a_1 C + \sigma_P a_2 F\big) \\
& - \theta a_4\big(\sigma_S a_1 B + \sigma_P a_2 E\big) - \theta a_3\big(\sigma_S a_1 C + \sigma_P a_2 F\big).
\end{aligned}
$$

The function $h(t, r, \hat{\beta})$ that solves (B.4) is given by

$$
h(t, r, \hat{\beta}) = \exp\big(\tilde{a}(t)\hat{\beta} + \tilde{b}(t)r + c(t)\big),
$$

where functions $\tilde{a}, \tilde{b}, c$ are the solution to the following system of ordinary differential equations

$$
\begin{cases}
\tilde{a}' - \phi - \lambda\tilde{a} = 0, & \tilde{a}(T) = 0, \\
\tilde{b}' + \phi - \kappa\tilde{b} = 0, & \tilde{b}(T) = 0, \\
c' + C_1 + C_2\tilde{b} + C_3\tilde{a} + (C_4 + C_7)\tilde{b}^2 + (C_6 + C_8)\tilde{a}^2 + (C_5 + C_9)\tilde{a}\tilde{b} = 0, & c(T) = 0.
\end{cases}
\tag{B.5}
$$

Solving this system of equations for functions $\tilde{a}(t)$ and $\tilde{b}(t)$, one can obtain

$$\tilde{a}(t) = \frac{\phi}{\lambda}\big(e^{-\lambda(T-t)} - 1\big),$$
$$\tilde{b}(t) = -\frac{\phi}{\kappa}\big(e^{-\kappa(T-t)} - 1\big).$$

One can also solve for function $c(t)$ and obtain the closed form solution. We do not state the solution here because it is not used in this paper.

C Detection-Error Probability

Define the conditional characteristic functions

$$\begin{aligned} f_1(\omega, t, N) &= E^{\mathbb{P}}[\exp(i\omega\xi_{1,N}) \mid \mathcal{F}_t^{S,Z,r}] = E^{\mathbb{P}}[\Xi_{1,N}^{i\omega} \mid \mathcal{F}_t^{S,Z,r}], \\ f_2(\omega, t, N) &= E^{\mathbb{Q}}[\exp(i\omega\xi_{1,N}) \mid \mathcal{F}_t^{S,Z,r}] = E^{\mathbb{Q}}[\Xi_{1,N}^{i\omega} \mid \mathcal{F}_t^{S,Z,r}] \\ &= E^{\mathbb{P}}[\exp(i\omega\xi_{1,N})\exp(\xi_{1,N})|\mathcal{F}_t^{S,Z,r}] = E^{\mathbb{P}}[\Xi_{1,N}^{i\omega+1}|\mathcal{F}_t^{S,Z,r}], \end{aligned}$$

where $i = \sqrt{-1}$ and $\xi_{1,t} = \ln \xi_t^{e^*}$.

Since the conditional characteristic functions are martingales, the Feyman - Kac theorem implies that functions f_1 and f_2 satisfy

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}\Xi_{1,t}^2 (e_t^*)^2 \frac{\partial^2 f_1}{\partial \Xi_{1,t}^2} = 0, \qquad f_1(\omega, N, N) = \Xi_{1,N}^{i\omega}, \tag{C.1}$$

$$\frac{\partial f_2}{\partial t} + \frac{1}{2}\Xi_{1,t}^2 (e_t^*)^2 \frac{\partial^2 f_2}{\partial \Xi_{1,t}^2} = 0, \qquad f_2(\omega, N, N) = \Xi_{1,N}^{i\omega+1}. \tag{C.2}$$

Let us look for a solution in form $f_1(\omega, t, N) = \Xi_{1,t}^{i\omega} e^{D(t)}$. Substituting the trial solution into (C.1) and dividing the result by $\Xi_{1,t}^{i\omega} e^{D(t)}$ yields

$$D'(t) + \tfrac{1}{2} i\omega(i\omega - 1)(e_t^*)^2 = 0, \;\; D(N) = 0.$$

Solving this equation we obtain

$$D(t) = -\frac{1}{2}\omega^2 \int_t^N (e_s^*)^2 ds - \frac{1}{2} i\omega \int_t^N (e_s^*)^2 ds.$$

Therefore,

$$\begin{aligned} &f_1(\omega, t, N) \\ &\quad = \exp\Big(i\omega\Big[-B_t - \int_t^N \frac{(e_s^*)^2}{2} ds - \frac{1 + k_S^2 + k_P^2}{2} \int_0^t (e_s^*)^2 ds\Big] - \omega^2 \int_t^N \frac{(e_s^*)^2}{2} ds\Big) \end{aligned}$$

where $B_t = \int_0^t \big(e_s^* d\hat{B}_s^Z + k_S e_s^* d\hat{B}_s^S + k_P e_s^* d\hat{B}_s^P\big)$

Similarly, we use $f_2(\omega, t, N) = \Xi_{1,t}^{i\omega+1} e^{E(t)}$ as a trial solution, which after substitution into (C.2) and division by $\Xi_{1,t}^{i\omega+1} e^{E(t)}$ yields

$$E'(t) + \tfrac{1}{2} i\omega(i\omega + 1)(e_t^*)^2 = 0, \;\; E(N) = 0.$$

Solving this ordinary differential equation we obtain

$$E(t) = -\frac{1}{2}\omega^2 \int_t^N (e_s^*)^2 ds + \frac{1}{2} i\omega \int_t^N (e_s^*)^2 ds.$$

Thus,

$$\begin{aligned} f_2(\omega, t, N) = \exp\Big(i\omega \Big[-B_t - \frac{1 + k_S^2 + k_P^2}{2} \int_0^t (e_s^*)^2 ds + \frac{1}{2} \int_t^N (e_s^*)^2 ds \Big] \\ - B_t - \frac{1 + k_S^2 + k_P^2}{2} \int_0^t (e_s^*)^2 ds - \frac{1}{2}\omega^2 \int_t^N (e_s^*)^2 ds \Big). \end{aligned}$$

It is obvious that

$$\begin{aligned} \mathrm{Re}\Big(\frac{f_1(\omega, 0, N)}{i\omega} \Big) &= -\frac{1}{\omega} \exp\Big(-\frac{1}{2}\omega^2 \int_0^N (e_s^*)^2 ds \Big) \sin\Big(\frac{1}{2}\omega \int_0^N (e_s^*)^2 ds \Big), \\ \mathrm{Re}\Big(\frac{f_2(\omega, 0, N)}{i\omega} \Big) &= \frac{1}{\omega} \exp\Big(-\frac{1}{2}\omega^2 \int_0^N (e_s^*)^2 ds \Big) \sin\Big(\frac{1}{2}\omega \int_0^N (e_s^*)^2 ds \Big). \end{aligned}$$

Therefore, the detection-error probability (see formula 861.22 in Dwight, 1973) is

$$\begin{aligned} \varepsilon_N(\theta) &= \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \Big(\mathrm{Re}\Big[\frac{f_2(\omega, 0, N)}{i\omega} \Big] - \mathrm{Re}\Big[\frac{f_1(\omega, 0, N)}{i\omega} \Big] \Big) d\omega \\ &= \tfrac{1}{2} - \tfrac{1}{2} \mathrm{erf}\Big(\frac{\sqrt{\tilde{K}}}{2} \Big), \end{aligned}$$

where $\tilde{K} = \frac{1}{2} \int_0^N (e_s^*)^2 ds$ and $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

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