Portfolio Optimization with unobservable Markov-modulated drift process

Ulrich Rieder
Department of Optimization
and Operations Research
University of Ulm, Germany
D-89069 Ulm, Germany
e-mail:
rieder@mathematik.uni-ulm.de

Nicole Bäuerle
Institute for Mathematical Stochastics
University of Hannover, Germany
D-30167 Hannover, Germany
e-mail:
baeuerle@stochastik.uni-hannover.de

Abstract We study portfolio optimization problems where the drift rate of the stock is Markov-modulated and the driving factors cannot be observed by the investor. Using results from filter theory we reduce this problem to one with complete observation. In the case of logarithmic and power utility we solve the problem explicitly with the help of stochastic control methods. It turns out that the value function is a classical solution of the corresponding Hamilton-Jacobi-Bellman equation. As a special case we investigate the so-called Bayesian case, i.e. the drift rate is unknown but does not change during time. In this case we prove a number of interesting properties of the optimal portfolio strategy. In particular, using the likelihood-ratio ordering, we can compare the optimal investment in the case of observable drift rate to the one in the case of unobservable drift rate. Thus, we also obtain the sign of the drift risk.

Key words: portfolio optimization, Markov-modulated drift, HJB equation, optimal investment strategies, Bayesian control, stochastic orderings
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1 Introduction

This paper investigates the problem of optimal portfolio choice in a financial market with one bond and one stock. The drift rate of the stock price process is modelled as a continuous-time Markov chain. The aim is to maximize the expected utility from terminal wealth. However, we are only able to observe the stock price process and have to base our decision on this observation. In particular we are not informed about the state of the drift process. Such a model is also called a *Hidden Markov Model*. For a general treatment of such models see e.g. Elliott et al. (1994).

It is well-known that a financial market with constant parameters can only serve for a relatively short period of time. Thus, there is the need to use stochastically varying parameters. One possibility is to introduce a continuous-time Markov chain, representing the general market direction. For simplicity we assume that only the drift rate of the stock price process depends on this market direction. Furthermore, it seems to be realistic that we cannot directly observe this market direction since not all the driving factors and their impact are known. Thus, we can only try to estimate this hidden factor by observing the stock price.

Portfolio optimization problems with partial observation, in particular with unknown drift process have been studied extensively over the last decade. Lakner (1995, 1998) and Rishel (1999) for example have treated the case where the drift rate follows a linear Gaussian model. Karatzas/Zhao (2001) investigated the Bayesian case, i.e. when the drift rate is constant but unknown. The papers Sass/Haussmann (2003) and Haussmann/Sass (2003) discuss a market model which is even more general than ours, e.g. in allowing a stochastic interest rate and assuming d risky assets. Since it is possible to reduce the problem to one in a complete financial market, these cited papers use the martingale approach to solve the portfolio problem. The only exceptions are Rishel (1999) and also Karatzas/Zhao (2001) who use a stochastic control approach. Portfolio optimization problems with observable Markov-modulated market data have been treated in Bäuerle/Rieder (2004).

In this paper we use a stochastic control approach for the portfolio optimization problem with unobservable Markov-modulated drift process. A first contribution of our paper is that we can show that this approach works very well in the case of logarithmic and power utility in the sense that we even get a classical solution of the corresponding Hamilton-Jacobi-Bellman equation. This is of practical interest since it gives an alternative way of computing the value function and the optimal portfolio strategy. As a special case we also investigate the so-called Bayesian case, i.e. the drift rate is unknown but does not change during time. This setting has already been investigated in Karatzas/Zhao (2001), however we give a self-contained approach to this problem, treating it as a special case of the hidden Markov-modulated drift model and derive a more explicit formula for the optimal investment strategy. We prove a number of interesting properties of the optimal portfolio strategy, in particular, when compared to the case of observable drift rate. For example when we have a power utility $u(x) = \frac{1}{\alpha}x^\alpha$ with $\alpha \in (0,1)$, it turns out that we have to invest more in the stock in the case of an unobservable drift rate, compared to the case where the drift rate is known and equal to our expectation. If $\alpha < 0$ the situation is vice versa. Thus, for $\alpha \in (0,1)$ the drift risk is positive, whereas for $\alpha < 0$ the drift risk is negative. This result is obtained by using the likelihood-ratio ordering in an appropriate way. Some numerical results are also presented.

The paper is organized as follows: in Section 2 we introduce the market model and define the optimization problem. In Section 3 we use the filtering theory to reduce the problem to one with complete observation. In the case of a logarithmic utility function, the problem is solved in Section 4. Section 5 treats the case of a power utility. With the help of a stochastic control approach we are able to solve the problem. In particular it turns out in Section 6 that the value function is a classical solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. The special Bayesian case is treated in Section 7 and properties of the optimal investment strategy are proven in Section 8.

2 The Model

Suppose that $(\Omega, \mathcal{F}, \mathfrak{F} = \{\mathcal{F}_t, 0 \le t \le T\}, P)$ is a filtered probability space and $T > 0$ is a fixed time horizon. We consider a financial market with one bond and one risky asset. The bond evolves according to

$$dB_t = rB_t dt \tag{1}$$

with $r > 0$ being the interest rate. The stock price process $S = (S_t)$ is given by

$$dS_t = S_t \left(\mu_t dt + \sigma dW_t\right) \tag{2}$$

where $\mu_t = \mu' Y_t$, $W = (W_t)$ is a Brownian motion and $Y = (Y_t)$ is a continuous-time Markov chain with state space $\{e_1, \dots, e_d\}$, where e_k is the k-th unit vector in $\mathbb{R}^d$ and (Y_t) has the generator $Q = (q_{ij})$. All processes are adapted w.r.t. $\mathfrak{F}$ and (W_t) and (Y_t) are independent. $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ and $\sigma > 0$.

The optimization problem is to find self-financing investment strategies in this market that maximize the expected utility from terminal wealth. We assume that our investor is only able to observe the stock price process and that he knows the initial distribution of Y_0. In particular he is not informed about the current state of the Markov chain. Let $\mathfrak{F}^S = \{\mathcal{F}_t^S, 0 \le t \le T\}$ be the filtration generated by the stock price process (S_t). In what follows we denote by $\pi_t \in I\!R$ the fraction of wealth invested in the stock at time t. $1-\pi_t$ is then the fraction of wealth invested in the bond at time t. If $\pi_t < 0$, then this means that the stock is sold short and $\pi_t > 1$ corresponds to a credit. The process $\pi = (\pi_t)$ is called *portfolio strategy*. An admissible portfolio strategy has to be an $\mathfrak{F}^S$-adapted process. The *wealth process* under an admissible portfolio strategy π is given by the solution of the stochastic differential equation (SDE)

$$d\tilde{X}_t^\pi = \tilde{X}_t^\pi \left[(r + (\mu_t - r)\pi_t)dt + \sigma \pi_t dW_t \right], \tag{3}$$

where we assume that $\tilde{X}_0^\pi = x_0$ is the given initial wealth. We denote by

$$\mathcal{U}[t,T] := \Big\{ \pi = (\pi_s)_{t \le s \le T} \mid \pi_s \in I\!R,\ \pi \text{ is } \mathfrak{F}^S - \text{adapted}, \\ (3) \text{ has a unique solution}, \int_t^T (\pi_s \tilde{X}_s^\pi)^2 ds < \infty \ a.s. \Big\}$$

the set of admissible portfolio strategies over the time horizon $[t,T]$. Let $U : I\!R_+ \to I\!R$ be an increasing, concave and differentiable utility function. The *value functions* for our problem are defined by

$$\tilde{V}_\pi(t,x) = E^{t,x}\left[U(\tilde{X}_T^\pi) \right] \quad \text{for all } \pi \in \mathcal{U}[t,T]$$
$$\tilde{V}(t,x) = \sup_{\pi \in \mathcal{U}[t,T]} \tilde{V}_\pi(t,x)$$

where $E^{t,x}$ is the conditional expectation, given $\tilde{X}_t^\pi = x$. A portfolio strategy $\pi^* \in \mathcal{U}[0,T]$ is *optimal* if

$$\tilde{V}(0,x_0) = \tilde{V}_{\pi^*}(0,x_0).$$

Note that $\tilde{V}(0,x_0)$ depends on the initial distribution of Y_0.

3 The Reduction

We can reduce the control problem with partial observation to one with complete observation as follows: denote by

$$p_k(t) := P(Y_t = e_k \mid \mathcal{F}_t^S), \quad k = 1, \dots, d$$

the *Wonham-filter* of the Markov chain and define $p_t = (p_1(t), \dots, p_d(t))$ (cf. Elliott et al. (1994)). The following statements hold:

Lemma 1: There exists a Brownian motion $(\hat{W}_t)$ w.r.t. $\mathfrak{F}^S$ such that

a) the filter processes $p_k(t)$ satisfy for $t \geq 0$

$$p_k(t) \quad = \quad p_k(0) + \int_0^t \sum_j q_{jk} p_j(s) ds + \int_0^t \frac{1}{\sigma}(\mu_k - \hat{\mu}_s) p_k(s) d\hat{W}_s$$

where $\hat{\mu}_t := \sum_{k=1}^d \mu_k p_k(t) = E[\mu_t \mid \mathcal{F}_t^S]$.

b) $\mu_t dt + \sigma dW_t = \hat{\mu}_t dt + \sigma d\hat{W}_t$.

c) $\mathcal{F}_t^S = \sigma(\hat{W}_s, s \leq t)$.

Note that part b) implies that $\hat{W}_t := W_t + \frac{1}{\sigma}\int_0^t (\mu_s - \hat{\mu}_s) ds$. The control model with *complete observation* is now characterized for $\pi \in \mathcal{U}[0,T]$ by the following $d+1$-dimensional state process:

$$dX_t^\pi \quad = \quad X_t^\pi \left[(r + (\hat{\mu}_t - r)\pi_t) dt + \sigma \pi_t d\hat{W}_t \right] \tag{4}$$

$$X_0^\pi \quad = \quad x_0 \tag{5}$$

$$dp_k(t) \quad = \quad \sum_j q_{jk} p_j(t) dt + \frac{1}{\sigma}(\mu_k - \hat{\mu}_t) p_k(t) d\hat{W}_t \tag{6}$$

$$p_k(0) \quad = \quad P(Y_0 = e_k), \quad k = 1, \ldots, d. \tag{7}$$

The wealth process is explicitly given by

$$X_t^\pi = x_0 \exp\left\{ \int_0^t (r + (\hat{\mu}_s - r)\pi_s - \frac{1}{2}\sigma^2 \pi_s^2) ds + \int_0^t \sigma \pi_s d\hat{W}_s \right\}.$$

The value functions in the reduced model are defined by

$$V_\pi(t,x,p) \quad = \quad E^{t,x,p}\left[U(X_T^\pi)\right] \quad \text{for all } \pi \in \mathcal{U}[t,T]$$

$$V(t,x,p) \quad = \quad \sup_{\pi \in \mathcal{U}[t,T]} V_\pi(t,x,p)$$

where $E^{t,x,p}$ is the conditional expectation, given $X_t^\pi = x, p_t = p$. The following result is often taken for granted, however has to be proved formally

Theorem 2: For all $\pi \in \mathcal{U}[t,T]$ and $x > 0$ it holds that $V_\pi(t,x,p_t) = \tilde{V}_\pi(t,x)$ and $V(t,x,p_t) = \tilde{V}(t,x)$.

The proof follows directly from Lemma 1. The reduced model is now one with complete observation. We will solve it with the help of the HJB equation.

4 Logarithmic Utility

In this section we assume that the utility function is given by $U(x) = \log(x)$. For $\pi \in \mathcal{U}[t,T]$ with the additional assumption $\pi_s \in [-M, M]$ for $M \in I\!R_+$ large, we obtain from the explicit solution for X_t^π

$$V_\pi(t,x,p) = \log(x) + h_\pi(t,p)$$

where

$$\begin{aligned} h_\pi(t,p) &= E^{t,p}\left[\int_t^T r + (\mu_t - r)\pi_s - \frac{1}{2}\sigma^2\pi_s^2 ds + \int_t^T \sigma\pi_s dW_s\right] \\ &= E^{t,p}\left[\int_t^T r + (\mu_t - r)\pi_s - \frac{1}{2}\sigma^2\pi_s^2 ds\right]. \end{aligned}$$

Note that we need $\pi_s \in [-M, M]$ in order to have $E[\int_0^T \pi_s dW_s] = 0$ and that h_π does not depend on x. By $\mathcal{S}$ we denote the probability simplex in $I\!R^d$. It is now obvious that

Lemma 3:

a) for all $t \in [0,T], x > 0, p \in \mathcal{S}$ we have

$$V(t,x,p) = \log(x) + h(t,p),$$

where $h(t,p) = \sup_{\pi \in \mathcal{U}[t,T]} h_\pi(t,p)$. And

b) if for all $t \in [0,T]$,

$$\pi_t^* := \frac{\hat{\mu}_t - r}{\sigma^2},$$

then $\pi^* = (\pi_t^*)$ is an optimal portfolio strategy for the given investment problem.

Proof: Part a) follows from the considerations preceding Lemma 3. b) follows directly from a pathwise maximization and the fact that $M \to \infty$ does not change the optimal investment strategy.

In the case of complete observation, i.e. when we can observe the drift process μ_t, then it is well-known that the optimal investment strategy at time t would be to invest a constant fraction $\frac{\mu_t - r}{\sigma^2}$ of the wealth in the stock. 3 b) shows that the so-called *certainty equivalence principle* holds, i.e. the unknown appreciation rate μ_t is replaced by the estimate $\hat{\mu}_t = E[\mu_t \mid \mathcal{F}_t^S]$ in the optimal portfolio strategy (cf. Kuwana (1991)).

5 Power Utility

Suppose that the utility function is given by $U(x) = \frac{1}{\alpha}x^\alpha, \ \ \alpha < 1, \alpha \neq 0$. It is well-known that in this case the value function can be written in the form $V(t,x,p) = \frac{1}{\alpha}x^\alpha g(t,p)^{1-\alpha}$. One of our main contributions in this paper is to show that the corresponding portfolio optimization problem has a smooth value function, where g can be identified as a classical solution of a linear parabolic differential equation. This is not the case when the drift process μ_t is more general (see Zariphopoulou (2001)). In particular we can circumvent the use of viscosity solutions. Our theorem is as follows:

Theorem 4:

a) The value function V of our investment problem is for all $(t,x,p) \in [0,T] \times I\!R_+ \times \mathcal{S}$ given by

$$V(t,x,p) = \frac{1}{\alpha}x^\alpha g(t,p)^{1-\alpha},$$

where $g \geq 0$ is a classical solution of the following linear parabolic differential equation

$$\begin{aligned} 0 \;\; = \;\; & g_t + \frac{\alpha}{1-\alpha}\left\{ r + \frac{1}{2(1-\alpha)}\frac{(\mu'p - r)^2}{\sigma^2}\right\} g \\ & + \sum_k \left\{ \sum_j q_{jk}p_j + \frac{\alpha}{1-\alpha}p_k(\mu_k - \mu'p)\frac{\mu'p - r}{\sigma^2}\right\} g_{p_k} \\ & + \frac{1}{2\sigma^2}\sum_{k,j}(\mu_k - \mu'p)(\mu_j - \mu'p)p_k p_j g_{p_k p_j}. \end{aligned} \tag{8}$$

with $g(T,p) = 1$ for all $p \in \mathcal{S}$.

b) The optimal portfolio strategy $\pi^* = (\pi_t^*) \in \mathcal{U}[0,T]$ is given in feedback form $\pi_t^* = u^*(t,p_t)$, where the function u^* is given by

$$u^*(t,p) = \frac{1}{1-\alpha} \cdot \frac{\mu'p - r}{\sigma^2} + \frac{\sum_k p_k(\mu_k - \mu'p)g_{p_k}(t,p)}{\sigma^2 g(t,p)}.$$

c) The Feynman-Kac formula yields the following stochastic representation of g:

$$g(t,p) = E\left[\exp\left(r\frac{\alpha}{1-\alpha}(T-t) + \int_t^T \frac{\alpha}{2(1-\alpha)^2}\frac{(\mu' Z_s - r)^2}{\sigma^2}ds\right) \Bigm| Z_t = p\right]$$

where the stochastic process $(Z_t) \in I\!R^d$ is a solution of the SDE

$$dZ_t^k = a_k(Z_t)dt + \sum_j b_{k,j}(Z_t)dW_t^j$$

with

$$\begin{aligned} a_k(p) &:= \sum_j q_{jk} p_j + \frac{\alpha}{1-\alpha} p_k(\mu_k - \mu' p) \frac{\mu' p - r}{\sigma^2} \\ b_{k,j}(p) &:= \frac{1}{\sigma^2}(\mu_k - \mu' p)(\mu_j - \mu' p) p_k p_j. \end{aligned}$$

Remark:

a) In the case when the investor is able to observe the driving Markov chain, the optimal fraction of wealth invested in the stock at time t, when the Markov chain is in state e_k is given by

$$u_o^*(t, e_k) = \frac{1}{1-\alpha} \cdot \frac{\mu_k - r}{\sigma^2}$$

(see e.g. Bäuerle/Rieder (2004)). Thus, in the unobservable case, the optimal fraction invested consists of the same *myopic part* and an additional term, which we call the *drift risk.* This term is sometimes also called *market risk* or *hedging demand*, but since it stems from the unknown drift rate only, we decided to call it drift risk.

b) Note that, since g and g_{p_k} are continuous and $[0, T] \times \mathcal{S}$ is compact, the optimal portfolio strategy (π_t^*) is bounded.

The proof of Theorem 4 is given in Section 6.

6 The HJB Equation and the Proof of Theorem 4

In order to solve the investment problem, a classical approach in stochastic control theory is to examine the so-called Hamilton-Jacobi-Bellman (HJB) equation. For our problem, it turns out to be

$$\begin{aligned} 0 = \sup_{u \in \mathbb{R}} \Big\{ & v_t + x[r + u(\mu' p - r)] v_x + \frac{1}{2} x^2 u^2 \sigma^2 v_{xx} + \\ & \sum_{k,j} q_{jk} p_j v_{p_k} + \sum_k x u p_k (\mu_k - \mu' p) v_{x p_k} + \\ & \frac{1}{2\sigma^2} \sum_{k,j} (\mu_k - \mu' p)(\mu_j - \mu' p) p_k p_j v_{p_k p_j} \Big\} \end{aligned} \tag{9}$$

with the boundary condition $v(T, x, p) = \frac{1}{\alpha} x^\alpha$ for all $x \in \mathbb{R}_+, p \in \mathcal{S}$. In what follows we abbreviate the expression in curly brackets by $\mathcal{A}v(t, x, p, u)$. For the proof of Theorem 4 we proceed as follows:

Theorem 5: The function $v(t,x,p) := \frac{1}{\alpha}x^\alpha g(t,p)^{1-\alpha}$ with g given in (8) is a solution of the HJB equation (9).

Proof: First note that, the boundary condition $v(T,x,p) = \frac{1}{\alpha}x^\alpha$ is satisfied. Moreover, since the coefficients of the linear parabolic differential equation for g are polynomials in p, the function g is sufficiently differentiable (see e.g. Kloeden/Platen (1995) p. 153). We first compute the derivatives of v:

$$\begin{aligned}
v_t &= \frac{1}{\alpha}x^\alpha(1-\alpha)g^{-\alpha}g_t \\
v_x &= x^{\alpha-1}g^{1-\alpha} \\
v_{xx} &= (\alpha-1)x^{\alpha-2}g^{1-\alpha} \\
v_{p_k} &= \frac{1}{\alpha}x^\alpha(1-\alpha)g^{-\alpha}g_{p_k} \\
v_{xp_k} &= x^{\alpha-1}(1-\alpha)g^{-\alpha}g_{p_k} \\
v_{p_kp_j} &= \frac{1}{\alpha}x^\alpha(1-\alpha)\left(g^{-\alpha}g_{p_kp_j} - \alpha g^{-\alpha-1}g_{p_j}g_{p_k}\right).
\end{aligned}$$

Plugging this into the HJB equation gives us after some simple algebra (note that we need $\alpha < 1$ here)

$$\begin{aligned}
0 = \sup_{u\in I\!R} \frac{1}{\alpha}\Big\{ & g_t + [r + u(\mu' p - r)]\frac{\alpha}{1-\alpha}g - \frac{1}{2}\alpha u^2\sigma^2 g + \\
& \sum_{k,j} q_{jk}p_j g_{p_k} + \sum_k u p_k(\mu_k - \mu' p)\alpha g_{p_k} + \\
& \frac{1}{2\sigma^2}\sum_{k,j}(\mu_k - \mu' p)(\mu_j - \mu' p)p_k p_j\left(g_{p_kp_j} - \alpha g^{-1}g_{p_j}g_{p_k}\right)\Big\}
\end{aligned}$$

Since $g \geq 0$ the maximum point is well-defined and given by

$$\frac{1}{1-\alpha}\cdot\frac{\mu' p - r}{\sigma^2} + \frac{\sum_k p_k(\mu_k - \mu' p)g_{p_k}}{\sigma^2 g}.$$

Inserting the maximum point and simplifying the expression, we obtain that g has to satisfy the partial differential equation (8) which is true due to our assumption. □

The power change of variable for the value function has already been used by Zariphopoulou (2001) and Pham (2002). Here it is shown that this trick also works for a multidimensional setting. The next theorem provides the verification that $v(t,x,p)$ given in Theorem 5 is indeed the value function of our investment problem.

Theorem 6: Suppose $v(t,x,p)$ is given as in Theorem 5. Then

a) $V(t,x,p) = v(t,x,p)$ for all $(t,x,p) \in [0,T] \times I\!R_+ \times \mathcal{S}$.

b) The optimal portfolio strategy $\pi^* = (\pi_t^*) \in \mathcal{U}[0,T]$ is given as in Theorem 4 b).

Proof: We restrict here to the case $\alpha \in (0,1)$. The case $\alpha < 0$ can be shown similarly. Let $\pi \in \mathcal{U}[t,T]$ be an arbitrary portfolio strategy and (X_t^π) the corresponding wealth process. We interpret $v(t,x,p)$ as a function on $I\!R_+ \times I\!R^{d+1}$. Since v is smooth enough we can apply Ito's formula and obtain:

$$\begin{aligned}
v(T, X_T^\pi, p_T) &= v(t,x,p) + \int_t^T \mathcal{A}v(s, X_s^\pi, p_s, \pi_s)\, ds \qquad (10)\\
&+ \int_t^T v_x(s, X_s^\pi, p_s, \pi_s) X_s^\pi \pi_s \sigma d\hat{W}_s \\
&+ \int_t^T \sum_k v_{p_k}(s, X_s^\pi, p_s, \pi_s) \frac{1}{\sigma} p_k(s)(\mu_k - \mu' p_s) d\hat{W}_s \\
&\le v(t,x,p) + \int_t^T \left(X_s^\pi\right)^\alpha g(s,p_s)^{1-\alpha} \pi_s \sigma d\hat{W}_s \\
&+ \int_t^T \sum_k \left(X_s^\pi\right)^\alpha \frac{g_{p_k}(s,p_s)}{g(s,p_s)^\alpha} \frac{(1-\alpha)}{\alpha\sigma} p_k(s)(\mu_k - \mu' p_s) d\hat{W}_s
\end{aligned}$$

where the inequality follows from the HJB equation. Since $v \geq 0$, the local martingale $(\int_t^T \ldots d\hat{W}_s)_{T \geq t}$ is bounded from below by $-v(t,x,p)$ and thus is a supermartingale. Taking the conditional expectation and using the boundary condition for v we obtain

$$E^{t,x,p}\left[\frac{1}{\alpha}(X_T^\pi)^\alpha\right] \le v(t,x,p).$$

Since π was arbitrary, we obtain $V(t,x,p) \le v(t,x,p)$. Now suppose that (π_t^*) is as in part b). Since (π_t^*) is a maximizer of the HJB equation, we obtain equality in equation (10) under (π_t^*). Note that, since (π_t^*) is bounded and since g, g_{p_k} are continuous and $[0,T] \times \mathcal{S}$ is compact, the local martingales $\int \ldots d\hat{W}_s$ are martingales. Taking expectation we obtain this time

$$E^{t,x,p}\left[\frac{1}{\alpha}(X_T^\pi)^\alpha\right] = v(t,x,p),$$

and the statement follows. □

Thus, part a) and b) of Theorem 4 are shown. The Feynman-Kac formula (Theorem 4 c)) is standard see e.g. Kloeden/Platen (1995) p. 153).

7 The Bayesian Case

In this section we consider a special case of the previously discussed model, namely the so-called *Bayesian case*. Here, the unobserved drift process (μ_t) is simply a random variable $\mu_t = \theta$ which does not change during time and the investor knows the initial distribution $P(\theta = \mu_k) =: p_k,\ k = 1,\dots,d.$ $\mu_1,\dots,\mu_d$ are the possible values θ can take. As before, we assume that θ and (W_t) are independent. This model has already been solved by Karatzas/Zhao (2001) via the martingale method and by stochastic control. We relate now their result to our model of Section 2 giving a self-contained proof. Formally we get the Bayesian case if we set the intensity matrix $Q \equiv 0$ in the model of Section 2. With this modification, the results of Sections 3-5 hold for the Bayesian case. However, we will see that the analysis can be simplified considerably in this setup. This is mainly due to the fact that instead of looking at the $I\!R^{d+1}$-valued state process $(X_t, p_1(t), \dots, p_d(t))$ in the reduced model, we can find a sufficient statistic for the unobserved parameter θ and can restrict to a 2-dimensional state space. A crucial step for this procedure is to look at the optimization problem under a change of measure. Since the logarithmic utility is quite simple, we restrict to the power utility $U(x) = \frac{1}{\alpha}x^\alpha,\ \alpha < 1,\ \alpha \neq 0$ here. In the sequel we will use the following results. Recall from Section 2 that $\hat{\mu}_t = \sum_{k=1}^d \mu_k p_k(t)$ with $p_k(t) = P(\theta = \mu_k \mid \mathcal{F}_t^S),\ k = 1,\dots,d$ and $\hat{W}_t = W_t + \frac{1}{\sigma}\int_0^t (\theta - \hat{\mu}_s)ds$. Let us now introduce the process

$$Y_t := \hat{W}_t + \frac{1}{\sigma}\int_0^t (\hat{\mu}_s - r)ds = W_t + \frac{\theta - r}{\sigma}t.$$

It is convenient to write $\gamma_k := \frac{\mu_k - r}{\sigma}$ and

$$L_t(\mu_k, y) := \begin{cases} \exp(\gamma_k y - \frac{1}{2}\gamma_k^2 t) & ,t > 0 \\ 1 & ,t = 0 \end{cases}$$

for $t \in [0,T], y \in I\!R$, and $k = 1,\dots,d$. It is well-known that $(L_t^{-1}(\theta, Y_t))$ is a martingale density process w.r.t. the filtration $\mathfrak{F}^{\theta,W}$ which is the filtration generated by θ and (W_t). Then we can define a new probability measure Q by $\frac{dQ}{dP} = L_T^{-1}(\theta, Y_T)$. Under Q the process (W_t^Q) with $W_t^Q := Y_t$ is a Brownian motion w.r.t. $\mathfrak{F}^{\theta,W}$. The process $(L_t(\theta, Y_t))$ is a Q-martingale w.r.t. $\mathfrak{F}^{\theta,W}$. Note that it can be shown that θ and (W_t^Q) are independent under Q. Finally, for $t \in [0,T], y \in I\!R$ we use the abbreviation

$$F(t,y) := \sum_{k=1}^d L_t(\mu_k, y)p_k.$$

Now we are able to state our first result.

Lemma 7: With the notations introduced in this section it holds that

a) $\mathfrak{F}^S = \mathfrak{F}^Y$, i.e. the filtration generated by (S_t) is the same as the one generated by $(Y_t) = (W_t^Q)$.

b) $p_k(t)$ depends only on Y_t. More precisely, it holds for $k = 1, \dots, d$ that

$$p_k(t) = P(\theta = \mu_k \mid \mathcal{F}_t^S) = \frac{L_t(\mu_k, Y_t) p_k}{F(t, Y_t)}.$$

Proof:

a) Follows from the definition of the stock price process (S_t) and $(\hat{W}_t)$.

b) The Bayes formula for conditional expectations reads

$$E[Z \mid \mathcal{F}_t^S] = \frac{E_Q[Z L_T(\theta, Y_T) \mid \mathcal{F}_t^S]}{E_Q[L_T(\theta, Y_T) \mid \mathcal{F}_t^S]}$$

where Z is a random variable defined on our probability space. Plugging in $Z = 1_{[\theta=\mu_k]}$ yields the desired result. □

In particular Lemma 7 implies that $\hat{\mu}_t = \sum_{k=1}^d \mu_k p_k(t)$ is a function of Y_t alone. Therefore, (Y_t) can be seen as a *sufficient statistic*. There is no need to consider the conditional probabilities $p_k(t)$ for all $k = 1, \dots, d$. More precisely, when we define

$$\mu(t, y) = \frac{\sum_{k=1}^d \mu_k L_t(\mu_k, y) p_k}{\sum_{k=1}^d L_t(\mu_k, y) p_k}$$

we have $\hat{\mu}_t = \mu(t, Y_t)$. It is now convenient to introduce the process $\gamma_t = \gamma(t, Y_t)$, where $\gamma(t, y) := \frac{\mu(t,y)-r}{\sigma}$. We can reduce our portfolio problem to a problem with complete observation and 2-dimensional state space in the following way:

$$\begin{aligned} dX_t^\pi &= X_t^\pi \left[(r + \sigma \gamma_t \pi_t) dt + \sigma \pi_t d\hat{W}_t \right], \quad X_0^\pi = x_0 \\ dY_t &= \gamma_t dt + d\hat{W}_t, \quad Y_0 = 0. \end{aligned}$$

As in Section 3 we define the value functions $V_\pi(t, x, y)$ and $V(t, x, y)$ for $(t, x, y) \in [0, T] \times I\!R_+ \times I\!R$ and obtain as in Theorem 2

$$\begin{aligned} V_\pi(t, x, Y_t) &= \tilde{V}_\pi(t, x) \\ V(t, x, Y_t) &= \tilde{V}(t, x). \end{aligned}$$

A proof of the following Theorem 8 a) and b) can also be found in Karatzas/Zhao (2001) Theorem 3.2.

Theorem 8:

a) The value function V of our investment problem with power utility is for all $(t,x,y) \in [0,T] \times I\!R_+ \times I\!R$ given by

$$V(t,x,y) = \frac{1}{\alpha} x^\alpha g(t,y)^{1-\alpha},$$

where $g(t,y) \geq 0$ is a classical solution of the following linear parabolic differential equation

$$0 \quad = \quad g_t + \frac{\alpha}{1-\alpha}\left\{r + \frac{1}{2}\frac{\gamma(t,y)^2}{(1-\alpha)}\right\} g + \frac{\gamma(t,y)}{1-\alpha} g_y + \frac{1}{2} g_{yy} \tag{11}$$

with $g(T,y) = 1$ for all $y \in I\!R$.

b) The following representation of g holds:

$$g(t,y) = E\left[\left(\frac{F(T,Y_T)B_T}{F(t,Y_t)B_t}\right)^{\frac{\alpha}{1-\alpha}} \middle| Y_t = y\right]$$

for $t \in [0,T], y \in I\!R$.

c) The optimal portfolio strategy $\pi^* = (\pi_t^*) \in \mathcal{U}[0,T]$ is given in feedback form $\pi_t^* = u^*(t,Y_t)$, where the function u^* is given by

$$u^*(t,y) = \frac{1}{1-\alpha} \cdot \frac{\mu(t,y)-r}{\sigma^2} + \frac{g_y(t,y)}{\sigma g(t,y)}.$$

Proof: It is straightforward to see that our portfolio problem is equivalent to the following optimization problem (w.r.t. the equivalent martingale measure Q):

$$E_Q\left[F(T,Y_T)\frac{1}{\alpha}(X_T^\pi)^\alpha\right] \to \max$$

$$dX_t^\pi = X_t^\pi(rdt + \sigma\pi_t dW_t^Q), \quad X_0^\pi = x_0$$

$$dY_t = dW_t^Q, \quad Y_0 = 0.$$

We denote the value function of this problem by $V^Q(t,x,y)$. It follows from the definition that $V(0,x_0,0) = V^Q(0,x_0,0)$.

a) Solving the HJB equation for the Q-problem we obtain as in Section 6

$$V^Q(t,x,y) = \frac{1}{\alpha} x^\alpha g^Q(t,y)^{1-\alpha},$$

where g^Q is a classical solution of the following linear parabolic differential equation

$$0 = g_t^Q + \frac{\alpha r}{1-\alpha} g^Q + \frac{1}{2} g_{yy}^Q$$

with $g^Q(T,y) = F(T,y)^{\frac{1}{1-\alpha}}$ for all $y \in I\!R$. Moreover, the Feynman-Kac formula gives the representation

$$g^Q(t,y) = \exp\left(r(T-t)\frac{\alpha}{1-\alpha}\right) E_Q\left[F(T,Y_T)^{\frac{1}{1-\alpha}} \;\middle|\; Y_t = y\right].$$

If we define $g(t,y) := F(t,y)^{\frac{1}{\alpha-1}} g^Q(t,y)$ then it is easy to see after some calculations that g is a classical solution of the HJB equation for the problem under P which is equivalent to (11).

b) Using the representation of g^Q in a) and applying the Bayes formula yields the statement for $g(t,y)$.

c) From the HJB equation we obtain that the optimal portfolio strategy is given by $\pi_t^* = u^*(t,Y_t)$ where the function u^* is given as stated.

□

Remark: Note that the optimal portfolio strategy (π_t^*) with $\pi_t^* = u^*(t,Y_t)$ can also be written as

$$u^*(t,y) = \frac{g_y^Q(t,y)}{\sigma g^Q(t,y)}.$$

This follows from the equivalent Q-problem formulated in the proof of Theorem 8.

8 Properties of the optimal Investment Strategy in the Bayesian Case

In this section we investigate the structural properties of the optimal investment fraction $u^*(t,y)$ given in Theorem 8 c). In particular we will compare the optimal investment strategy with the one we obtain when the drift rate is known. In the observable case the problem has been solved by Merton (1971, 1973). Suppose that

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

is the dynamics of the stock price and $\mu \in I\!R$ is observable, then it is well-known that it is optimal to invest the constant fraction

$$\frac{1}{1-\alpha} \cdot \frac{\mu - r}{\sigma^2}$$

of the wealth in the stock. Let us now assume that we are in the Bayesian case and observe at time t that the state of Y_t is y. Then we expect that the drift rate of the stock is $\mu(t,y)$. If we would know for sure that the drift rate is $\mu(t,y)$ then we would invest the fraction

$$u_o^*(t,y,\alpha) = \frac{1}{1-\alpha} \cdot \frac{\mu(t,y) - r}{\sigma^2} = \frac{1}{(1-\alpha)\sigma} \cdot \gamma(t,y)$$

of the wealth in the stock. Recall that $u^*(t,y) = \frac{g_y^Q(t,y)}{\sigma g^Q(t,y)}$ and u^* depends on α. Hence we will write $u^*(t,y) = u^*(t,y,\alpha)$. We obtain now the following comparisons, where part d) is non-trivial and of particular interest for practical applications:

Theorem 9:

a) At time T the optimal fraction is equal to the myopic part:

$$\lim_{t \to T} u^*(t,y,\alpha) = u_o^*(T,y,\alpha)$$

for all $y \in I\!R, \alpha < 1$.

b) As $\alpha \to 0$, the optimal fraction tends to the myopic part:

$$\lim_{\alpha \to 0} u^*(t,y,\alpha) = u_o^*(t,y,0)$$

for all $y \in I\!R, t \in [0,T]$.

c) Suppose that $\mu_1 \leq \ldots \leq \mu_d$, then we obtain the following bounds:

$$\frac{1}{1-\alpha} \cdot \frac{\mu_1 - r}{\sigma^2} \leq u^*(t,y,\alpha) \leq \frac{1}{1-\alpha} \cdot \frac{\mu_d - r}{\sigma^2}$$

for all $y \in I\!R, t \in [0,T]$.

d) Suppose that $r \leq \mu_1 \leq \ldots \leq \mu_d$. If $\alpha \in (0,1)$ then

$$u^*(t,y,\alpha) \geq u_o^*(t,y,\alpha)$$

and if $\alpha < 0$ then

$$u^*(t,y,\alpha) \leq u_o^*(t,y,\alpha)$$

for all $y \in I\!R, t \in [0,T]$.

Proof: Let us define $\gamma^*(t,y,\alpha) := (1-\alpha)\frac{g^Q_y(t,y)}{g^Q(t,y)}$. Then it is sufficient to prove the statements for $\gamma^*(t,y,\alpha)$ and $\gamma(t,y)$. From Section 7 we know that $\gamma^*(t,y,\alpha) = \frac{\Gamma_D(t,y,\alpha)}{\Gamma_N(t,y,\alpha)}$ with

$$\Gamma_N(t,y,\alpha) = \int_{\mathbb{R}} F(T,y+x)^{\frac{1}{1-\alpha}} \phi_{T-t}(x)\, dx$$
$$\Gamma_D(t,y,\alpha) =$$
$$\int_{\mathbb{R}} F(T,y+x)^{\frac{\alpha}{1-\alpha}} \left(\sum_{k=1}^{d} p_k \gamma_k L_T(\mu_k, y+x) \right) \phi_{T-t}(x)\, dx,$$

where ϕ_{T-t} is the density of the normal distribution with expectation 0 and variance $T-t$.

a) In this case the following representation of $\Gamma_D(t,y,\alpha)$ and $\Gamma_N(t,y,\alpha)$ are useful:

$$\begin{aligned} \Gamma_N(t,y,\alpha) &= E\left[F(T,y+W_{T-t})^{\frac{1}{1-\alpha}}\right] \\ \Gamma_D(t,y,\alpha) &= E\left[F(T,y+W_{T-t})^{\frac{\alpha}{1-\alpha}} \left(\sum_{k=1}^{d} p_k \gamma_k L_T(\mu_k, y+W_{T-t}) \right)\right]. \end{aligned}$$

Since $\lim_{t\to T} W_{T-t} = 0$ a.s. we obtain due to the continuity of the involved functions

$$\begin{aligned} \lim_{t\to T} \Gamma_N(t,y,\alpha) &= F(T,y)^{\frac{1}{1-\alpha}} \\ \lim_{t\to T} \Gamma_D(t,y,\alpha) &= F(T,y)^{\frac{\alpha}{1-\alpha}} \left(\sum_{k=1}^{d} p_k \gamma_k L_T(\mu_k, y) \right). \end{aligned}$$

Altogether this yields

$$\lim_{t\to T} \gamma^*(t,y,\alpha) = \frac{\sum_{k=1}^d p_k \gamma_k L_T(\mu_k,y)}{\sum_{k=1}^d p_k L_T(\mu_k,y)} = \gamma(T,y).$$

b) We obtain

$$\begin{aligned} \lim_{\alpha\to 0} \Gamma_N(t,y,\alpha) &= \sum_{k=1}^{d} p_k \int_{\mathbb{R}} L_T(\mu_k, y+x)\phi_{T-t}(x)\, dx \\ \lim_{\alpha\to 0} \Gamma_D(t,y,\alpha) &= \sum_{k=1}^{d} p_k \gamma_k \int_{\mathbb{R}} L_T(\mu_k, y+x)\phi_{T-t}(x)\, dx. \end{aligned}$$

Moreover, it holds that

$$\int_{\mathbb{R}} L_T(\mu_k, y+x)\phi_{T-t}(x)\, dx = L_t(\mu_k, y)$$

which implies the result.

c) Since $L_T, p_k \geq 0$ we obtain

$$\gamma_1 F(T,y+x) \leq \sum_{k=1}^{d} p_k \gamma_k L_T(\mu_k, y+x) \leq \gamma_d F(T,y+x).$$

Hence we can bound the denominator of $\gamma^*(t,y,\alpha)$ by

$$\gamma_1 \Gamma_N(t,y,\alpha) \leq \Gamma_D(t,y,\alpha) \leq \gamma_d \Gamma_N(t,y,\alpha)$$

and the result follows.

d) Suppose $\alpha \in (0,1)$. We have to show that $\gamma(t,y) \le \gamma^*(t,y,\alpha)$. Both sides can be interpreted as expectations in the following way:

$$\gamma(t,y) = \sum_{k=1}^{d} \gamma_k p_k(t,y),$$

where

$$p_k(t,y) = \frac{p_k L_t(\mu_k, y)}{F(t,y)}$$

and

$$\gamma^*(t,y,\alpha) = \sum_{k=1}^{d} \gamma_k q_k(t,y,\alpha),$$

where

$$q_k(t,y,\alpha) = \frac{\int_{I\!R} p_k L_T(\mu_k, y+x) F(T, y+x)^{\frac{\alpha}{1-\alpha}} \phi_{T-t}(x)\, dx}{\Gamma_N(t,y,\alpha)}.$$

We will show now that the densities satisfy

$$(p_k(t,y), k = 1,\dots,d) \le_{lr} (q_k(t,y,\alpha), k = 1,\dots,d)$$

where $\le_{lr}$ is the likelihood ratio order, i.e. we show that

$$\frac{q_k(t,y,\alpha)}{p_k(t,y)}$$

is increasing in k for all $t \in [0,T]$ and $y \in I\!R$. Then it is well-known that the expectations are ordered as stated. Obviously it holds that

$$\frac{q_k(t,y,\alpha)}{p_k(t,y)} = C \cdot \frac{\int_{I\!R} L_T(\mu_k, y+x) F(T, y+x)^{\frac{\alpha}{1-\alpha}} \phi_{T-t}(x)\, dx}{L_t(\mu_k, y)}$$

where $C > 0$ is a constant. Since $L_T(\mu_k, y+x) = L_t(\mu_k, y) L_{T-t}(\mu_k, x)$ it follows that

$$\frac{q_k(t,y,\alpha)}{p_k(t,y)} = C \cdot E\left[F(T, y+X_k)^{\frac{\alpha}{1-\alpha}}\right],$$

where $X_k \sim \mathcal{N}(\gamma_k(T-t), T-t)$. Thus, we have to show that

$$E\left[F(T, y+X_k)^{\frac{\alpha}{1-\alpha}}\right] \le E\left[F(T, y+X_{k+1})^{\frac{\alpha}{1-\alpha}}\right].$$

This inequality is of the form

$$E[f(X_k)] \le E[f(X_{k+1})]$$

where the function $f(x)$ is increasing in x since $\alpha \in (0,1)$ and $\gamma_k \ge 0$. Thus, the statement is true since $X_k \le_{st} X_{k+1}$ where $\le_{st}$ is the usual stochastic order. If $\alpha \in (-\infty, 0)$, then $f(x)$ is decreasing and we obtain the reverse inequality. □

Remark:

1. The optimal fraction $u_o^*(t,y,0)$ of Theorem 9 b) is the optimal fraction we obtain in the case of a logarithmic utility function (cf. Section 4). Thus, the portfolio problem with logarithmic utility can be seen as the limiting problem, when $\alpha \to 0$ in the power utility case.

2. Part d) of Theorem 9 tells us, that we have to invest more in the stock in the case of an unobservable drift rate, compared to the case where we know that $\mu(t,y)$ is the drift rate when $\alpha \in (0,1)$. If $\alpha < 0$ the situation is vice versa. A heuristic explanation of this phenomenon is as follows: though in all cases our investor is risk averse, the degree of risk aversion changes with α. Formally the degree of risk aversion is defined by the *Arrow-Pratt absolute risk aversion coefficient*, which is
$$-\frac{U''(x)}{U'(x)} = (1-\alpha)\frac{1}{x}$$
in case of the power utility $U(x) = \frac{1}{\alpha}x^\alpha$. Thus, the risk aversion decreases for all wealth levels with α. In particular if $\alpha \in (0,1)$, the investor is less risk averse than in the logarithmic utility case ($\alpha = 0$) and thus invests more in the stock.

3. Theorem 9 c) implies that $\lim_{\alpha\to-\infty} u^*(t,y,\alpha) = 0$ (see also the preceding remark).

In the following figures we have computed the optimal fractions $u^*(t,y,\alpha)$ and $u_o^*(t,y,\alpha)$ in the case of partial and complete observation for the following data: $d = 3, r = 0.04, \sigma = 0.2, \mu_2 = 0.1, \mu_3 = 0.2, t = 0, y = 0$ and $p_1 = 0.2, p_2 = 0.4, p_3 = 0.4$. Figure 1 shows the optimal fractions that have to be invested in the stock in the observed case $u_o^*(t,y,\alpha)$ and in the unobserved case $u^*(t,y,\alpha)$ as a function of α, when $\mu_1 = 0.04$ and $T = 1$. For $\alpha = 0$ both fractions coincide according to Theorem 9 b). Our conjecture is that $u^*(t,y,\alpha)$ and the difference $u^*(t,y,\alpha) - u_o^*(t,y,\alpha)$ are increasing in α if $\mu_k \geq r$ for all k. Figure 2 shows the same situation with $\mu_1 = -0.2$, i.e. the first stock has a negative appreciation rate. In this case we can see that Theorem 9 d) does not hold anymore. Figure 3 and 4 show the optimal fractions as functions of the planning horizon T with $\alpha = 0.5$. As shown in Theorem 9 a), the fractions coincide for $T = t = 0$. Figure 3 is computed with $\mu_1 = 0.04$ and figure 4 with $\mu_1 = -0.2$. In the case $\alpha > 0$ and $\mu_k \geq r$ for all k we conjecture that $u^*(t,y,\alpha)$ is increasing in T and converges against the upper bound $\frac{1}{1-\alpha} \cdot \frac{\mu_d - r}{\sigma^2}$. Figure 4 shows that there is no monotonicity w.r.t. T in general.

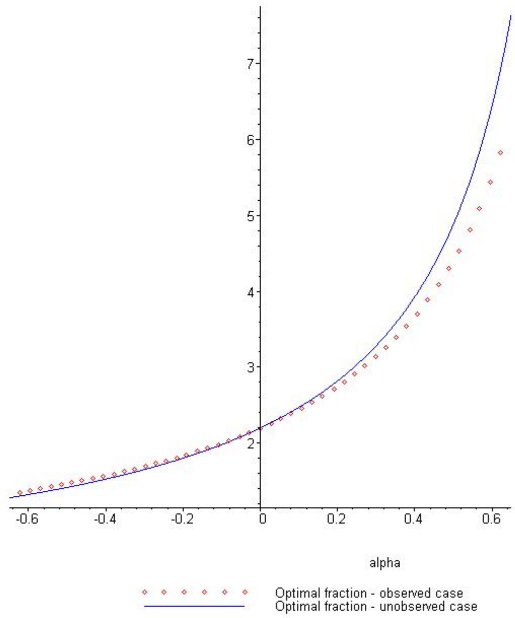
7
6
5
4
3
2
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
alpha
Optimal fraction - observed case
Optimal fraction - unobserved case

fig. 1

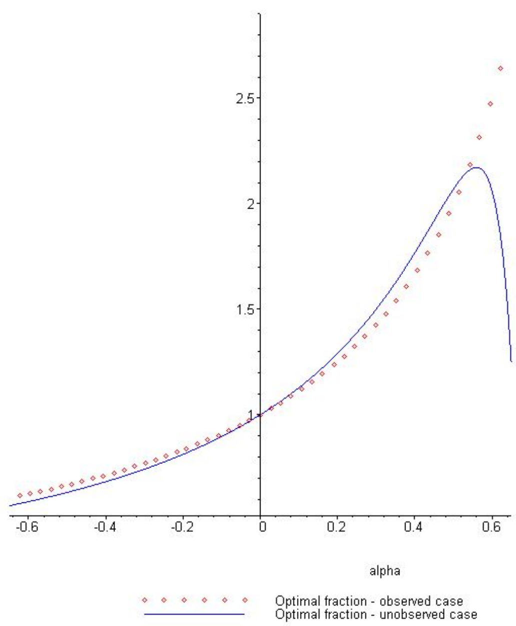
2.5
2
1.5
1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
alpha
Optimal fraction - observed case
Optimal fraction - unobserved case

fig. 2

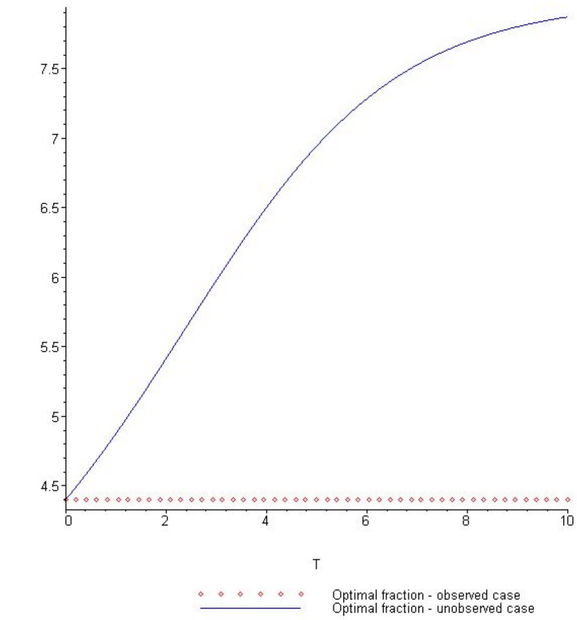
7.5
7
6.5
6
5.5
5
4.5
0
2
4
6
8
10
T
Optimal fraction - observed case
Optimal fraction - unobserved case

fig. 3

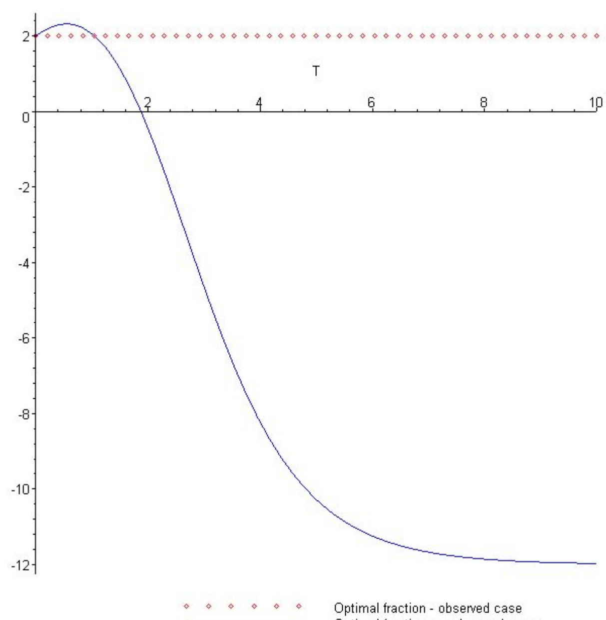
2
T
2
4
6
8
10
0
-2
-4
-6
-8
-10
-12
Optimal fraction - observed case
Optimal fraction - unobserved case

fig. 4

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