Portfolio optimization with Quasiconvex Risk Measures

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(joint work with Elisa Mastrogiacomo)

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Agenda

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- preliminaries

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- formulation of the problem and motivation

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- characterization of the solution of the optimization problem with quasiconvex risk measures
- analysis of the efficient frontier in the quasiconvex case

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- basic idea:
  given n assets with returns (or Profit & Losses) given by $X_1, X_2, ..., X_n$ (and the corresponding vector X), choose the optimal portfolio's weights $w = (w_1, ..., w_n)$ solving

$$\min_{(w_1,...,w_n)\in W} \text{"risk associated" to } X \cdot w$$

  where $X \cdot w = w_1 X_1 + ... + w_n X_n$ and W is a subset of $\mathbb{R}^n$.

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  - $W = \{w \in \mathbb{R}^n : w \geq 0; \sum_{i=1}^n w_i = 1\}$
  - $W_1 = \{w \in W : E[w \cdot X] = \mu\}$, with μ target return

State of the art

- Markowitz: risk = variance
- risk measured by a coherent risk measure:
  - VaR and CVaR: Gaivoronski and Pflug (2005)
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GOAL:

extension of the portfolio optimization problem to quasiconvex risk measures and study of the related efficient frontier

Review on risk measures

It is well known that a risk measure ρ is a functional

$$\rho : \mathcal{X} \to \overline{\mathbb{R}},$$

quantifying the riskiness of financial positions whose returns (or P&L's) are represented by random variables in the space $\mathcal{X}$.

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$$\text{if } \rho(X), \rho(Y) \leq \rho(Z) \quad \Rightarrow \rho(\alpha X + (1-\alpha)Y) \leq \rho(Z), \forall \alpha \in (0,1)$$

Risk measures

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Any monotone quasiconvex cash-subadditive risk measures ρ on L^∞ can be represented as

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} K(E_Q[-X], Q),$$

where $\mathcal{M}_{1,f}$ denotes the set of (finitely additive) probabilities and K is a suitable functional

see Cerreia-Vioglio et al. (2011), Drapeau and Kupper (2010), Frittelli and Maggis (2011) (and Penot and Volle (1990))

with quasiconvex risk measures, the optimization problem becomes a min-max problem:

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... the problem above reduces to

$$\min_{w \in W} \max_{Q \in \mathcal{M}} \{E_Q[-X \cdot w] - G(Q)\}$$

for convex risk measures (with extra assumptions)!

Useful notions of Quasiconvex analysis

Let $\mathcal{X}$ be a topological vector space and $\mathcal{X}^*$ its dual space.

A function $f : \mathcal{X} \to \mathbb{R}$ is quasiconvex if

$$\{X \in \mathcal{X} : f(X) \leq c\} \text{ is a convex set (for any } c \in \mathbb{R})$$

or, equivalently, if

$$f(\alpha X + (1-\alpha)Y) \leq \max\{f(X); f(Y)\}, \quad \forall \alpha \in (0,1), X, Y \in \mathcal{X}.$$

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- Greenberg-Pierskalla subdifferential of f at $\bar{X}$:

$$\partial^{GP} f(\bar{X}) \triangleq \left\{X^* \in \mathcal{X}^* : \ \langle X^*, X - \bar{X} \rangle < 0, \forall X \text{ s.t. } f(X) < f(\bar{X})\right\}$$

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- normal cone at $\bar{X} \in C$ to a convex subset C of $\mathcal{X}$:

$$N(C, \bar{X}) \triangleq \{X^* \in \mathcal{X}^* : \langle X^*, X - \bar{X} \rangle \leq 0 \text{ for any } X \in C\}$$

see Penot and Zalinescu (2003) and Penot (2003)

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- $\mathcal{X}^*$ its dual space
- $\mathcal{P}$ set of all probability measures $Q \ll P$ such that $\frac{dQ}{dP} \in \mathcal{X}^*$

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e.g.
$\mathcal{Z} = \mathbb{R}^n$, $F(Z) = Z \cdot X$ with $Z = (Z_1, ..., Z_n)$ (portfolio weights) and $X = (X_1, ..., X_n)$ (assets' vector)

For a convex risk measure...

Let ρ be a monotone convex risk measure represented by

$$\rho(X) = \sup_{Q \in \mathcal{P}_0} \{E_Q[-X] - G(Q)\}, \tag{2}$$

for some convex lsc penalty functional G and for some convex, closed and compact set $\mathcal{P}_0$.

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$(\bar{Z}, \bar{Q})$ is a saddle point of $E_{\bar{Q}}[-F(\bar{Z})] - G(Q)$ iff

$$\partial E_{\bar{Q}}[-F(\bar{Z})] \cap (-N(C, \bar{Z})) \neq \{0\} \qquad \text{and} \quad \bar{Q} \in \partial\rho(\bar{X}).$$

If the condition above is satisfied, then $(\bar{Z}, \bar{Q})$ is an optimal solution of the optimization problem.

see Proposition 6.4 of Ruszczynski and Shapiro (2006)

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Hence, $\bar{Z}$ could not be a local minimizer for $E_{\bar{Q}}[-F(\cdot)]$.

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where:

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- $L(X, Q) \triangleq K\left(E_Q[X], Q\right)$ is quasi-convex and lsc in X and quasi-concave and upper semi-continuous in Q.

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- vice versa: if $\rho : L^p \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is a quasiconvex and monotone risk measure satisfying $\rho(0) = 0$ and continuity from above, then it can be represented as in (4) for some suitable functional R

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Hence: any risk measure satisfying Assumption (A) is quasiconvex!

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- Assumption (A) generalizes the one true in the convex case. For convex risk measures satisfying monotonicity, cash-additivity and lsc:

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- an example of L (not reducing to the one of convex case) and satisfying hypothesis in (A):

$$L(X, Q) = E_Q[X] \wedge \gamma - G(Q)$$

for a given $\gamma \in \mathbb{R}$ and for a convex and lsc G

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If C is a convex, closed and compact subset of $\mathcal{Z}$ and $F : \mathcal{Z} \to \mathcal{X}$ is a concave and continuous from above functional, then

$$\min_{Z\in C} \sup_{Q\in\mathcal{P}_0} K\left(E_Q[-F(Z)], Q\right) = \sup_{Q\in\mathcal{P}_0} \inf_{Z\in C} K\left(E_Q[-F(Z)], Q\right). \quad (5)$$

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Proof: application of Minimax Theorem of Sion (1958) (revisited by Tuy (2004)).

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Proof: application of Minimax Theorem of Sion (1958) (revisited by Tuy (2004)).

Consequence: existence of a saddle point of $K(E_Q[-F(Z)], Q)$ if $\mathcal{P}_0$ is (weakly-)compact.

Main result

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Let $\bar{Z} \in \mathcal{Z}$, $\bar{Q} \in \mathcal{P}_0$, $\bar{X} = F(\bar{Z})$ and suppose that $\bar{Z}$ is not a local minimizer for $K_{\bar{Q}}(E_{\bar{Q}}[-F(\cdot)])$.

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$(\bar{Z}, \bar{Q})$ is a saddle point of $K_Q(E_Q[-F(Z)])$ iff

$$\partial^{(*)} E_{\bar{Q}}[-F(\bar{Z})] \cap (-N(C, \bar{Z})) \neq \{0\} \qquad \text{and} \quad \bar{Q} \in \partial^{GP} \rho(\bar{X}). \tag{7}$$

If the condition above is satisfied, then $(\bar{Z}, \bar{Q})$ is an optimal solution of the optimization problem.

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Moreover, it is easy to prove that:

Proposition

If $E_{\bar{Q}}[-F(\cdot)]$ is continuous at $\bar{Z}$ and $\bar{Z}$ is not a minimizer of $E_{\bar{Q}}[-F(\cdot)]$, and ρ is continuous at $\bar{X} = F(\bar{Z})$,
the following conditions are equivalent:

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If $E_{\bar{Q}}[-F(\cdot)]$ is continuous at $\bar{Z}$ and $\bar{Z}$ is not a minimizer of $E_{\bar{Q}}[-F(\cdot)]$, and ρ is continuous at $\bar{X} = F(\bar{Z})$,
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Example

Take $\rho(X) = f\left(E[-X]\right)$, with

$$f(x) = \begin{cases} -1; & x < -\frac{1}{2} \\ 1 - 4^{-x}; & x \geq -\frac{1}{2} \end{cases}$$

and $X = (X_1, X_2)$ such that $E[X_1] < \frac{1}{4} < \frac{1}{2} < E[X_2]$.

The efficient frontier (wrt $\tilde{C}$) is not convex.
Consider, for instance, $r_{p_1} = \frac{1}{2}, r_{p_2} = \frac{1}{4}$ and $\alpha = \frac{1}{2}$.

Thank you for your attention!!!

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