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POSETS HAVING A SELFDUAL INTERVAL POSET

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Dedicated to Professor Ján Jakubík on the occasion of his seventieth birthday

1. INTRODUCTION

The lattice of all intervals of a lattice has been studied by many authors, cf. [1]–[10]. In [1] the selfduality of this lattice was investigated. The author proved that in the case of a finite lattice the lattice of all intervals is selfdual if and only if either card $L \leqslant 2$ or card $L = 4$ and L has two atoms. He also proposed the problem whether there exists an infinite lattice with the selfdual lattice of intervals. Negative answer to this problem follows from the following result presented in [8]: If P is any partially ordered set with card $P > 4$, then the partially ordered system of all intervals of P is not selfdual.

In all papers mentioned above the empty set has been included into the system of all intervals. In the present paper this is not the case. We characterize partially ordered sets satisfying the condition that every interval of P contains a finite maximal chain and having a selfdual system of intervals (cf. 2.7 and 2.8).

Let $(P, \leqslant)$ be any partially ordered set. By an interval of P a set $\langle a, b\rangle = \{x \in P\colon a \leqslant x \leqslant b\}$ where $a, b \in P$, $a \leqslant b$, is meant. If $a = b$, we use the notation $\langle a\rangle$ instead of $\langle a, a\rangle$. The system of all intervals of P is denoted by $\operatorname{Int} P$. This system is partially ordered by the set-theoretical inclusion $\subseteq$. $\operatorname{Min} P$ and $\operatorname{Max} P$ is the set of all minimal and maximal elements of P, respectively. The symbol $\prec$ indicates the covering relation (not only in $(P, \leqslant)$ but also in $(\operatorname{Int} P, \subseteq)$). If U is an equivalence relation on P, instead of $(a, b) \in U$ we will also write aUb. For the equivalence class containing a the notation $[a]U$ will be used.

A partially ordered set $(Q, \leqslant)$ is said to be selfdual if there exists a dual automorphism of $(Q, \leqslant)$.

Lemma 1.1. *Let* $(P, \leqslant)$ *be any partially ordered set,* $\langle a,b\rangle$, $\langle a_1,b_1\rangle$, $\langle a_2,b_2\rangle \in \operatorname{Int} P$. *Then*

a) $\langle a,b\rangle = \inf\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$ *(in the partially ordered system* $(\operatorname{Int} P, \subseteq)$*) if and only if* $\langle a,b\rangle = \langle a_1,b_1\rangle \cap \langle a_2,b_2\rangle$;

b) $\langle a,b\rangle = \sup\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$ *if and only if* $a = \inf\{a_1,a_2\}$, $b = \sup\{b_1,b_2\}$.

Proof. Let $\langle a,b\rangle = \inf\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$. Then evidently $\langle a,b\rangle \subseteq \langle a_1,b_1\rangle \cap \langle a_2,b_2\rangle$. But the converse inclusion holds, too, because if $x \in \langle a_1,b_1\rangle \cap \langle a_2,b_2\rangle$, then $\langle x\rangle$ is a lower bound of $\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$ so that $\langle x\rangle \subseteq \langle a,b\rangle$ by assumption.

Now let $\langle a,b\rangle = \sup\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$. Then a is a lower bound of $\{a_1,a_2\}$ and b is an upper bound of $\{b_1,b_2\}$. Suppose that b' is any upper bound of $\{b_1,b_2\}$. Then $\langle a,b'\rangle$ is an upper bound of $\{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle\}$ and the assumption yields $\langle a,b\rangle \subseteq \langle a,b'\rangle$. Hence $b' \geqslant b$. We have proved $b = \sup\{b_1,b_2\}$. The relation $a = \inf\{a_1,a_2\}$ can be proved analogously.

The converse implications are evident. □

2. SUFFICIENT CONDITION

In this section $(P, \leqslant)$ will be a partially ordered set satisfying the condition that for any $a,b \in P$, $a \leqslant b$, there exists a finite maximal chain in $\langle a,b\rangle$.

Let U, V be equivalence relations on P. Consider the following conditions:

(i) for every $a \in P$ there is $[a]U = \langle u_1,v_1\rangle$, $[a]V = \langle u_2,v_2\rangle$ for some $u_1,u_2 \in \operatorname{Min} P$, $v_1,v_2 \in \operatorname{Max} P$;

(ii) $U \cap V$ is the least equivalence relation (i.e. the equality);

(iii) for every $a,b \in P$, $a \leqslant b$, there exist $z_1,z_2 \in \langle a,b\rangle$ satisfying aUz_1Vb, aVz_2Ub.

We will show that if there exists a couple of equivalence relations U, V on P satisfying (i), (ii), (iii), then the partially ordered system $(\operatorname{Int} P, \subseteq)$ is selfdual.

Evidently, the condition (ii) is equivalent to

(ii′) for any $a,b \in P$, $[a]U \cap [b]V$ is either empty or a one-element set;

and also to

(ii″) for any $a \in P$, $[a]U \cap [a]V = \{a\}$.

It is also easy to see that if U, V satisfy (ii), (iii), then U, V satisfy the following condition, too.

(iv) for every $a,b \in P$, $a \leqslant b$, there exists a unique element $z_1 \in \langle a,b\rangle$ satisfying aUz_1Vb and a unique element $z_2 \in \langle a,b\rangle$ with aVz_2Ub.

Lemma 2.1. *Let* U, V *be equivalence relations on* P *satisfying* (iii). *If* $a,b \in P$, $a \prec b$, *then either* aUb *or* aVb.

The proof is evident.

Now suppose that U, V are equivalence relations on P satisfying (i)–(iii). We will construct a dual automorphism of $(\operatorname{Int} P, \subseteq)$.

Let $\langle a, b\rangle \in \operatorname{Int} P$. By (i) there exist $u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$ such that u is the least element of $[a]V$, v is the greatest element of $[b]U$. Since $a \leqslant b$, by (iii) there exists $z_1 \in \langle a, b\rangle$ with aUz_1Vb. Using again (iii) we can find $c \in \langle u, z_1\rangle$, $d \in \langle z_1, v\rangle$ satisfying $uUcVz_1$, z_1UdVv. Now $c \in [u]U \cap [b]V$, $d \in [a]U \cap [v]V$, so that c, d are uniquely determined by a, b, as follows from (ii). Since $c \leqslant d$, we can set $\varphi(\langle a, b\rangle) = \langle c, d\rangle$. We have defined a mapping $\varphi\colon \operatorname{Int} P \to \operatorname{Int} P$. In 2.2–2.6 the properties of this mapping φ are discussed.

Notice that $\varphi(\langle a\rangle) = \langle u, v\rangle$, where u is the least element of $[a]V$, v is the greatest element of $[a]U$. If $u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$ and $u \leqslant v$, then $\varphi(\langle u, v\rangle) = \langle z\rangle$, where $z \in [u]U \cap [v]V$.

To prove that φ is a dual automorphism of $(\operatorname{Int} P, \subseteq)$, it is sufficient to show that φ is one-to-one, onto and satisfies

$$\begin{aligned} \langle a, b\rangle \prec \langle a', b'\rangle &\implies \varphi(\langle a, b\rangle) \supseteq \varphi(\langle a', b'\rangle), \\ \varphi(\langle a, b\rangle) \prec \varphi(\langle a', b'\rangle) &\implies \langle a, b\rangle \supseteq \langle a', b'\rangle, \end{aligned}$$

thanks to the assumption that for any $x, y \in P$, $x \leqslant y$, there exists a finite maximal chain in $\langle x, y\rangle$.

Lemma 2.2. *The mapping φ is one-to-one.*

Proof. Let $\varphi(\langle a, b\rangle) = \varphi(\langle a', b'\rangle) = \langle c, d\rangle$. Then $c \in [u]U \cap [b']V$, $d \in [a]U \cap [v]V$ and simultaneously $c \in [u']U \cap [b']V$, $d \in [a']U \cap [v']V$, where $u, u' \in \operatorname{Min} P$, $v, v' \in \operatorname{Max} P$, u and u' are the least elements of $[a]V$ and $[a']V$, respectively, v and v' are the greatest elements of $[b]U$ and $[b']U$, respectively. The fact that $d \in [a]U \cap [a']U$ ensures $[a]U = [a']U$, hence aUa'. Further, $c \in [u]U \cap [u']U$ yields $[u]U = [u']U$, but since $[u]U$ is an interval and u, u' are minimal elements of P belonging to $[u]U$, we have $u = u'$. Now $u = u' \in [a]V \cap [a']V$, hence aVa'. We have proved $aU \cap Va'$. By (ii) this implies $a = a'$. The relation $b = b'$ can be proved analogously. □

Lemma 2.3. *The mapping φ is onto* $\operatorname{Int} P$.

Proof. Take any $\langle c, d\rangle \in \operatorname{Int} P$. There exists $z_1 \in \langle c, d\rangle$ such that cVz_1Ud by (iii). Further, (iii) ensures the existence of elements $a \in \langle u, z_1\rangle$, $b \in \langle z_1, v\rangle$ satisfying $uVaUz_1$, z_1VbUv for u the least element of $[c]U$, v the greatest element of $[d]V$. It is easy to see that $\varphi(\langle a, b\rangle) = \langle c, d\rangle$. □

Observe that interchanging the roles of U and V in the foregoing definition of φ we get the description of φ^{-1}, as the proof of the last lemma shows.

Lemma 2.4. *We have*

$$\langle a,b\rangle \prec \langle a',b'\rangle \implies \varphi(\langle a,b\rangle) \supseteq \varphi(\langle a',b'\rangle).$$

Proof. The relation $\langle a,b\rangle \prec \langle a',b'\rangle$ implies that either $a=a'$, $b\prec b'$ or $a\prec a'$, $b=b'$ holds. Let us analyse the first possibility, the other can be treated analogously. By 2.1 we have either bUb' or bVb'.

First suppose bUb'. Let $\varphi(\langle a,b\rangle)=\langle c,d\rangle$, $\varphi(\langle a,b'\rangle)=\langle c',d'\rangle$. We will show that $d=d'$. We have $d\in[a]U\cap[v]V$, $d'\in[a]U\cap[v']V$, where v and v' are the greatest elements of $[b]U$ and $[b']U$, respectively. But $[b]U=[b']U$, hence $v=v'$, so that $d=d'$ by (ii'). We have to prove that $\langle c,d\rangle\supseteq\langle c',d'\rangle$, which is equivalent to $c\leqslant c'$. From the definition of φ one can see that $c\leqslant b$ and since $b\prec b'$, we have $c<b'$. In view of (iii) there exists $t\in\langle c,b'\rangle$ such that $cUtVb'$. Therefore $t\in[c]U\cap[b']V$. Further, $c'\in[u]U\cap[b']V$ with u being the least element of $[a]V$. But $[c]U=[u]U$, so that $t=c'$ by (ii'). We have $c\leqslant t=c'$.

Now let bVb'. Again let $\varphi(\langle a,b\rangle)=\langle c,d\rangle$, $\varphi(\langle a,b'\rangle)=\langle c',d'\rangle$. It is easy to see that in this case $c=c'$. We have to show that $d'\leqslant d$. Denote by v and v' the greatest elements of $[b]U$ and $[b']U$, respectively. By (iii) there exists $r\in\langle b,v'\rangle$ such that $bUrVv'$. Further, (iii) ensures also the existence of an element $s\in\langle a,r\rangle$ with $aUsVr$. We have $s\in[a]U\cap[r]V$, $d'\in[a]U\cap[v']V$ but $[r]V=[v']V$, so that $s=d'$ by (ii'). Since $d'=s\leqslant r\leqslant v$, there exists $p\in\langle d',v\rangle$ satisfying $d'UpVv$. Since $p\in[d']U\cap[v]V$, $d\in[a]U\cap[v]V$ and $[d']U=[s]U=[a]U$, using again (ii') we obtain $p=d$. So $d'\leqslant p=d$.

The proof is complete. □

Lemma 2.5. *We have*

$$\varphi(\langle a,b\rangle)\prec\varphi(\langle a',b'\rangle) \implies \langle a,b\rangle\supseteq\langle a',b'\rangle.$$

Proof. The implication which has to be proved can be rewritten as

$$\langle c,d\rangle\prec\langle c',d'\rangle \implies \varphi^{-1}(\langle c,d\rangle)\supseteq\varphi^{-1}(\langle c',d'\rangle).$$

In view of the remark following 2.3 it is evident that the proof of the last implication would be quite similar to that of 2.4. □

As a consequence of 2.2–2.5 we get

Theorem 2.6. *The mapping* φ *is a dual automorphism of* $(\operatorname{Int} P, \subseteq)$.

Corollary 2.7. *Let* $(P, \leqslant)$ *be a partially ordered set such that for any* $a, b \in P$, $a \leqslant b$, *there exists a finite maximal chain in* $\langle a, b\rangle$. *If there exists a couple of equivalence relations* U, V *on* P *satisfying* (i)–(iii), *then the partially ordered system* $(\operatorname{Int} P, \subseteq)$ *is selfdual.*

If we have a dual automorphism of $(\operatorname{Int} P, \subseteq)$, then by means of automorphisms of $(P, \leqslant)$ other dual automorphisms of $(\operatorname{Int} P, \subseteq)$ can be obtained.

Theorem 2.8. *Let* φ *be any dual automorphism of* $(\operatorname{Int} P, \subseteq)$, Φ *any automorphism of* $(P, \leqslant)$. *Define* $\psi\colon \operatorname{Int} P \to \operatorname{Int} P$ *by*

$$\psi(\langle a, b\rangle) = \varphi(\langle \Phi(a), \Phi(b)\rangle).$$

Then ψ *is also a dual automorphism of* $(\operatorname{Int} P, \subseteq)$.

The proof is obvious.

3. NECESSARY CONDITION

In this section we will show that every dual automorphism of $(\operatorname{Int} P, \subseteq)$ is obtained from a dual automorphism φ corresponding to some equivalence relations U, V on P satisfying (i)–(iii), by means of an automorphism Φ of $(P, \leqslant)$ in the manner described in 2.8.

In 3.1–3.12 ψ will be a fixed dual automorphism of $(\operatorname{Int} P, \subseteq)$. The assumption that every interval in $(P, \leqslant)$ contains a finite maximal chain will not be needed before 3.13.

Lemma 3.1. *Let* $u \in \operatorname{Min} P$. *The interval* $\psi(\langle u\rangle)$ *contains as subsets just those intervals expressible as* $\psi(\langle u, x\rangle)$ *for some* $x \in P$.

Proof. Evidently $\psi(\langle u, x\rangle) \subseteq \psi(\langle u\rangle)$. Conversely, if $\langle a, b\rangle = \psi(\langle r, s\rangle) \subseteq \psi(\langle u\rangle)$, then $\langle r, s\rangle \supseteq \langle u\rangle$, but since u is a minimal element of P, $r = u$ necessarily holds. □

For any $x \in P$, $\langle x\rangle$ is a minimal element of $\operatorname{Int} P$. Hence $\langle x\rangle$ is the image of a maximal element of $\operatorname{Int} P$, i.e. $\langle x\rangle = \psi(\langle u, v\rangle)$ for some $u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$, $u \leqslant v$. Using this fact and 3.1 we get

Lemma 3.2. *If* $u \in \operatorname{Min} P$, *then* $\psi(\langle u \rangle) = \{t \in P \colon \langle t \rangle = \psi(\langle u, v \rangle)$ *for some* $v \in \operatorname{Max} P,\ v \geqslant u\}$.

Lemma 3.3. *The system* $\{\psi(\langle u \rangle) \colon \in \operatorname{Min} P\}$ *is a decomposition of* P.

Proof. We are going to show that every $t \in P$ is contained in a single $\psi(\langle u \rangle)$. As we have noted above, for every $t \in P$ there exist $u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$ such that $\langle t \rangle = \psi(\langle u, v \rangle)$. These u, v are uniquely determined by t and t belongs to $\psi(\langle u \rangle)$ only for this unique minimal element u. □

The following three lemmas can be verified analogously.

Lemma 3.4. *Let* $v \in \operatorname{Max} P$. *Then* $I(\in \operatorname{Int} P) \subseteq \psi(\langle v \rangle)$ *if and only if* $I = \psi(\langle y, v \rangle)$ *for some* $y \in P$.

Lemma 3.5. *If* $v \in \operatorname{Max} P$, *then* $\psi(\langle v \rangle) = \{t \in P \colon \langle t \rangle = \psi(\langle u, v \rangle)$ *for some* $u \in \operatorname{Min} P,\ u \leqslant v\}$.

Lemma 3.6. *The system* $\{\psi(\langle v \rangle) \colon v \in \operatorname{Max} P\}$ *is a decomposition of* P.

Let U and V be the equivalence relations on P corresponding to the decompositions of P mentioned in 3.3 and 3.5, respectively.

Theorem 3.7. *The equivalence relations* U, V *satisfy the conditions* (i)–(iii).

Proof. Evidently, (i) holds. To verify (ii'), let $r, t \in \psi(\langle u \rangle) \cap \psi(\langle v \rangle)$ for some $u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$. Then $\langle r \rangle = \langle t \rangle = \psi(\langle u, v \rangle)$ by 3.2 and 3.5, hence $r = t$. It remains to show that (iii) is valid. Let $a, b \in P$, $a \leqslant b$. There exist $u, u_1 \in \operatorname{Min} P$, $v, v_1 \in \operatorname{Max} P$, $r, s \in P$ such that $u \leqslant v$, $u_1 \leqslant v_1$, $r \leqslant s$, $\langle a \rangle = \psi(\langle u, v \rangle)$, $\langle b \rangle = \psi(\langle u_1, v_1 \rangle)$, $\langle a, b \rangle = \psi(\langle r, s \rangle)$. Since $\langle a \rangle, \langle b \rangle \subseteq \langle a, b \rangle$, we have $\langle u, v \rangle, \langle u_1, v_1 \rangle \supseteq \langle r, s \rangle$. Now we take $z_1, z_2 \in P$ satisfying $\langle z_1 \rangle = \psi(\langle u, v_1 \rangle)$, $\langle z_2 \rangle = \psi(\langle u_1, v \rangle)$. The inclusions $\langle u, v_1 \rangle, \langle u_1, v \rangle \supseteq \langle r, s \rangle$ imply $\langle z_1 \rangle, \langle z_2 \rangle \subseteq \langle a, b \rangle$. Further, $z_1, a \in \psi(\langle u \rangle)$, $z_1, b \in \psi(\langle v_1 \rangle)$, hence aUz_1Vb. Analogously $z_2, a \in \psi(\langle v \rangle)$, $z_2, b \in \psi(\langle u_1 \rangle)$ give aVz_2Ub.

The proof is complete. □

Corollary 3.8. *Let* $(P, \leqslant)$ *be any partially ordered set. If the partially ordered system* $(\operatorname{Int} P, \subseteq)$ *is selfdual, then there exists a couple of equivalence relations* U, V *on* P *satisfying* (i)–(iii).

Now we are going to define a mapping $\Phi \colon P \to P$ with the aim to prove that Φ is an automorphism of $(P, \leqslant)$.

Let $x \in P$. Then $\psi(\langle x\rangle) = \langle u', v'\rangle$ for some $u' \in \operatorname{Min} P$, $v' \in \operatorname{Max} P$. In view of 3.7 there exists a unique $z \in \langle u', v'\rangle$ satisfying $u'VzUv'$. Set $\Phi(x) = z$.

Lemma 3.9. *Φ is a one-to-one mapping.*

Proof. Let $\Phi(x) = \Phi(y) = z$. There exist $u', \bar{u} \in \operatorname{Min} P$, $v', \bar{v} \in \operatorname{Max} P$ such that $u' \leqslant v'$, $\bar{u} \leqslant \bar{v}$, $\psi(\langle x\rangle) = \langle u', v'\rangle$, $\psi(\langle y\rangle) = \langle \bar{u}, \bar{v}\rangle$. By the definition of Φ we have $u'VzUv'$, $\bar{u}VzU\bar{v}$. It follows that u', $\bar{u}$ and v', $\bar{v}$ belong to the same V-class and U-class, respectively. Taking into consideration the facts that U, V fulfil the condition (i) and $u', \bar{u} \in \operatorname{Min} P$, $v', \bar{v} \in \operatorname{Max} P$, we obtain $u' = \bar{u}$, $v' = \bar{v}$. Hence $\psi(\langle x\rangle) = \psi(\langle y\rangle)$, which implies $x = y$. □

Lemma 3.10. *The mapping Φ is onto.*

Proof. Take any $z \in P$. Then z belongs to a V-class $\psi(\langle v\rangle)$ and to a U-class $\psi(\langle u\rangle)$ ($u \in \operatorname{Min} P$, $v \in \operatorname{Max} P$). Let u' be the least element of $\psi(\langle v\rangle)$, v' the greatest element of $\psi(\langle u\rangle)$. Now if $\psi^{-1}(\langle u', v'\rangle) = \langle x\rangle$, then evidently $\Phi(x) = z$. □

Now we are going to show that $x \prec y$ if and only if $\Phi(x) \prec \Phi(y)$.

Lemma 3.11. *If $x \prec y$ in P, then $\Phi(x) \prec \Phi(y)$.*

Proof. Let $x \prec y$, $\psi(\langle x\rangle) = \langle u, v\rangle$. Since $\langle x, y\rangle$ is an interval covering the minimal ones $\langle x\rangle$, $\langle y\rangle$, $\psi(\langle x, y\rangle)$ is an interval covered by the maximal ones $\psi(\langle x\rangle) = \langle u, v\rangle$, $\psi(\langle y\rangle)$. Hence as to $\psi(\langle y\rangle)$, we have either $\psi(\langle y\rangle) = \langle u, v'\rangle$ for some $v' \in \operatorname{Max} P$ or $\psi(\langle y\rangle) = \langle u', v\rangle$ for some $u' \in \operatorname{Min} P$. Without loss of generality we can suppose that the latter possibility occurs. Then $\psi(\langle x, y\rangle) = \langle t, v\rangle$ for an element t covering both u and u'. Since $\langle u, v\rangle = \sup\{\langle u, t\rangle, \langle t, v\rangle\}$ and $\langle t\rangle = \inf\{\langle u, t\rangle, \langle t, v\rangle\}$, we have $\psi^{-1}(\langle u, v\rangle) = \langle x\rangle = \inf\{\psi^{-1}(\langle u, t\rangle), \langle x, y\rangle\}$, $\psi^{-1}(\langle t\rangle) = \sup\{\psi^{-1}(\langle u, t\rangle), \langle x, y\rangle\}$, where $\psi^{-1}(\langle u, t\rangle)$ is an interval covered just by two maximal intervals, namely by $\psi^{-1}(\langle u\rangle)$ and $\psi^{-1}(\langle t\rangle)$. It is easy to see that the case of $\psi^{-1}(\langle u, t\rangle)$ being an interval with a maximal element of P as the greatest element is impossible. Hence $\psi^{-1}(\langle u, t\rangle) = \langle \bar{u}, s\rangle$ for some $\bar{u} \in \operatorname{Min} P$ and $s \in P$ covered by a maximal element of P. Then $x \in \langle \bar{u}, s\rangle$ and $\psi^{-1}(\langle t\rangle) = \sup\{\langle \bar{u}, s\rangle, \langle x, y\rangle\} = \langle \bar{u}, \bar{v}\rangle$ for an element $\bar{v} \in \operatorname{Max} P$ such that $\bar{v} = \sup\{y, s\}$, by 1.1. Since $\langle u\rangle \prec \langle u, t\rangle$, we have $\psi^{-1}(\langle u\rangle) \succ \psi^{-1}(\langle u, t\rangle) = \langle \bar{u}, s\rangle$ and this implies $\psi^{-1}(\langle u\rangle) = \langle \bar{u}, \bar{v}_1\rangle$ for a maximal element $\bar{v}_1$ of P, $\bar{v}_1 \neq \bar{v}$, $\bar{v}_1 \succ s$. Now consider u' instead of u. Analogously as before we can show that there exists $p \in P$ covering $\bar{u}$ such that $y \in \psi^{-1}(\langle u', t\rangle) = \langle p, \bar{v}\rangle$, $\bar{u} = \inf\{x, p\}$, and there exists $\bar{u}_1 \in \operatorname{Min} P$, $\bar{u}_1 \neq \bar{u}$, $\bar{u}_1 \prec p$ with $\psi^{-1}(\langle u'\rangle) = \langle \bar{u}_1, \bar{v}\rangle$.

Further, let us investigate $\psi^{-1}(\langle v\rangle)$. This interval is a maximal one, let us denote it by $\langle\bar{\bar{u}},\bar{\bar{v}}\rangle$. Then

$$\begin{aligned}\langle x,y\rangle &= \psi^{-1}(\langle t,v\rangle) = \psi^{-1}(\sup\{\langle t\rangle,\langle v\rangle\})\\ &= \inf\{\psi^{-1}(\langle t\rangle),\psi^{-1}(\langle v\rangle)\} = \inf\{\langle\bar{u},\bar{v}\rangle,\langle\bar{\bar{u}},\bar{\bar{v}}\rangle\},\end{aligned}$$

hence $\langle x,y\rangle = \langle\bar{u},\bar{v}\rangle\cap\langle\bar{\bar{u}},\bar{\bar{v}}\rangle$, by 1.1. In view of 3.2 and 3.5 the relations $\langle u\rangle = \psi(\langle\bar{u},\bar{v}_1\rangle)$, $\langle v\rangle = \psi(\langle\bar{\bar{u}},\bar{\bar{v}}\rangle)$, $\langle u'\rangle = \psi(\langle\bar{u}_1,\bar{v}\rangle)$ imply $u\in\psi(\langle\bar{v}_1\rangle)$, $v\in\psi(\langle\bar{\bar{u}}\rangle)$, $u'\in\psi(\langle\bar{v}\rangle)$. By the definition of Φ we have $\Phi(x)\in\psi(\langle\bar{v}_1\rangle)\cap\psi(\langle\bar{\bar{u}}\rangle)$, $\Phi(y)\in\psi(\langle\bar{v}\rangle)\cap\psi(\langle\bar{\bar{u}}\rangle)$ and using again 3.2 and 3.5 we obtain $\langle\Phi(x)\rangle = \psi(\langle\bar{\bar{u}},\bar{v}_1\rangle)$, $\langle\Phi(y)\rangle = \psi(\langle\bar{\bar{u}},\bar{v}\rangle)$. Since $\langle\bar{\bar{u}},s\rangle\prec\langle\bar{\bar{u}},\bar{v}_1\rangle$, $\langle\bar{\bar{u}},\bar{v}\rangle$, applying ψ we get $\psi(\langle\bar{\bar{u}},s\rangle)\succ\langle\Phi(x)\rangle$, $\langle\Phi(y)\rangle$. Hence one of the elements $\Phi(x)$, $\Phi(y)$ covers the other. Suppose $\Phi(y)\prec\Phi(x)$. Applying ψ to the inclusions $\langle x,y\rangle\subseteq\langle\bar{\bar{u}},\bar{v}\rangle$, $\langle x\rangle\subseteq\langle\bar{\bar{u}},s\rangle$ we get $\langle t,v\rangle\supseteq\langle\Phi(y)\rangle$, $\langle u,v\rangle\supseteq\psi(\langle\bar{\bar{u}},s\rangle) = \langle\Phi(y),\Phi(x)\rangle$. Hence $\langle\Phi(y),\Phi(x)\rangle\subseteq\langle t,v\rangle$ and applying ψ^{-1} we obtain $\langle\bar{\bar{u}},s\rangle\supseteq\langle x,y\rangle$. It follows that $y\leqslant s$. But we have proved $\bar{v} = \sup\{y,s\}$, so $\bar{v} = s$, a contradiction. Therefore $\Phi(x)\prec\Phi(y)$ and the proof is complete. □

It remains to prove that $\Phi(x)\prec\Phi(y)$ implies $x\prec y$. This implication is equivalent to $r\prec s \implies \Phi^{-1}(r)\prec\Phi^{-1}(s)$. To prove the last implication, we could proceed analogously as in the previous lemma. However, we choose another way. By assumption ψ is a dual automorphism of $(\operatorname{Int}P,\subseteq)$. Then evidently ψ^{-1} is also a dual automorphism of the same system. Then $\{\psi^{-1}(\langle u\rangle)\colon u\in\operatorname{Min}P\}$, $\{\psi^{-1}(\langle v\rangle)\colon v\in\operatorname{Max}P\}$ are decompositions of P. Let U' and V' be the equivalence relations on P corresponding to the first and to the second decomposition, respectively. As above, U' and V' fulfil (i)–(iii). Now define $\Phi'\colon P\to P$ analogously as in the case of Φ. That is, if $z\in P$ then take $\psi^{-1}(\langle z\rangle)$ which is a maximal interval, say $\langle u,v\rangle$. Further, take the unique element $x\in\langle u,v\rangle$ satisfying $uV'xU'v$ and set $\Phi'(z) = x$. Evidently Φ', just as Φ, is a one-to-one mapping, onto and satisfies $r\prec s \implies \Phi'(r)\prec\Phi'(s)$.

We will show that $\Phi' = \Phi^{-1}$. From $\psi^{-1}(\langle z\rangle) = \langle u,v\rangle$ we obtain $\langle z\rangle = \psi(\langle u,v\rangle)$. Let u' and v' be the least element of $\psi(\langle v\rangle)$ and the greatest element of $\psi(\langle u\rangle)$, respectively. Then $\langle u'\rangle\subseteq\psi(\langle v\rangle)$, $\langle v'\rangle\subseteq\psi(\langle u\rangle)$ and applying ψ^{-1} we get $\psi^{-1}(\langle u'\rangle)\supseteq\langle v\rangle$, $\psi^{-1}(\langle v'\rangle)\supseteq\langle u\rangle$. Hence v belongs to the U'-class $\psi^{-1}(\langle u'\rangle)$, u belongs to the V'-class $\psi^{-1}(\langle v'\rangle)$. But for $x = \Phi'(z)$ we have $uV'xU'v$, so x belongs to the same V'-class as u and to the same U'-class as v. We have $x\in\psi^{-1}(\langle u'\rangle)\cap\psi^{-1}(\langle v'\rangle)$, which gives $\langle x\rangle = \psi^{-1}(\langle u',v'\rangle)$. Consequently, $\psi(\langle x\rangle) = \langle u',v'\rangle$. Now it is clear that $\Phi(x) = z$, so that $\Phi^{-1}(z) = x$. □

In this way we have proved

Lemma 3.12. *If $x,y\in P$ and $\Phi(x)\prec\Phi(y)$, then $x\prec y$.*

The following theorem is a direct consequence of 3.9–3.12.

Theorem 3.13. *Let $(P, \leqslant)$ be a partially ordered set satisfying the condition that its every interval contains a finite maximal chain, and let ψ be a dual automorphism of* $(\operatorname{Int} P, \subseteq)$. *Then the above defined mapping Φ is an automorphism of $(P, \leqslant)$.*

Theorem 3.14. *Let $(P, \leqslant)$ be a partially ordered set satisfying the condition that its every interval contains a finite maximal chain, and let ψ be a dual automorphism of* $(\operatorname{Int} P, \subseteq)$. *If U, V are the equivalence relations on P corresponding to ψ as in* 3.7, *φ is the dual automorphism of* $(\operatorname{Int} P, \subseteq)$ *corresponding to U, V as in* 2.6 *and Φ is the automorphism of $(P, \leqslant)$ as in* 3.13, *then*

$$\psi(\langle a, b\rangle) = \varphi(\langle \Phi(a), \Phi(b)\rangle)$$

for every $a, b \in P$, $a \leqslant b$.

Proof. First we will show that $\psi(\langle x\rangle) = \varphi(\langle \Phi(x)\rangle)$ for every $x \in P$. In the previous section we have remarked that $\varphi(\langle \Phi(x)\rangle) = \langle u', v'\rangle$, where u' is the least element of $[\Phi(x)]V$ and v' is the greatest element of $[\Phi(x)]U$. Hence $u'V\Phi(x)Uv'$ and by the definition of $\Phi(x)$ this means that $\psi(\langle x\rangle) = \langle u', v'\rangle$.

Now take arbitrary $a, b \in P$, $a \leqslant b$. Then

$$\begin{aligned}\varphi(\langle \Phi(a), \Phi(b)\rangle) &= \varphi(\sup\{\langle \Phi(a)\rangle, \langle \Phi(b)\rangle\}) = \inf\{\varphi(\langle \Phi(a)\rangle), \varphi(\langle \Phi(b)\rangle)\} \\ &= \inf\{\psi(\langle a\rangle), \Psi(\langle b\rangle)\} = \psi(\sup\{\langle a\rangle, \langle b\rangle\}) = \psi(\langle a, b\rangle).\end{aligned}$$

The proof is complete. □

4. EXAMPLES

We give some examples of partially ordered sets with selfdual systems of intervals. The simplest examples are antichains. The infinite fence shown in Fig. 1 and the crowns shown in Fig. 2 serve as further simple examples.

Now let $(P, \leqslant)$ be as in Fig. 3. If U and V are the equivalence relations on P corresponding to the decompositions $\{\{a_i, b_i, c_i, d_i\} : i \in \mathbb{Z}\}$ and $\{\{d_i, c_{i+1}, b_{i+2}, a_{i+3}\} : i \in \mathbb{Z}\}$, respectively, then evidently U, V satisfy (i)–(iii). Therefore $(P, \leqslant)$ has a selfdual system of intervals. Supposing that for an $n \in N$ and every $i \in \mathbb{Z}$ we have $a_{i+n} = a_i$, $b_{i+n} = b_i$, $c_{i+n} = c_i$, $d_{i+n} = d_i$, we obtain a finite partially ordered set with a selfdual system of intervals. Varying the length we can get further examples.

In the above examples all U-classes and V-classes are chains. Nonetheless, we can easily construct examples which do not satisfy this condition. Fig. 4 represents such an example.

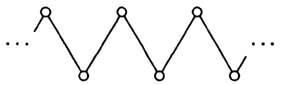

Fig. 1

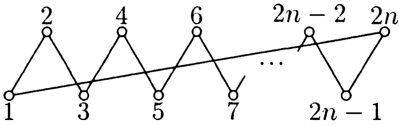

Fig. 2

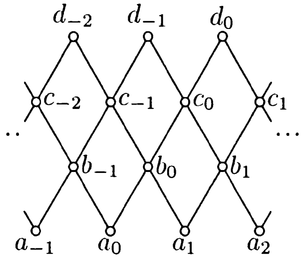

Fig. 3

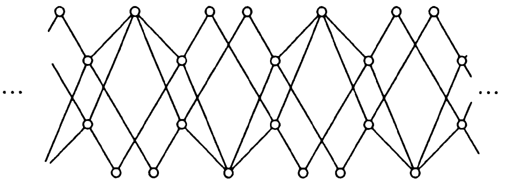

Fig. 4

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