PORTFOLIO OPTIMIZATION WITH DRAWDOWN CONSTRAINTS[1]

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Abstract

We propose a new one-parameter family of risk measures, which is called Conditional Drawdown-at-Risk (CDaR). These measures of risk are functionals of the portfolio drawdown (underwater) curve considered in an active portfolio management. For some value of the tolerance parameter β, the CDaR is defined as the mean of the worst (1-β)*100% drawdowns. The CDaR risk measure contains the Maximal Drawdown and Average Drawdown as its limiting cases. For a particular example, we find the optimal portfolios for a case of Maximal Drawdown, a case of Average Drawdown, and several intermediate cases between these two. The CDaR family of risk measures is similar to Conditional Value-at-Risk (CVaR), which is also called Mean Shortfall, Mean Access loss, or Tail Value-at-Risk. Some recommendations on how to select the optimal risk measure for getting practically stable portfolios are provided. We solved a real life portfolio allocation problem using the proposed measures.

1. Introduction

Optimal portfolio allocation is a longstanding issue in both practical portfolio management and academic research on portfolio theory. Various methods have been proposed and studied (for a recent review, see, for example, [6]). All of them, as a starting point, assume some measure of portfolio risk.

From a standpoint of a fund manager, who trades clients' or bank's proprietary capital, and for whom the clients' accounts are the only source of income coming in the form of management and

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[1] The views expressed herein are those of the authors and are not necessarily those of TrendLogic Associates, Inc. or the University of Florida.

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incentive fees, losing these accounts is equivalent to the death of his business. This is true with no regard to whether the employed strategy is long-term valid and has very attractive expected return characteristics. Such fund manager's primary concern is to keep the existing accounts and to attract the new ones in order to increase his revenues. A particular client who was persuaded into opening an account with the manager through reading the disclosure document, listening to the manager's attractive story, knowing his previous returns, etc., will decide on firing the manager based, most likely, on his account's drawdown sizes and duration. In particular, it is highly uncommon, for a Commodity Trading Advisor (CTA) to still hold a client whose account was in a drawdown, even of small size, for longer than 2 years. By the same token, it is unlikely that a particular client will tolerate a 50% drawdown in an account with an average- or small-risk CTA. Similarly, in an investment bank setup, a proprietary system trader will be expected to make money in 1 year at the longest, i.e., he cannot be in a drawdown for longer than a year. Also, he/she may be shut down if a certain maximal drawdown condition will be breached, which, normally, is around 20% of his backing equity. Additionally, he will be given a warning drawdown level at which he will be reviewed for letting him keep running the system (around 15%). Obviously, these issues make managed accounts practitioners very concerned about both the size and duration of their clients' accounts drawdowns.

First, we want to mention paper [7], where an assumption of log-normality of equity statistics and use of dynamic programming theory led to an exact analytical solution of a maximal drawdown problem for a one-dimensional case. A subsequent generalization of this work for multiple dimensions was done in [3]. In difference to these works, which were looking to find a time-dependent fraction of "capital at risk", we will be looking to find a constant set of weights, which will satisfy a certain risk condition over a period of time. We make no assumption about the underlying probability distribution, which allows considering variety of practical applications. We primarily concentrate on the portfolio equity curves over a particular past history path, which, effectively, makes the risk measures not stochastic but historical. Being perfectly aware of this insufficiency, we leave the issue of predictive power of a constant set of weights for future research, trying to introduce and test the new approach in this simplified version. To some extend we consider a setup similar to the index tracking problem [4] where an index historical performance is replicated by a portfolio with constant weights.

In this paper, we have introduced and studied a one-parameter family of risk measures called Conditional Drawdown-at-Risk (CDaR). This measure of risk quantifies in aggregated format the number and magnitude of the portfolio drawdowns over some period of time. By definition, a drawdown is the drop in the portfolio value comparing to the maximum achieved in the past. We can define drawdown in absolute or relative (percentage) terms. For example, if at the present time the portfolio value equals $9M and the maximal portfolio value in the past was $10M, we can say that the portfolio drawdown in absolute terms equals $1M and in relative terms equals 10%. For some value of the tolerance parameter β, the β-CDaR is defined as the mean of the worst $(1-\beta)*100\%$ drawdowns experienced over some period of time. For instance, 0.95-CDaR (or 95%- CDaR) is the average of the worst 5% drawdowns over the considered time interval. The CDaR risk measure contains the average drawdown and maximal drawdown as its limiting cases. The CDaR takes into account both the size and duration of the drawdowns, whereas the maximal drawdown measure concentrates on a single event – maximal account's loss from its previous peak.

CDaR is related to Value-at-Risk (VaR) risk measure and to Conditional Value-at-Risk (CVaR) risk measure studied in paper [13]. By definition, with respect to a specified probability level β, the β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α in a specified time τ (see, for instance, [5]), whereas the β-CVaR is the conditional expectation of losses above that amount α. The CDaR risk measure is similar to CVaR and can be viewed as a modification of the CVaR to the case when the loss-function is defined as a drawdown. CDaR and CVaR are conceptually closely related percentile-based risk performance measures. Optimization approaches developed for CVaR can be directly extended to CDaR. The paper [11] considers several equivalent approaches for generating return-CVaR efficient frontiers; in particular, it considers an approach, which maximizes return with CVaR constraints. A nice feature of this approach is that the threshold, which is exceeded $(1-\beta)*100\%$, is calculated automatically using an additional variable (see details in [11,13]) and the resulting problem is linear. CVaR is known also as Mean Excess Loss, Mean Shortfall [4,10], or Tail Value-at-Risk [2]. A case study on the hedging of a portfolio of options using the CVaR minimization technique is included in [11]. Also, the CVaR minimization approach was applied to credit risk management of a portfolio of bonds [1]. A case study on optimization of a portfolio of stocks with CVaR constraints is considered in [11].

Similar to the Markowitz mean-variance approach [9], we formulate and solve the optimization problem with the return performance function and CDaR constraints. The return-CDaR optimization problem is a piece-wise linear convex optimization problem (see definition of convexity in [12]), which can be reduced to a linear programming problem using auxiliary variables. Explanation of the procedure for reducing the piece-wise linear convex optimization problems to linear programming problems is beyond the scope of this paper. In formulating the optimization problems with CDaR constraints and reducing it to a linear programming problem, we follow ideas presented in the paper [11]. Linear programming allows solving large optimization problems with hundreds of thousands of instruments. The algorithm is fast, numerically stable, and provides a solution during one run (without adjusting parameters like in genetic algorithms or neural networks). Linear programming approaches are routinely used in portfolio optimization with various criteria, such as mean absolute deviation [8], maximum deviation [14], and mean regret [4]. The reader interested in other applications of optimization techniques in the finance area can find relevant papers in [15].

2. General Setup

Denote by function $w(\vec{x},t)$ the uncompounded portfolio value at time t, where portfolio vector $\vec{x}=(x_1,x_2,\dots,x_m)$ consists of weights of m instruments in the portfolio. The drawdown function at time t is defined as the difference between the maximum of the function $w(\vec{x},t)$ over the history preceding the point t and the value of this function at time t

$$f(\vec{x},t)=\max_{0\le\tau\le t}\{w(\vec{x},\tau)\}-w(\vec{x},t)\,. \tag{1}$$

We consider three risk measures: (i) maximum drawdown (MaxDD), (ii) average drawdown (AvDD), and (iii) conditional drawdown-at-risk (CDaR). The last risk measure, Conditional Drawdown-at-Risk, is actually a family of performance measures depending upon a parameter β. It is defined similar to Conditional Value-at-Risk studied in [2] and, as special cases, contains the Maximum Drawdown and the Average Drawdown risk measures.

Maximum drawdown on an the interval $[0,T]$, is calculated by maximizing the drawdown function $f(\vec{x},t)$, i.e.,

$$\mathbf{M}(\vec{x}) = \max_{0 \le t \le T} \{ f(\vec{x},t) \}. \tag{2}$$

The average drawdown is equal to

$$\mathbf{A}(\vec{x}) = \frac{1}{T} \int_0^T f(\vec{x},t)\, dt . \tag{3}$$

For some value of the parameter $\beta \in [0,1]$, the CDaR, is defined as the mean of the worst $(1-\beta)*100\%$ drawdowns. For instance, if β=0, then CDaR is the average drawdown, and if $\beta = 0.95$, then CDaR is the average of the worst 5% drawdowns. Let us denote by $\alpha(\vec{x},\beta)$ a threshold such that $(1-\beta)*100\%$ of drawdowns exceed this threshold. Then, CDaR with tolerance level β can be expressed as follows

$$\Delta_\beta(\vec{x}) = \frac{1}{(1-\beta)T} \int_\Omega f(\vec{x},t)\, dt, \quad \Omega = \{ t \in [0,T] : f(\vec{x},t) \ge \alpha(\vec{x},\beta) \} . \tag{4}$$

Here, when β tends to 1, CDaR tends to the maximum drawdown, i.e. $\Delta_1(\vec{x}) = \mathbf{M}(\vec{x})$.
To limit possible risks, depending upon our risk preference, we can impose constraints on the maximum drawdown given by (2)

$$\mathbf{M}(\vec{x}) \le \nu_1 C,$$

on average drawdown given by (3)

$$\mathbf{A}(\vec{x}) \le \nu_2 C ,$$

on DVaR given by (4)

$$\Delta_\beta(\vec{x}) \le \nu_3 C,$$

or combine several constraints together

$$\mathbf{M}(\vec{x}) \le \nu_1 C, \quad \mathbf{A}(\vec{x}) \le \nu_2 C, \quad \Delta_\beta(\vec{x}) \le \nu_3 C , \tag{5}$$

where the constant C represents the available capital and the coefficients ν_1, ν_2 and ν_3 define the proportion of this capital which is "allowed to be lost". Usually,

$$0 \le \nu_1, \nu_2, \nu_3 \le 1 . \tag{6}$$

Suppose that the historical returns for m portfolio instruments on interval $[0,T]$ are available. Let vector $\vec{y}(t) = (y_1(t), y_2(t), \ldots, y_m(t))$ be a set of uncompounded cumulative net profits for m portfolio instruments at a time moment t. The cumulative portfolio value then equals $w(\vec{x},t) = \vec{y}(t) \cdot \vec{x}$.

The average annualized return $R(\vec{x})$ over a period $[0,T]$, which is a linear function of $\vec{x}$, is defined as follows

$$R(\vec{x}) = \tfrac{1}{dC} w(\vec{x},t) = \tfrac{1}{dC} \vec{y}(t) \cdot \vec{x} , \tag{7}$$

where d is the number of years in the time interval $[0,T]$.

For the case considered, the so-called technological constraints on the vector $\vec{x}$ need to be imposed. Here, we assume that they are given by the set of box constraints:

$$X = \left\{ \vec{x} : x_{\min} \le x_k \le x_{\max}, \forall\, k = \overline{1,m} \right\}. \tag{8}$$

for some constant values of $x_{\min}$ and $x_{\max}$.

Our objective is to maximize the return $R(\vec{x})$ subject to constraints on various risk performance measures and technological constraints (8) on the portfolio positions.

3. Problem Statement

Maximization of the average return with constraints on maximum drawdown can be formulated as the following mathematical programming problem

$$\begin{aligned} &\max_{\vec{x}} R(\vec{x}) \\ &\textit{subject to} \\ &\begin{cases} \mathbf{M}(\vec{x}) \le \nu_1 C \\ \vec{x} \in X. \end{cases} \end{aligned} \tag{9}$$

Maximization of the average return with constraints on the average drawdown can be formulated as follows

$$\begin{aligned} &\max_{\vec{x}} R(\vec{x}) \\ &\textit{subject to} \\ &\begin{cases} \mathbf{A}(\vec{x}) \le \nu_2 C \\ \vec{x} \in X. \end{cases} \end{aligned} \tag{10}$$

Analogously, maximization of the average return with constraints on CDaR can be formulated as follows

$$\begin{aligned} &\max_{\vec{x}} R(\vec{x}) \\ &\textit{subject to} \\ &\begin{cases} \Delta_\beta(\vec{x}) \le \nu_3 C \\ \vec{x} \in X. \end{cases} \end{aligned} \tag{11}$$

Similar to [2], the problems (9), (10), (11) can be reduced to linear programming problems using some auxiliary variables.

4. Discretization

By dividing interval $[0,T]$ into N equal intervals (for instance, trading days)

$$t_i = i \cdot \frac{T}{N}, \quad i = \overline{1, N}, \tag{12}$$

we create the discrete approximations of the vector function $\vec{y}(t)$

$$\vec{y}(t_i) = \vec{y}_i, \tag{13}$$

the drawdown function

$$f_i(\vec{x}) = \max_{1 \le j \le i}\{\vec{y}_j \cdot \vec{x}\} - \vec{y}_i \cdot \vec{x}, \tag{14}$$

and the average annualized return function

$$R(\vec{x}) = \tfrac{1}{dC}\, \vec{y}_N \cdot \vec{x}. \tag{15}$$

For the discrete time case, problems (9), (10) and (11) can be accordingly reformulated. The optimization problem with constraint on maximum drawdown is given below

$$\begin{aligned} &\max_{\vec{x}} \left\{\tfrac{1}{dC}\, \vec{y}_N \cdot \vec{x}\right\} \\ &\textit{subject to} \\ &\begin{cases} \max\limits_{1 \le i \le N}\{\max\limits_{1 \le j \le i}\{\vec{y}_j \cdot \vec{x}\} - \vec{y}_i \cdot \vec{x}\} \le \nu_1 C \\ \mathrm{x}_{\min} \le x_k \le x_{\max}, \quad \forall\, k = \overline{1, m}. \end{cases} \end{aligned} \tag{16}$$

The optimization problem with constraint on average drawdown can be written as follows

$$\begin{aligned} &\max_{\vec{x}} \left\{\tfrac{1}{dC}\, \vec{y}_N \cdot \vec{x}\right\} \\ &\textit{subject to} \\ &\begin{cases} \tfrac{1}{N} \sum\limits_{i=1}^{N} \{\max\limits_{1 \le j \le i}\{\vec{y}_j \cdot \vec{x}\} - \vec{y}_i \cdot \vec{x}\} \le \nu_2 C \\ \mathrm{x}_{\min} \le x_k \le x_{\max}, \quad \forall\, k = \overline{1, m}. \end{cases} \end{aligned} \tag{17}$$

Following the approach for Conditional Value-at-Risk (CVaR) [2], it can be proved that the discrete version of the optimization problem with constraint on CDaR may be stated as follows

$$\begin{aligned} &\max_{\vec{x}} \left\{\tfrac{1}{dC}\, \vec{y}_N \cdot \vec{x}\right\} \\ &\textit{subject to} \\ &\begin{cases} \alpha + \tfrac{1}{(1-\beta)N} \sum\limits_{i=1}^{N} (\{\max\limits_{1 \le j \le i}\{\vec{y}_j \cdot \vec{x}\} - \vec{y}_i \cdot \vec{x}\} - \alpha)^+ \le \nu_3 C \\ \mathrm{x}_{\min} \le x_k \le x_{\max}, \quad \forall\, k = \overline{1, m}, \end{cases} \end{aligned} \tag{18}$$

where we use the notation $(g)^+ = \max\{0, g\}$. An important feature of this formulation is that it does not involve the threshold function $\alpha(\vec{x}, \beta)$. An optimal solution to the problem (18) with respect to $\vec{x}$ and α gives the optimal portfolio and the corresponding value of the threshold function.

The problems (16), (17), and (18) have been reduced to linear programming problems using auxiliary variables and have been solved by the CPLEX solver (inputs are prepared with C++ programming language). An alternative verification of the solutions was obtained via solving similar optimization problems using a more general Genetic Algorithm method implemented in VB6, discussion of which is beyond the present scope.

5. Results

As the starting equity curves, we have used the equity curves generated by a characteristic futures technical trading system in $m = 32$ different markets, covering a wide range of major liquid markets (currencies, currency crosses, U.S. treasuries both short- and long-term, foreign long-term treasuries, international equity indices, and metals). The list of market ticker symbols, provided in the results below, is mnemonic and corresponds to the widely used data provider, FutureSource.

It is necessary to mention several issues related to technological constraints (8). In our case, we chose $x_{\min} = 0.2$ and $x_{\max} = 0.8$. This choice was dictated by the need to have the resultant margin-to-equity ratio in the account within admissible bounds, which are specific for a particular portfolio. These constraints, in this futures trading setup is analogous to the "fully-invested" condition from classical Sharpe-Markowitz theory [1], and it is namely this condition, which makes the efficient frontier concave. In the absence of these constraints, the efficient frontier would be a straight line passing through (0,0), due to the virtually infinite leverage of these types of strategies. Another subtle issue has to do with the stability of the optimal portfolios if the constraints are "too lax". It is a matter of empirical evidence that the more lax the constraints are – the better portfolio equity curve you can get through optimal mixing – and the less stable with respect to walk-forward analysis these results would be. The above set of constraints was empirically found to be both leading to sufficiently stable portfolios and allowing enough mixing of the individual equity curves.

The individual equity curves, when the market existed at the time, covered a time span of 1/1/1988 through 9/1/1999. The equity curves were based on $20M backing equity in a margin account and were uncompounded, i.e. it was assumed that the amount of risk being taken, was always based of the original $20M, not taking the money being made or lost into account.

The problem, then, is to find a set of weights $\vec{x} = (x_1, x_2, \dots, x_m)$, such that it solves the minimization problems (16), (17), or (18). Let us denote the problem (16) as the MaxDD problem, the problem (17) as the AvDD problem, and the problem (18) as the β -CDaR problem. We have solved the above optimization problems for cases of $(1-\beta) = 0\%$, 5%, 10%, 20%, 40%, 60%, 80% and 100%. As we have noted before, cases of $(1-\beta) = 0\%$ and $(1-\beta) = 100\%$ correspond to MaxDD and AvDD problems, accordingly.

Tables 1 and 2 below provide the list of markets and corresponding sets of optimal weights for MaxDD and AvDD problems. Table 3 provides the weights for the case with $(1-\beta) = 5\%$ CDaR. In these tables, the solution achieving maximal Reward/Risk ratio is boldfaced. Note that the smallest value of risk is chosen in such a way that the solutions to the optimization problem still exist. This means that each problem does not have a solution beyond the upper and lower bounds of the risk range covered (the whole efficient frontier is shown). Notions of risk and rate of return are expressed in percent with respect to the original account size, i.e. $20M.

Efficient frontiers for problems reward-MaxDD and reward-AvDD, are shown in Figures 1 and 2, respectively. We do not show efficient frontiers for CDaR measure on separate graphs (except for MaxDD and AvDD). However, we show on Figure 3 the reward-MaxDD graphs for portfolios optimal with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints. As it is expected, the case with $(1-\beta)=0\%$ CDaR corresponding to MaxDD has a concave efficient frontier majorating other graphs. The reward is not maximal for each level of MaxDD when we solved the optimization problems with $(1-\beta)=5\%$, 40% and 100% CDaR constraints. Viewed from the reference point of MaxDD problem, $(1-\beta)<100\%$ solutions are uniformly "worse". However, none of these solutions are truly better or worse than others from a mathematical standpoint. Each of them provides the optimal solution in its own sense. Some thoughts on which might be a better solution from a practical standpoint are provided below. Similar to Figure 3, Figure 4 depicts the reward-AvDD graphs for portfolios optimal with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints. The case with $(1-\beta)=100\%$ CDaR corresponding to AvDD has a concave efficient frontier majorating other graphs.

As in classical portfolio theory, we are interested in a portfolio with a maximal Reward/Risk ratio, i.e., the portfolio where the straight line coming through (0,0) becomes tangent to the efficient frontier. We will call the Reward/Risk ratios for Risk defined in terms of problems (16), (17), and (18) as MaxDDRatio, AvDDRatio, and CDaRRatio which, by definition, are:

$$MaxDDRatio=\frac{R(\vec{x})}{M(\vec{x})},\ AvDDRatio=\frac{R(\vec{x})}{A(\vec{x})}\text{ , and } CDaRRatio=\frac{R(\vec{x})}{\Delta_\beta(\vec{x})}.$$

The charts of MaxDDRatio and AvDDRatio quantities are shown in Figures 5 and 6 for the same cases of $(1-\beta)$ as in Figures 3 and 4.

We have solved optimization problem (18) for cases of $(1-\beta)=0\%$, 5%, 10%, 20%, 40%, 60%, 80% and 100%. Let us note that already the case of $(1-\beta)=5\%$ (see Table 3), which considers minimization of the worst 5% part of the underwater curve, is producing a set of weights significantly different from the $(1-\beta)=0\%$ case (MaxDD problem), and $(1-\beta)=5\%$ CDaR case contains several tens of events over which the averaging was performed. We consider that optimization with $(1-\beta)=5\%$ or 10% constraints produces a more robust portfolio than the optimization with MaxDD or AvDD constraints. CDaR solution takes into account many significant drawdowns, comparing to the case with MaxDD constraints, which considers only the largest drawdown. Also, CDaR solution is not dominated by many small drawdowns like the case with AvDD constraints.

We have also made an alternative check of our results via solving the related nonlinear optimization problems corresponding to problems (16-18). These problems have optimized the corresponding drawdown ratios defined above within the same set of constraints. Verification was done using Genetic Algorithm-based search software. We were satisfied to find that this procedure has produced the same sets of weights for the optimal solutions.

6. Conclusions

We have introduced a new CDaR risk measure, which, we believe, is useful for the practical portfolio management. This measure is similar to CVaR risk measure and has the MaxDD and

AvDD risk measures as its limiting cases. We have studied Reward/Risk ratios implied by these measures of risk, namely MaxDDRatio, AvDDRatio, and CDaRRatio. We have shown that the portfolio allocation problem with CDaR, MaxDD and AvDD risk measures can be efficiently solved. We have posed and for a real-life example, solved a portfolio allocation problem. These developments, if implemented in a managed accounts' environment will allow a trading or risk manager to allocate risk according to his personal assessment of extreme drawdowns and their duration on his portfolio equity.

We believe that however attractive the MaxDD approach is, the solutions produced by this optimization may contain a significant amount of statistical error because the decision is based on a single observation of maximal loss. Having a CDaR family of risk measures allows a risk manager to have control over the worst $(1-\beta)*100\%$ of drawdowns, and due to statistical averaging within that range, to get a better predictive power of this risk measure in the future, and therefore a more stable portfolio. Our studies indicate that when considering CDaR with an appropriate level (e.g., $\beta = 0.95$, i.e., optimizing over the 5% of the worst drawdowns), one can get a more stable weights allocation than that produced by the MaxDD problem. A detailed study of this issue calls for a separate publication.

Table 1. List of markets and corresponding sets of optimal weights for the MaxDD problem. The solution achieving maximal Reward/Risk ratio is boldfaced.

Risk, %	4.0	5.0	**6.0**	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0
Reward, %	25.0	36.3	**44.5**	51.4	57.3	63.0	67.7	71.7	75.2	78.0	80.4	81.9	82.9	83.0
Reward/Risk	6.26	7.27	**7.42**	7.34	7.16	7.00	6.77	6.52	6.27	6.00	5.74	5.46	5.18	4.88

OPTIMAL PORTFOLIO CONFIGURATION

AAO	0.20	0.25	**0.25**	0.28	0.21	0.39	0.68	0.80	0.69	0.80	0.80	0.80	0.80	0.80
AD	0.20	0.40	**0.74**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
AXB	0.20	0.37	**0.32**	0.47	0.63	0.80	0.55	0.64	0.80	0.80	0.80	0.80	0.80	0.80
BD	0.20	0.20	**0.20**	0.20	0.62	0.41	0.53	0.56	0.80	0.80	0.80	0.80	0.80	0.80
BP	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.22	0.51	0.77	0.80	0.80	0.80
CD	0.25	0.59	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.62	0.80	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DGB	0.20	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.63	0.80	0.80	0.80	0.80	0.80
ED	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.35	0.74	0.80	0.80	0.80
EU	0.20	0.20	**0.20**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FV	0.20	0.20	**0.39**	0.58	0.52	0.50	0.54	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.27	0.58	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXBPJY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.53	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.28	**0.29**	0.32	0.34	0.65	0.72	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.20	0.20	**0.41**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.33	0.20	**0.25**	0.30	0.73	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.27	0.80	0.80
FXUSSG	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.28	0.21	0.43	0.72	0.80	0.80	0.80
FXUSSK	0.20	0.80	**0.80**	0.65	0.73	0.70	0.60	0.35	0.20	0.20	0.20	0.80	0.80	0.80
GC	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.57	0.80	0.80
JY	0.20	0.23	**0.34**	0.25	0.37	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LBT	0.20	0.35	**0.62**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LFT	0.20	0.20	**0.20**	0.20	0.39	0.63	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LGL	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.37	0.80	0.80	0.80	0.80	0.80	0.80
LML	0.20	0.27	**0.36**	0.46	0.51	0.60	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80
MNN	0.20	0.30	**0.42**	0.45	0.44	0.80	0.80	0.80	0.77	0.80	0.80	0.80	0.80	0.80
SF	0.20	0.20	**0.37**	0.39	0.52	0.52	0.63	0.75	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.40	0.80
SJB	0.49	0.74	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SNI	0.20	0.56	**0.67**	0.69	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	**0.23**	0.32	0.60	0.69	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 2. List of markets and corresponding sets of optimal weights for the AvDD problem. The solution achieving maximal Reward/Risk ratio is boldfaced.

Risk, %	0.77	1.00	**1.23**	1.46	1.50	1.69	1.92	2.15	2.38	2.61	2.84	3.07
Reward, %	21.7	35.6	**45.3**	53.3	54.5	59.9	65.7	70.6	74.8	78.2	81.2	83.0
Reward/risk	28.2	35.6	**36.8**	36.5	36.3	35.4	34.2	32.9	31.4	30.0	28.6	27.0

OPTIMAL PORTFOLIO CONFIGURATION

AAO	0.20	0.46	**0.61**	0.77	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
AD	0.21	0.57	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
AXB	0.20	0.20	**0.23**	0.55	0.62	0.80	0.80	0.80	0.80	0.80	0.80	0.80
BD	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.52	0.80	0.80
BP	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.43	0.80	0.80	0.80
CD	0.20	0.37	**0.54**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.24	0.60	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DGB	0.33	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.30	0.71	0.80
ED	0.20	0.30	**0.35**	0.33	0.32	0.21	0.31	0.44	0.70	0.75	0.80	0.80
EU	0.20	0.20	**0.20**	0.20	0.20	0.20	0.46	0.80	0.80	0.80	0.80	0.80
FV	0.20	0.20	**0.37**	0.50	0.53	0.76	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.20	**0.20**	0.31	0.33	0.42	0.57	0.73	0.80	0.80	0.80	0.80
FXBPJY	0.20	0.20	**0.32**	0.49	0.50	0.69	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	**0.29**	0.53	0.58	0.77	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.20	0.59	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.29	0.62	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	**0.20**	0.20	0.20	0.27	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.20	0.20	**0.20**	0.40	0.48	0.71	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.20	0.74	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
GC	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.79
JY	0.20	0.38	**0.62**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LBT	0.20	0.52	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LFT	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.36	0.46	0.80
LGL	0.20	0.20	**0.20**	0.20	0.20	0.20	0.29	0.48	0.65	0.80	0.80	0.80
LML	0.20	0.20	**0.21**	0.34	0.34	0.49	0.64	0.80	0.80	0.80	0.80	0.80
MNN	0.20	0.20	**0.20**	0.20	0.20	0.20	0.42	0.80	0.80	0.80	0.80	0.80
SF	0.20	0.20	**0.38**	0.50	0.54	0.67	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.80
SJB	0.23	0.67	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SNI	0.20	0.33	**0.47**	0.62	0.66	0.72	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.32	0.69	0.77	0.80	0.80

Table 3. List of markets and corresponding sets of optimal weights for the CDaR problem with $(1-\beta) = 5\%$. The solution achieving maximal Reward/Risk ratio is boldfaced.

Risk, %	3.0	3.2	3.7	3.8	3.9	4.0	**5.0**	6.0	7.0	8.0	9.0	10.0	11.0	12.0
Reward, %	24.2	27.2	33.3	34.4	35.5	36.6	**46.3**	54.7	62.1	68.4	73.9	78.6	82.0	83.0
α, %	2.55	2.64	3.10	3.18	3.27	3.36	**4.26**	5.13	6.02	6.81	7.66	8.61	9.57	9.98
Reward/Risk	8.06	8.50	8.99	9.04	9.09	9.14	**9.26**	9.12	8.86	8.55	8.21	7.86	7.45	6.92

OPTIMAL PORTFOLIO CONFIGURATION

AAO	0.20	0.21	0.30	0.32	0.33	0.34	**0.49**	0.54	0.69	0.80	0.80	0.80	0.80	0.80
AD	0.24	0.36	0.60	0.64	0.68	0.69	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
AXB	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.33	0.46	0.80	0.80	0.80	0.80
BD	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.60	0.69	0.67	0.80	0.80	0.80	0.80
BP	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80	0.80
CD	0.20	0.20	0.29	0.31	0.32	0.33	**0.49**	0.64	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.23	0.34	0.41	0.44	0.46	0.51	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DGB	0.50	0.71	0.80	0.80	0.80	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.31	0.80	0.80	0.80
ED	0.20	0.20	0.20	0.20	0.20	0.20	**0.26**	0.27	0.31	0.28	0.28	0.48	0.64	0.80
EU	0.20	0.20	0.23	0.26	0.30	0.31	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FV	0.20	0.20	0.20	0.23	0.25	0.30	**0.47**	0.47	0.56	0.73	0.80	0.80	0.80	0.80
FXADJY	0.20	0.22	0.33	0.34	0.35	0.36	**0.49**	0.69	0.80	0.80	0.80	0.80	0.80	0.80
FXBPJY	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.32	0.50	0.73	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	0.29	0.31	0.34	0.34	**0.43**	0.39	0.46	0.76	0.80	0.80	0.80	0.80
FXEUJY	0.20	0.35	0.68	0.72	0.74	0.77	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.20	0.28	0.30	0.31	0.29	**0.38**	0.59	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.77	0.80
FXUSSG	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.37	0.59	0.75	0.80	0.80	0.80	0.80
FXUSSK	0.20	0.20	0.22	0.22	0.24	0.25	**0.61**	0.80	0.80	0.80	0.79	0.80	0.80	0.80
GC	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.80
JY	0.31	0.35	0.42	0.43	0.45	0.47	**0.75**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LBT	0.20	0.20	0.20	0.20	0.20	0.20	**0.47**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LFT	0.20	0.20	0.20	0.20	0.20	0.20	**0.25**	0.28	0.43	0.58	0.66	0.76	0.80	0.80
LGL	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.27	0.66	0.80	0.80
LML	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.31	0.52	0.69	0.74	0.80	0.80
MNN	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.34	0.74	0.80	0.80	0.80	0.80	0.80
SF	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.54	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.58	0.80
SJB	0.47	0.57	0.71	0.74	0.77	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SNI	0.21	0.22	0.29	0.29	0.30	0.33	**0.58**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	0.20	0.20	0.20	0.20	**0.20**	0.20	0.20	0.39	0.70	0.80	0.80	0.80

Figure 1. Efficient frontier for the MaxDD problem (rate of return versus MaxDD).

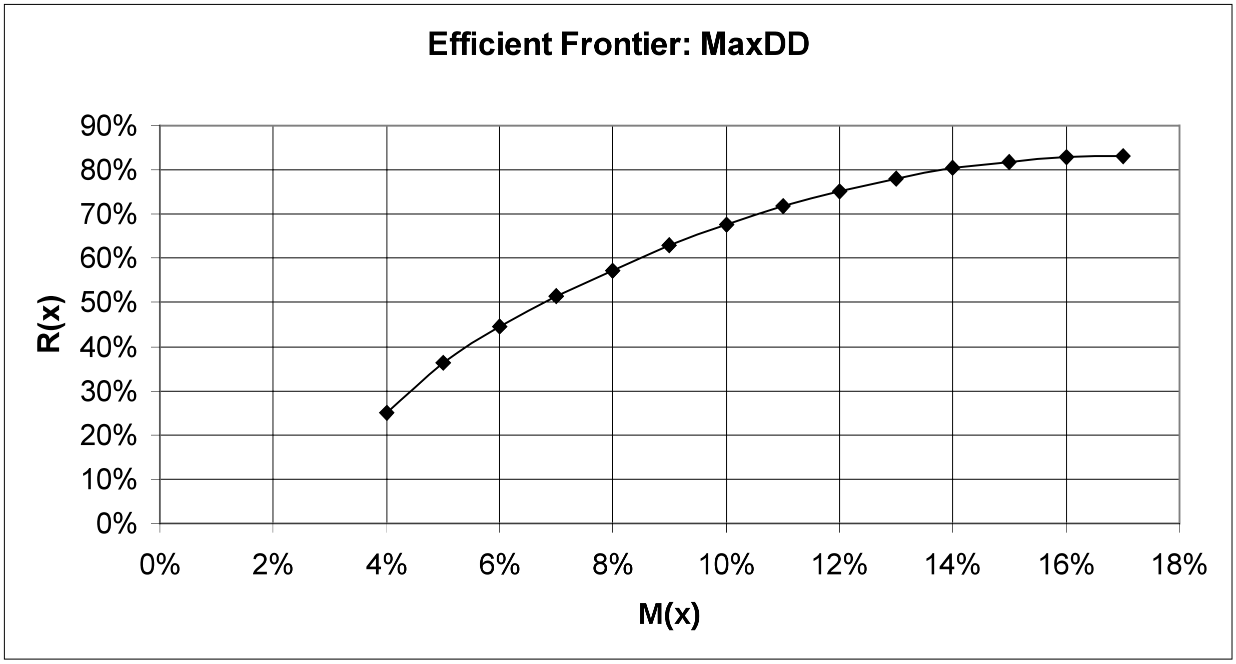


Figure 2. Efficient frontier for the AvDD problem (rate of return versus AvDD).

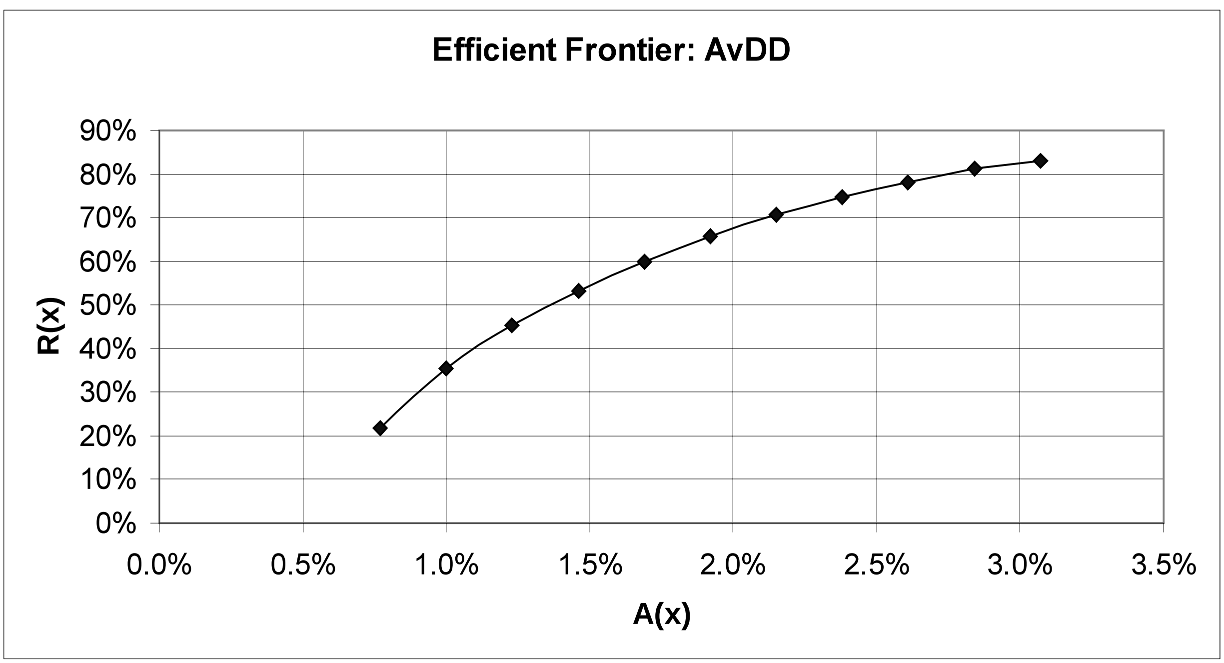


Figure 3. Reward-MaxDD graphs for optimal portfolios with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints (rate of return versus MaxDD). The frontier is efficient only for the case with $(1-\beta)=0\%$ CDaR constraints, which corresponds to the MaxDD risk measure.

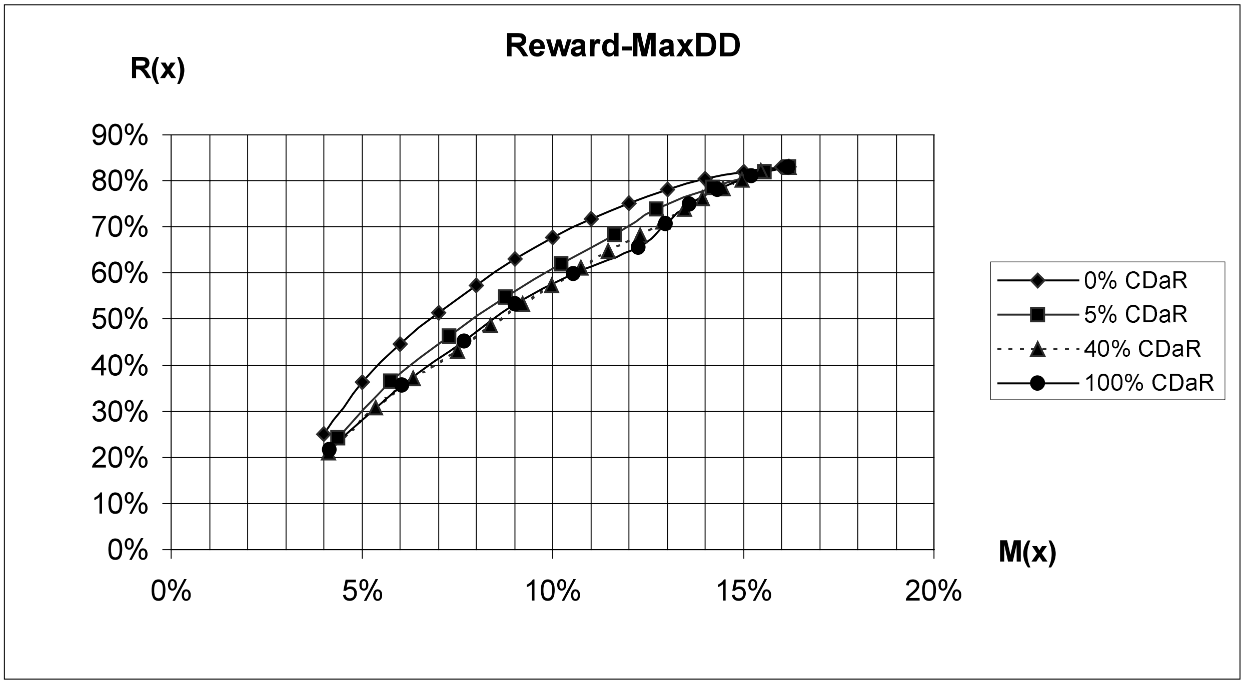


Figure 4. Reward-AvDD graphs for optimal portfolios with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints (rate of return versus AvDD). The frontier is efficient only for the case with $(1-\beta)=100\%$ CDaR constraints, which corresponds to the AvDD risk measure.

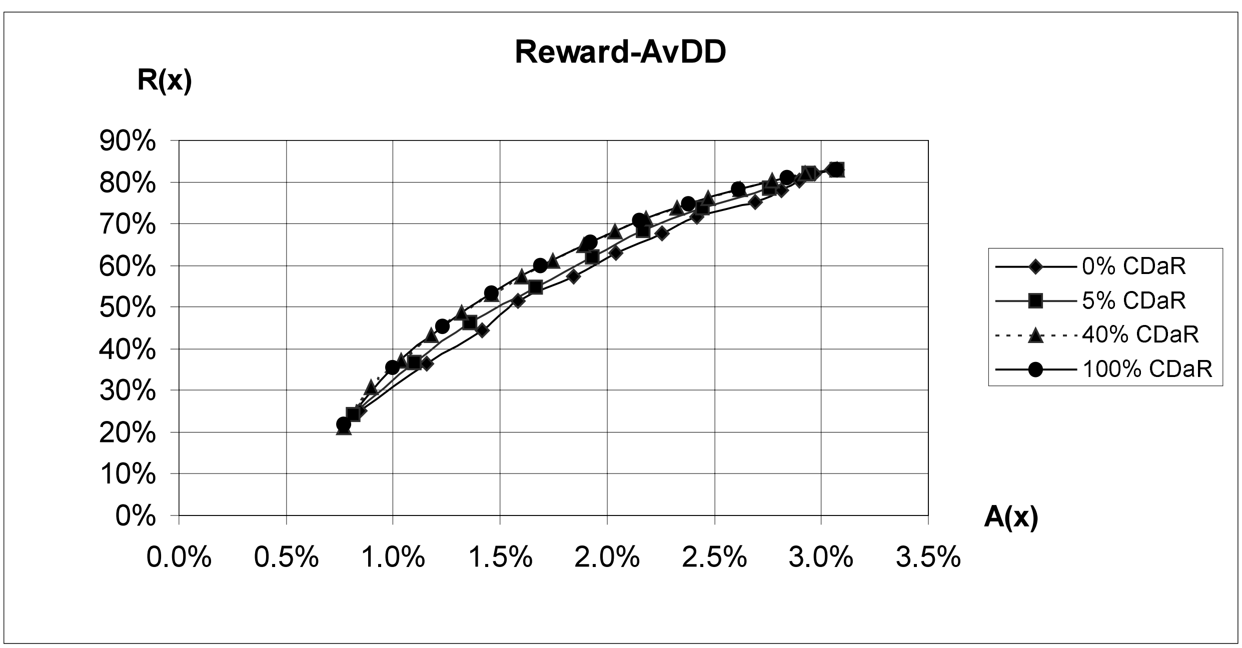


Figure 5. MaxDDRatio graphs for optimal portfolios with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints (MaxDDRatio versus MaxDD). The maximum MaxDDRatio is achieved in the case with $(1-\beta)=0\%$ CDaR constraints, which corresponds to the MaxDD risk measure.

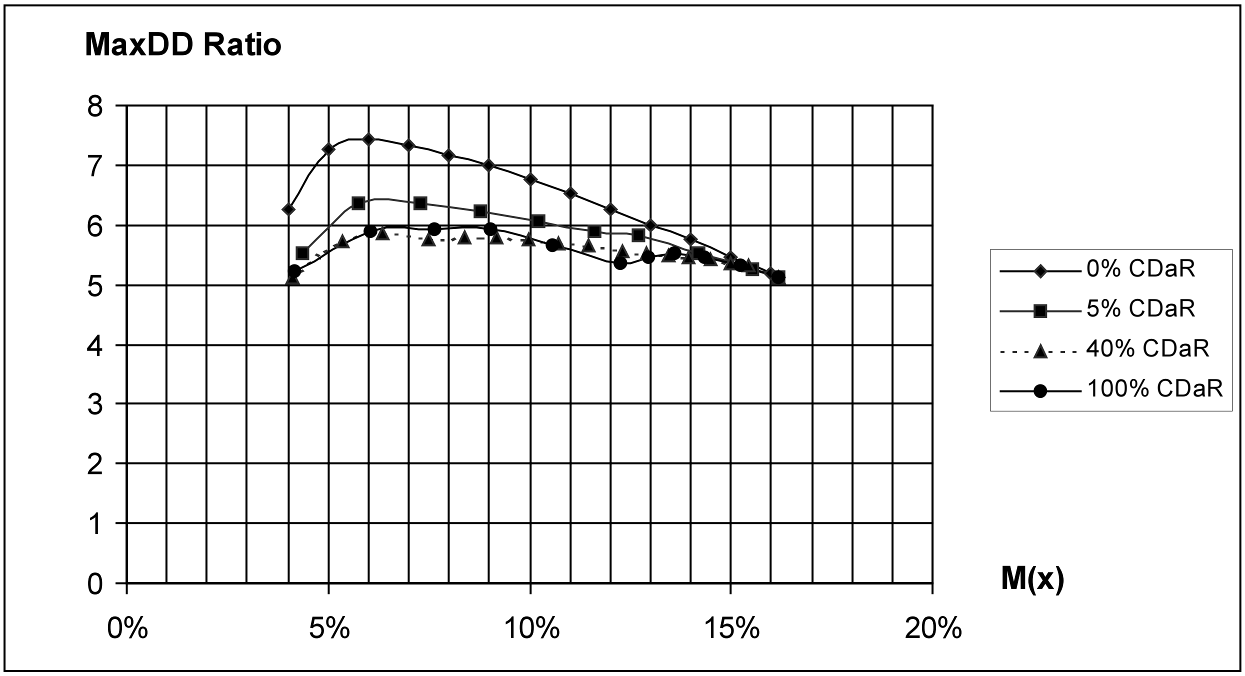
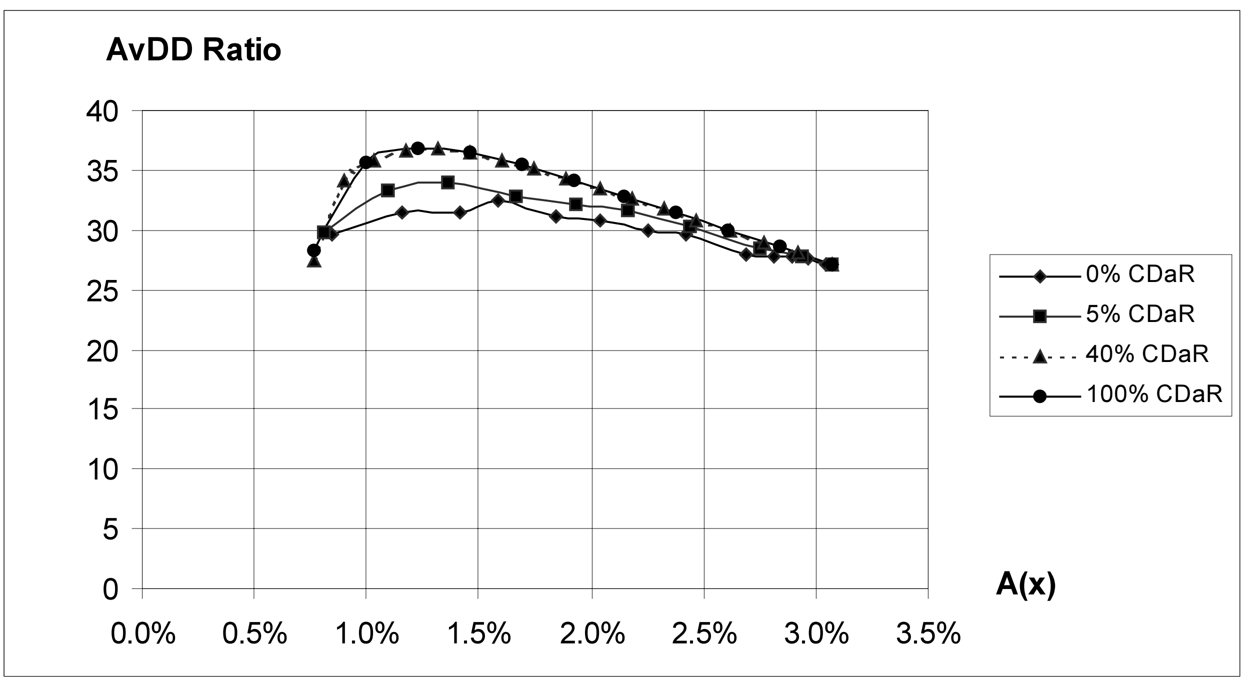


Figure 6. AvDDRatio graphs for optimal portfolios with $(1-\beta)=0\%$, 5%, 40% and 100% CDaR constraints (AvDDRatio versus AvDD). The maximum AvDDRatio is achieved in the case with $(1-\beta)=100\%$ CDaR constraints, which corresponds to the AvDD risk measure.



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