

Positive and Nodal Solutions For a Nonlinear Schrödinger Equation with Indefinite Potential

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Abstract

We deal with the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where V is a (possible) sign changing potential satisfying mild assumptions and the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ is a subcritical and superlinear function. By combining variational techniques and the concentration-compactness principle we obtain a positive ground state solution and also a nodal solution. The proofs rely in localizing the infimum of the associated functional constrained to Nehari type sets.

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1 Introduction

The existence of stationary solutions of the form $\psi(x, t) = u(x)e^{-\frac{iEt}{\hbar}}$ for the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi - f(|\psi|) \frac{\psi}{|\psi|}$$

has been extensively studied in the past twenty years involving the use of variational methods (see for instance [8, 14, 23] and references therein). Substituting in the equation and making suitable changes of variables leads to the following semilinear elliptic equation

$$(P) \quad -\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N.$$

Works on the existence of positive solutions for (P) when the potential V is bounded from below by a positive constant and f has subcritical growth, using mountain pass arguments are well known ([23, 14, 2]). Existence of standing waves of problem (P) by minimization of constrained variational problems was shown in [20, 8, 18] among many others. Several new results concerning sign changing potentials V have appeared lately (e.g. [24, 11, 13, 15, 16, 17]). Furthermore, nodal solutions of (P) in a bounded domain are proved to exist in [22] and in unbounded domain in [29, 3], for instance.

In this paper we are concerned with the existence of a positive ground state and additionally a nodal solution for the problem (P) with $V : \mathbb{R}^N \rightarrow \mathbb{R}$ being a (possible) sign changing potential satisfying very mild assumptions and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ being a superlinear function of class C^1 with subcritical growth.

In order to impose precise conditions on V , let us denote $V = V^+ - V^-$ with $V^\pm := \max\{\pm V, 0\}$. We list below the basic assumptions on V .

$$(V_0) \quad V \in L^t_{loc}(\mathbb{R}^N) \text{ for some } t > N/2;$$

$$(V_1) \quad V_\infty^+ := \lim_{|x| \rightarrow \infty} V(x) > 0;$$

$$(V_2) \quad \text{if we denote by } S \text{ the best constant to the Sobolev embedding, namely}$$

$$S := \inf \left\{ \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 : u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and } \|u\|_{L^{2^*}(\mathbb{R}^N)} = 1 \right\},$$

then

$$\|V^-\|_{L^{N/2}(\mathbb{R}^N)} < S.$$

After the work of P.L. Lions [20], conditions like (V_1) have appeared in many works with positive or sign changing potentials [29, 23, 9, 11, 7]. Some conditions related to (V_2) have already appeared in [5, 24, 11, 12, 6] where the authors considered problem (P) or some of its variants.

Concerning the nonlinearity f , we start by assuming that

$$(f_0) \quad f \in C^1(\mathbb{R}, \mathbb{R});$$

(f_1) there exist $2 \leq q+1 < \eta+1 < 2^* := 2N/(N-2)$ such that

$$\lim_{|s| \rightarrow 0} \frac{|f'(s)|}{|s|^{q-1}} = 0 \quad \text{and} \quad \limsup_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{\eta-1}} < +\infty.$$

Conditions (f_0) – (f_1) and (V_1) – (V_2) show that problem (P) has a variational structure. More specifically, the weak solutions of problem (P) are precisely the critical points of the C^1 -functional $I : W^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx - \int_{\mathbb{R}^N} F(u) \, dx,$$

where $F(s) := \int_0^s f(\tau) \, d\tau$. In order to obtain such critical points, we use minimax theorems and the concentration-compactness principle. The main idea is to use (V_1) to correctly localize the infimum of I constrained to Nehari type sets. To achieve this objective, we also assume that f satisfies the well-known Ambrosetti-Rabinowitz superlinear condition and a monotonicity condition. More specifically, we shall impose the following

(f_2) there exists $\theta > 2$ such that

$$0 < \theta F(s) \leq sf(s) \quad \text{for all } s \neq 0;$$

(f_3) the function $s \mapsto f(s)/s$ is increasing in $(0, +\infty)$.

We recall that a solution u_1 of (P) is called ground state solution if it possesses minimum energy among all solutions, that is,

$$I(u_1) = \min \{I(u) : u \neq 0 \text{ is a solution of } (P)\}.$$

Our first existence result is

Theorem 1.1 *Suppose that f satisfies (f_0) – (f_3) and V satisfies (V_0) – (V_2). Then problem (P) has a positive ground state solution provided V satisfies*

$$(V_3) \quad V(x) \leq V_\infty^+ \quad \text{for all } x \in \mathbb{R}^N, \quad V \not\equiv V_\infty^+.$$

In our second result we are interested in the question of multiple solutions for (P). In order to obtain a solution of (P) which changes sign in \mathbb{R}^N , we need a condition stronger than (f_3), namely

(\hat{f}_3) there exist $\eta \leq \sigma \leq 2^* - 1$ and $C > 0$ such that

$$f'(s)s - f(s) \geq C|s|^{\sigma-1}s, \quad \text{for all } s \in \mathbb{R}.$$

We can state now our multiplicity results as follows.

Theorem 1.2 *Suppose that f is odd and satisfies $(f_0) - (f_2)$ and (\hat{f}_3) . Suppose also that V satisfies $(V_0) - (V_2)$ and, for some $\gamma < V_\infty^+ q / (q + 1)$, there holds*

$$(\widehat{V}_3) \quad V(x) \leq V_\infty^+ - C e^{-\gamma|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Then problem (P) has in addition to a positive ground state solution a solution which changes sign.

As far as a positive solution is concerned, our result is complementary to previous results since the potential we consider may change sign and satisfy mild integral conditions. Moreover, we may have $V \notin L^{N/2}(\mathbb{R}^N)$ and therefore Theorem 1.1 also complements the results of [23, 7].

On the other hand, very little is known about existence of a sign changing solution for this problem and hence our interest. We notice that, unlike in [24], we do not assume that V is bounded from below. Moreover, we do not assume any kind of regularity for V^- at the origin and the set $\{x \in \mathbb{R}^N : V^-(x) > 0\}$ has finite measure. Thus, our results are not contained in [11]. More generally, the potential $V_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$V_\varepsilon(x) := \begin{cases} \frac{|x|^2}{1+|x|^2}, & \text{for } |x| > 1, \\ -\frac{\varepsilon}{|x|^\alpha} & \text{for } |x| \leq 1, \end{cases}$$

where $\varepsilon > 0$ is small and $0 < \alpha < 2$, satisfies our hypotheses but the arguments of [23, 7, 24, 11] do not apply for this potential. Finally, we would like to cite the recent papers [15, 16], where the authors have considered a singularly perturbed version of (P) and have obtained some results related, but not comparable, with ours.

As previously said, the proof of the results rely in applying variational methods together with a concentration-compactness argument, as in [26], in order to overcome the problem of lack of compactness of Sobolev embeddings in unbounded domains. Some of the ideas and calculations are inspired by the papers [10, 9, 1].

The paper is organized as follows. In Section 2 we present the variational framework and a version of the well known "splitting lemma" which is going to be the main tool in posterior compactness arguments. The existence of a ground state solution is proved in Section 3. Finally, Section 4 is devoted to the proof of existence of a solution for (P) which changes sign.

2 The variational framework

In this section we present the variational framework to deal with problem (P) and also give some preliminaries which are useful later. Since (\hat{f}_3) implies (f_3) , throughout the paper we assume that f satisfies $(f_0) - (f_3)$ and V satisfies $(V_0) - (V_2)$. We denote by $\|u\|_p$ the $L^p(\mathbb{R}^N)$ -norm of $u \in L^p(\mathbb{R}^N)$. If $u \in L^1(\mathbb{R}^N)$, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x) dx$.

We start with a straightforward consequence of our hypotheses on V .

Lemma 2.1 *The quadratic form*

$$u \mapsto \int (|\nabla u|^2 + V^+(x)u^2)$$

defines a norm in $W^{1,2}(\mathbb{R}^N)$ which is equivalent to the usual one.

Proof. In view of condition (V_1) there exists $R > 0$ such that

$$\nu := \frac{V_\infty^+}{2} \leq V^+(x) \leq \frac{3V_\infty^+}{2} = 3\nu, \quad \text{for all } x \in \mathbb{R}^N \setminus B_R(0),$$

where $B_R(0) := \{x \in \mathbb{R}^N : |x| < R\}$. This, Hölder's inequality and the definition of S provide

$$\begin{aligned} \int (|\nabla u|^2 + V^+(x)u^2) &= \int |\nabla u|^2 + \int_{B_R(0)} V^+(x)u^2 dx + \int_{|x| \geq R} V^+(x)u^2 dx \\ &\leq \int |\nabla u|^2 + \|V^+\|_{L^{N/2}(B_R(0))} \|u\|_{2^*}^2 + 3\nu \int_{|x| \geq R} u^2 dx \\ &\leq (1 + C_1) \int |\nabla u|^2 + 3\nu \int u^2 \\ &\leq \max\{1 + C_1, 3\nu\} \int (|\nabla u|^2 + u^2), \end{aligned}$$

with $C_1 := S^{-1} \|V^+\|_{L^{N/2}(B_R(0))}$.

On the other hand, we have that

$$\int_{B_R(0)} u^2 dx \leq |B_R(0)|^{2/N} \left(\int_{B_R(0)} |u|^{2^*} dx \right)^{2/2^*} \leq C_2 \int |\nabla u|^2,$$

where $C_2 := S^{-1} |B_R(0)|^{2/N}$ and $|B_R(0)|$ denotes the Lebesgue measure of $B_R(0)$. Hence,

$$\begin{aligned} \int (|\nabla u|^2 + u^2) &= \int |\nabla u|^2 + \int_{B_R(0)} |u|^2 dx + \int_{|x| \geq R} |u|^2 dx \\ &\leq (1 + C_2) \int |\nabla u|^2 + \frac{1}{\nu} \int_{|x| \geq R} V^+(x)u^2 dx \\ &\leq \max\{1 + C_2, \nu^{-1}\} \int (|\nabla u|^2 + V^+(x)u^2), \end{aligned}$$

and the lemma is proved.

In view of the above result we can set X as being the Hilbert space $W^{1,2}(\mathbb{R}^N)$ endowed with the inner product

$$\langle u, v \rangle_X := \int (\nabla u \cdot \nabla v + V^+(x)uv), \quad \text{for all } u, v \in X$$

and associated norm given by

$$\|u\|_X := \left(\int |\nabla u|^2 + V^+(x)u^2 \right)^{1/2}, \quad \text{for all } u \in X.$$

Since $\|\cdot\|_X$ is equivalent to the usual norm of $W^{1,2}(\mathbb{R}^N)$, the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for any $2 \leq p \leq 2^*$.

By (f_1) , for any given $\delta > 0$ there exists $C_\delta > 0$ such that

$$|f(s)| \leq \delta|s|^q + C_\delta|s|^\eta, \quad \text{for all } s \in \mathbb{R}. \tag{2.1}$$

Recalling that $2 \leq q + 1 < \eta + 1 < 2^*$, we can integrate the above inequality to conclude that the functional $u \mapsto \int F(u)$ is well defined in X . Moreover, since $V^- \in L^{N/2}(\mathbb{R}^N)$, we have that

$$\int V^-(x)u^2 \leq \|V^-\|_{N/2} \|u\|_{2^*}^2 \leq S^{-1} \|V^-\|_{N/2} \|u\|_X^2 < \infty, \tag{2.2}$$

for any $u \in X$. Thus, the functional $I : X \rightarrow \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \|u\|_X^2 - \frac{1}{2} \int V^-(x)u^2 - \int F(u) \tag{2.3}$$

is well defined. By using standard arguments (see [8, Theorem A.VI]) we can show that $I \in C^1(X, \mathbb{R})$ with

$$I'(u)\varphi = \int (\nabla u \nabla \varphi + V(x)u\varphi) - \int f(u)\varphi, \quad \text{for all } u, \varphi \in X.$$

Consequently, critical points of the functional I are precisely the weak solutions of problem (P) .

We now recall a well known compactness condition: we say that I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ $((PS)_c$ for short) if any sequence $(u_n) \subset X$ such that $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ possesses a convergent subsequence.

Lemma 2.2 *If $(u_n) \subset X$ is a $(PS)_c$ sequence for I , then (u_n) is bounded in X .*

Proof. Since $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$, we can use (f_2) and (2.2) to get

$$\begin{aligned} c + o_n(1)\|u_n\|_X &= I(u_n) - \frac{1}{\theta} I'(u_n)u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \left(1 - \frac{\|V^-\|_{N/2}}{S}\right) \|u_n\|_X^2, \end{aligned}$$

where $o_n(1)$ denotes a quantity approaching zero as $n \rightarrow \infty$. The above inequality implies that (u_n) is bounded in X .

In order to get compactness, it is important to consider the limit problem associated to (P) , namely the autonomous problem

$$(P_\infty) \quad -\Delta u + V_\infty^+ u = f(u) \quad \text{in } \mathbb{R}^N,$$

whose solutions are the critical points of the functional $I_\infty : X \rightarrow \mathbb{R}$ given by

$$I_\infty(u) := \frac{1}{2} \int (|\nabla u|^2 + V_\infty^+ u^2) - \int F(u).$$

Let \mathcal{N}_∞ be the Nehari manifold of I_∞ , that is

$$\mathcal{N}_\infty := \{u \in X \setminus \{0\} : I'_\infty(u)u = 0\}$$

and consider the related minimization problem

$$c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

The proof of the next result can be found in Berestycki-Lions [8].

Proposition 2.1 *Problem (P_∞) has a positive and radially symmetrical solution $\bar{u} \in X$ such that $I_\infty(\bar{u}) = c_\infty$. Moreover, if the function f is odd, for any $0 < \delta < \sqrt{V_\infty^+}$, there exists a constant $C = C(\delta) > 0$ such that*

$$\bar{u}(x) \leq Ce^{-\delta|x|}, \quad \text{for all } x \in \mathbb{R}^N. \tag{2.4}$$

In order to prove that the functional I satisfies some compactness condition we shall need the following version of a result due to Struwe [26] (see also [4]).

Lemma 2.3 (Splitting Lemma) *Let $(u_n) \subset X$ be such that*

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0$$

and $u_n \rightharpoonup u_0$ weakly in X . Then $I'(u_0) = 0$ and we have either

- (a) $u_n \rightarrow u_0$ strongly in X , or
- (b) *there exists $k \in \mathbb{N}$, $(y_n^j) \in \mathbb{R}^N$ with $|y_n^j| \rightarrow \infty$, $j = 1, \dots, k$, and nontrivial solutions u^1, \dots, u^k of the problem (P_∞) , such that*

$$I(u_n) \rightarrow I(u_0) + \sum_{j=1}^k I_\infty(u^j) \tag{2.5}$$

and

$$\left\| u_n - u_0 - \sum_{j=1}^k u^j(\cdot - y_n^j) \right\|_X \rightarrow 0.$$

The proof of this kind of compactness global lemma is by now standard, and therefore we only provide a sketch of the proof. Since we are not dealing with homogeneous nonlinearities, we shall need the following technical result, proof of which can be found in [1, Lemma 3.1].

Lemma 2.4 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $s \geq 2$ and $(g_n) \subset L^s(\Omega) \cap L^{2^*}(\Omega)$ be a sequence bounded in $L^{2^*}(\Omega)$ such that $g_n(x) \rightarrow 0$ a.e in Ω .*

(i) If f satisfies (f_1) , then

$$\int_{\Omega} |F(g_n + w) - F(g_n) - F(w)| = o_n(1),$$

for each $w \in L^{\eta+1}(\Omega) \cap L^{q+1}(\Omega)$.

(ii) If f satisfies $(f_1) - (f_3)$, then

$$\int_{\Omega} |f(g_n + w) - f(g_n) - f(w)|^r = o_n(1), \quad \text{for all } 1 < r \leq 2,$$

and $w \in L^2(\Omega) \cap L^{2^*}(\Omega)$.

We present below the main ideas of the proof of Lemma 2.3.

Proof of Lemma 2.3. We follow the proof presented in [28, Section 8.1]. Firstly, the compactness of the embedding $X \hookrightarrow L^p_{loc}(\mathbb{R}^N)$, (2.1) and straightforward calculations show that $I'(u_0) = 0$.

Step 1: By setting $u_n^1 := u_n - u_0$, we have that

$$(a.1) \quad \|u_n^1\|_X^2 = \|u_n\|_X^2 - \|u_0\|_X^2 + o_n(1),$$

$$(b.1) \quad I_{\infty}(u_n^1) \rightarrow c - I(u_0),$$

$$(c.1) \quad I'_{\infty}(u_n^1) \rightarrow 0.$$

Indeed, (a.1) follows from the weak convergence of (u_n) . In order to prove (b.1) we first notice that, since $u_0 \in L^2(\mathbb{R}^N)$ and $u_n \rightarrow u_0$ in $L^2_{loc}(\mathbb{R}^N)$, we have that

$$\int V_{\infty}^+ u_n u_0 = \int V_{\infty}^+ u_0^2 + o_n(1).$$

This equality, the weak convergence of (u_n) and Lemma 2.4(i) imply that

$$I_{\infty}(u_n^1) - I(u_n) + I(u_0) = \frac{1}{2} \int (V_{\infty}^+ - V)((u_n)^2 - (u_0)^2) + o_n(1). \quad (2.6)$$

Now, for each $\varepsilon > 0$ there exists $R > 0$ such that $|V_{\infty}^+ - V(x)| < \varepsilon$ on $\mathbb{R}^N \setminus B_R(0)$. Setting $B_R := B_R(0)$, we can use the boundedness of (u_n) and Hölder's inequality to get

$$\begin{aligned} \left| \int (V_{\infty}^+ - V)((u_n)^2 - (u_0)^2) \right| &\leq \int_{B_R} |V_{\infty}^+ - V| |(u_n)^2 - (u_0)^2| \\ &\quad + \varepsilon \int_{\mathbb{R}^N \setminus B_R} |(u_n)^2 - (u_0)^2| \\ &\leq C_1 \|u_n - u_0\|_{L^2(B_R)} + C_2 \varepsilon = o_n(1). \end{aligned}$$

The above expression and (2.6) imply that (b.1) holds. For the proof of (c.1) it suffices to argue as above and use Lemma 2.4(ii). We omit the details.

Let us define

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n^1|^2 \, dx.$$

If $\delta = 0$, it follows from a result due to P.L. Lions [20, Lemma I.1] that $u_n^1 \rightarrow 0$ in $L^t(\mathbb{R}^N)$ for any $2 < t < 2^*$. Since $I'_\infty(u_n^1) \rightarrow 0$, it follows that $u_n^1 \rightarrow 0$ in X and the proof is complete. If $\delta > 0$, we obtain a sequence $(y_n^1) \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n^1)} |u_n^1|^2 \, dx > \frac{\delta}{2}.$$

We now define a new sequence $(v_n^1) \subset X$ by setting $v_n^1 := u_n^1(\cdot + y_n^1)$. Notice that (v_n^1) is bounded and therefore we may assume that $v_n^1 \rightharpoonup u^1$ in X and $v_n^1 \rightarrow u^1$ a.e on \mathbb{R}^N . Since

$$\int_{B_1(0)} |v_n^1|^2 \, dx > \frac{\delta}{2}$$

it follows from the Sobolev embedding that $u^1 \not\equiv 0$. Moreover, since $u_n^1 \rightarrow 0$ in X , we have that (y_n^1) is unbounded. Hence, going to a subsequence if necessary, we may assume that $|y_n^1| \rightarrow \infty$. Furthermore, we can check that $I'_\infty(u^1) = 0$.

Step 2: We now define $u_n^2 := u_n^1 - u^1(\cdot - y_n^1)$. As done for (u_n^1) , we can check that

(a.2) $\|u_n^2\|_X^2 = \|u_n\|_X^2 - \|u_0\|_X^2 - \|u^1\|_X^2 + o_n(1),$

(b.2) $I_\infty(u_n^2) \rightarrow c - I(u_0) - I_\infty(u^1),$

(c.2) $I'_\infty(u_n^2) \rightarrow 0.$

We now proceed by iteration. Notice that if u is a nontrivial critical point of I_∞ and \bar{u} is a ground state of problem (P_∞) , then we have by (f_2) that

$$I_\infty(u) \geq I_\infty(\bar{u}) = \int \left(\frac{1}{2} f(\bar{u})\bar{u} - F(\bar{u}) \right) = \beta > 0,$$

and therefore it follows from (b.2) above that the iteration must finish at some index $k \in \mathbb{N}$. This concludes the proof.

We present below the compactness result which will be used in the proofs of our main theorems.

Corollary 2.1 *The functional I satisfies $(PS)_c$ for any $c < c_\infty$.*

Proof. Let $(u_n) \subset X$ be such that

$$I(u_n) \rightarrow c < c_\infty \quad \text{and} \quad I'(u_n) \rightarrow 0.$$

Lemma 2.2 implies that (u_n) is bounded in X and therefore, going to a subsequence if necessary, we can suppose that $u_n \rightharpoonup u_0$ weakly in X . By Lemma 2.3 we have $I'(u_0) = 0$. Hence, we conclude from (f_2) that

$$I(u_0) = \int \left(\frac{1}{2} f(u_0)u_0 - F(u_0) \right) \geq 0.$$

If $u_n \not\rightarrow u_0$ in X , we can invoke Lemma 2.3 again to obtain $k \in \mathbb{N}$ and nontrivial solutions u^1, \dots, u^k of (P_∞) satisfying

$$\lim_{n \rightarrow \infty} I(u_n) = c = I(u_0) + \sum_{j=1}^k I_\infty(u^j) \geq kc_\infty \geq c_\infty,$$

contrary to the hypothesis. Hence $u_n \rightarrow u_0$ strongly in X and the corollary is proved.

3 Positive ground state solution

We devote this section to the proof of Theorem 1.1. As stated before, we are looking for critical points of the functional I .

We start by introducing the Nehari manifold of I defined as

$$\mathcal{N} := \{u \in X \setminus \{0\} : I'(u)u = 0\}.$$

Let

$$c_1 := \inf_{u \in \mathcal{N}} I(u).$$

In what follows we present some properties of c_1 and \mathcal{N} . For the proofs we refer to [28, Chapter 4]. First we observe that hypothesis (f_3) gives that, for any $u \in X \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. The maximum of the function $t \mapsto I(tu)$ for $t \geq 0$ is achieved at $t = t_u$.

We now note that, in view of (2.1) and the Sobolev embeddings, the origin is a local minimum of I . Moreover, condition (f_2) provides $C > 0$ such that

$$F(s) \geq C|s|^\theta, \quad \text{for all } s \in \mathbb{R}.$$

Without loss of generality, we can assume that $\theta \in (2, 2^*)$. Thus, if $u \neq 0$ and $t > 0$, we have

$$I(tu) \leq \frac{t^2}{2} \|u\|_X^2 - \frac{t^2}{2} \int V^-(x)u^2 - Ct^\theta \int |u|^\theta$$

and we conclude that $I(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. These observations show that I has the Mountain Pass geometry. By using (f_3) and the same arguments presented in the proof of [28, Theorem 4.2], we can prove that c_1 is positive, it coincides with the Mountain Pass level of I and has the following characterization

$$c_1 = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) = \inf_{u \in X \setminus \{0\}} \max_{t \geq 0} I(tu) > 0, \tag{3.1}$$

where $\Gamma := \{g \in C([0, 1], X) : g(0) = 0, I(g(1)) < 0\}$.

In what follows we use the above remarks to obtain a relation between c_1 and c_∞ .

Proposition 3.1 *Suppose that V satisfies (V_3) . Then*

$$0 < c_1 < c_\infty.$$

Proof. Let $\bar{u} \in \mathcal{N}_\infty$ be given by Proposition 2.1 and $t_{\bar{u}} > 0$ be the unique number such that $t_{\bar{u}}\bar{u} \in \mathcal{N}$. We claim that $t_{\bar{u}} < 1$. Indeed, by using condition (V_3) we deduce that

$$\begin{aligned} \int f(t_{\bar{u}}\bar{u})t_{\bar{u}}\bar{u} &= t_{\bar{u}}^2 \int (|\nabla\bar{u}|^2 + V(x)\bar{u}^2) \\ &< t_{\bar{u}}^2 \int (|\nabla\bar{u}|^2 + V_\infty^+\bar{u}^2) = t_{\bar{u}}^2 \int f(\bar{u})\bar{u}, \end{aligned}$$

that is

$$\int \left(\frac{f(t_{\bar{u}}\bar{u})}{t_{\bar{u}}\bar{u}} - \frac{f(\bar{u})}{\bar{u}} \right) \bar{u}^2 < 0.$$

This inequality and (f_3) imply that $t_{\bar{u}} < 1$.

It follows from (3.1) and its previous remarks that

$$c_1 \leq \max_{t \geq 0} I(t\bar{u}) = I(t_{\bar{u}}\bar{u}) = \int \left(\frac{1}{2}f(t_{\bar{u}}\bar{u})t_{\bar{u}}\bar{u} - F(t_{\bar{u}}\bar{u}) \right).$$

By (f_3) , we have that $h : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$h(t) := \int \left(\frac{1}{2}f(t\bar{u})t\bar{u} - F(t\bar{u}) \right)$$

is strictly increasing. Hence, we conclude that

$$c_1 \leq h(t_{\bar{u}}) < h(1) = \int \left(\frac{1}{2}f(\bar{u})\bar{u} - F(\bar{u}) \right) = c_\infty$$

and the proposition is proved.

We are now ready to prove our first result.

Proof of Theorem 1.1. Since I satisfies the geometry of the Mountain Pass Theorem there exists a sequence $(u_n) \subset X$ such that

$$I(u_n) \rightarrow c_1 \quad \text{and} \quad I'(u_n) \rightarrow 0.$$

Proposition 3.1 and Corollary 2.1 imply that the sequence (u_n) strongly converges to a function $u \in X$ such that $I(u) = c_1 > 0$ and $I'(u) = 0$. Clearly $u \neq 0$ and therefore u is a ground state solution of (P) .

In order to show that u is nonnegative we first note that, since we are interested in positive solutions, we can suppose that $f(s) = 0$ for any $s \leq 0$. Thus, since $I'(u)u^- = 0$, we get

$$\|u^-\|_X^2 = \int V^-(x)(u^-)^2 \leq \frac{\|V^-\|_{N/2}}{S} \int |\nabla u^-|^2$$

and therefore

$$\left(1 - \frac{\|V^-\|_{N/2}}{S} \right) \|u^-\|_X^2 = 0,$$

from which follows that $u^- \equiv 0$. Elliptic regularity and the strong maximum principle imply that $u > 0$ in \mathbb{R}^N .

Remark 3.1 Let $u_1 \in X$ be the solution given by Theorem 1.1. In view of Proposition 2.1, we can argue as in [19, Theorem 3.1] to conclude that u_1 has the same decay of the solution \bar{u} of the limit problem, that is, for any $0 < \delta < \sqrt{V_\infty^+}$, there exists $C = C(\delta) > 0$ such that

$$u_1(x) \leq Ce^{-\delta|x|}, \quad \text{for all } x \in \mathbb{R}. \tag{3.2}$$

4 Nodal solution

We start by introducing the closed set

$$\mathcal{N}_\pm := \{u \in X : u^+ \neq 0, u^- \neq 0, I'(u^+)u^+ = 0 = I'(u^-)u^-\}.$$

Note that any solution of (P) which belongs to \mathcal{N}_\pm changes sign. It is easy to check that I is bounded from below in \mathcal{N}_\pm and there exists $\rho > 0$ such that

$$\|u^\pm\|_X \geq \rho, \quad \text{for all } u \in \mathcal{N}_\pm. \tag{4.1}$$

Let us consider the following minimization problem

$$c_2 := \inf_{u \in \mathcal{N}_\pm} I(u).$$

In our next result we establish the relation between c_2 and the other minimizers c_1 and c_∞ . For its proof, we follow the approach of [9]. Since f is nonhomogeneous, the calculations are more involved.

Proposition 4.1 *Suppose that V satisfies (V_1) and (\widehat{V}_3) . Then*

$$0 < c_2 < c_1 + c_\infty. \tag{4.2}$$

Proof. Let \bar{u} be given by Proposition 2.1 and define $\bar{u}_n(x) := \bar{u}(x - x_n)$, where $x_n := (0, \dots, 0, n)$. From now on we denote by u_1 a positive ground state of (P) given by Theorem 1.1. For any $\alpha, \beta > 0$ we consider the functions

$$h^\pm(\alpha, \beta, n) := \int |\nabla(\alpha u_1 - \beta \bar{u}_n)^\pm|^2 + V(x)|(\alpha u_1 - \beta \bar{u}_n)^\pm|^2 - \int f((\alpha u_1 - \beta \bar{u}_n)^\pm)(\alpha u_1 - \beta \bar{u}_n)^\pm.$$

Recalling that $I'(u_1)u_1 = 0$ and using (f₃) we get

$$\begin{aligned} & \int \left(|\nabla(u_1/2)|^2 + V(x)(u_1/2)^2 \right) - \int f(u_1/2)(u_1/2) \\ &= \int \left(\frac{f(u_1)}{u_1} - \frac{f(u_1/2)}{(u_1/2)} \right) \left(\frac{u_1}{2} \right)^2 > 0. \end{aligned} \tag{4.3}$$

Analogously,

$$\int \left(|\nabla (2u_1)|^2 + V(x) (2u_1)^2 \right) - \int f(2u_1)(2u_1) < 0. \tag{4.4}$$

Claim 1. For n sufficiently large there holds

$$\int \left(|\nabla (\bar{u}_n/2)|^2 + V(x) (\bar{u}_n/2)^2 \right) - \int f(\bar{u}_n/2) (\bar{u}_n/2) > 0, \tag{4.5}$$

$$\int \left(|\nabla (2\bar{u}_n)|^2 + V(x) (2\bar{u}_n)^2 \right) - \int f(2\bar{u}_n)(2\bar{u}_n) < 0. \tag{4.6}$$

We only prove (4.5), since the other inequality can be proved in the same way. First, notice that

$$\int \left(|\nabla (\bar{u}_n/2)|^2 + V(x) (\bar{u}_n/2)^2 \right) - \int f(\bar{u}_n/2) (\bar{u}_n/2) = \gamma_1 + J_n, \tag{4.7}$$

where

$$\gamma_1 := \int \left(|\nabla (\bar{u}_n/2)|^2 + V_\infty^+ (\bar{u}_n/2)^2 \right) - \int f(\bar{u}_n/2) (\bar{u}_n/2)$$

and

$$J_n := \frac{1}{4} \int (V(x) - V_\infty^+) \bar{u}_n^2.$$

It follows from (f_3) that $\gamma_1 > 0$. Thus, it suffices to check that $J_n \rightarrow 0$ as $n \rightarrow \infty$. Given $\varepsilon > 0$ we can use (V_1) to obtain $R > 0$ such that $|V(x) - V_\infty^+| \leq \varepsilon$ for $|x| > R$. Hence,

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} (V(x) - V_\infty^+) \bar{u}_n^2 \right| \leq \varepsilon \|\bar{u}_n\|_2 = \varepsilon \|\bar{u}\|_2. \tag{4.8}$$

We now recall that, since \bar{u} is radially symmetric, by the Radial Lemma (see [25]) there exists $C > 0$ such that

$$|\bar{u}(x)| \leq C|x|^{(1-N)/2} \|\bar{u}\|_{W^{1,2}(\mathbb{R}^N)}, \quad \text{a.e. } x \in \mathbb{R}^N.$$

Since $|x - x_n| \geq |x_n| - |x| \geq n - R$ on $B_R(0)$, we get

$$\begin{aligned} \left| \int_{B_R(0)} (V(x) - V_\infty^+) \bar{u}_n^2 \right| &\leq \|V - V_\infty^+\|_{L^{N/2}(B_R(0))} \left(\int_{B_R(0)} (\bar{u}(x - x_n))^{2^*} \right)^{(N-2)/N} \\ &\leq C_1 \left(\frac{1}{n-R} \right)^{(N-1)(N-2)/2N} \|\bar{u}\|_{W^{1,2}(\mathbb{R}^N)}^2. \end{aligned}$$

The above estimate and (4.8) imply that $J_n \rightarrow 0$ as $n \rightarrow \infty$. This proves (4.5) and establishes the claim.

Since $\bar{u}(x) \rightarrow 0$ as $|x| \rightarrow \infty$, it follows from (4.3)-(4.6) that there exists $n_0 > 0$ such that

$$\begin{cases} h^+(1/2, \beta, n) > 0 \\ h^+(2, \beta, n) < 0, \end{cases}$$

for $n \geq n_0$ and $\beta \in [1/2, 2]$. Now, for all $\alpha \in [1/2, 2]$, we have

$$\begin{cases} h^-(\alpha, 1/2, n) > 0 \\ h^-(\alpha, 2, n) < 0. \end{cases}$$

Hence, we can apply a variant of the Mean Value Theorem due to Miranda [21] (see also [27]), to obtain $\alpha^*, \beta^* \in [1/2, 2]$ such that $h^\pm(\alpha^*, \beta^*, n) = 0$, for any $n \geq n_0$. Thus,

$$\alpha^* u_1 - \beta^* \bar{u}_n \in \mathcal{N}_\pm \text{ for } n \geq n_0.$$

In view of the definition of c_2 , it suffices to show that

$$\sup_{\frac{1}{2} \leq \alpha, \beta \leq 2} I(\alpha u_1 - \beta \bar{u}_n) < c_1 + c_\infty,$$

for some $n \geq n_0$.

In order to do this, we compute

$$\begin{aligned} I(\alpha u_1 - \beta \bar{u}_n) &= \frac{1}{2} \int (|\nabla(\alpha u_1)|^2 + |\nabla(\beta \bar{u}_n)|^2) + \frac{1}{2} \int V(x) ((\alpha u_1)^2 + (\beta \bar{u}_n)^2) \\ &\quad - \alpha \beta \int (\nabla u_1 \nabla \bar{u}_n + V(x) u_1 \bar{u}_n) - \int F(\alpha u_1 - \beta \bar{u}_n) \pm \int F(\alpha u_1). \end{aligned}$$

Since u_1 is a positive solution of (P) we have that

$$\int (\nabla u_1 \nabla \bar{u}_n + V(x) u_1 \bar{u}_n) \geq 0,$$

and therefore

$$\begin{aligned} I(\alpha u_1 - \beta \bar{u}_n) &\leq \left\{ I(\alpha u_1) + \int F(\alpha u_1) \right\} + \frac{1}{2} \int (|\nabla(\beta \bar{u}_n)|^2 + V(x)(\beta \bar{u}_n)^2) \\ &\quad - \int F(\alpha u_1 - \beta \bar{u}_n) \pm \int V_\infty^+(\beta \bar{u}_n)^2 \pm \int F(\beta \bar{u}_n), \\ &= I(\alpha u_1) + I_\infty(\beta \bar{u}_n) + \frac{1}{2} \int (V(x) - V_\infty^+) (\beta \bar{u}_n)^2 - J_{\alpha, \beta, n}, \end{aligned} \tag{4.9}$$

where

$$J_{\alpha, \beta, n} := \int (F(\alpha u_1 - \beta \bar{u}_n) - F(\alpha u_1) - F(\beta \bar{u}_n)).$$

Claim 2. For some $n \geq n_0$, we have

$$\frac{1}{2} \int (V(x) - V_\infty^+) (\beta \bar{u}_n)^2 - J_{\alpha, \beta, n} < 0. \tag{4.10}$$

If this is true we can use (4.9) and $I_\infty(\beta \bar{u}_n) = I_\infty(\beta \bar{u})$ to get

$$\sup_{\frac{1}{2} \leq \alpha, \beta \leq 2} I(\alpha u_1 - \beta \bar{u}_n) < \sup_{\alpha \geq 0} I(\alpha u_1) + \sup_{\beta \geq 0} I_\infty(\beta \bar{u}) = c_1 + c_\infty,$$

which concludes the proof of the lemma.

It remains to prove Claim 2. We start by noting that, in view of (\widehat{V}_3) and (2.4), we have

$$\begin{aligned} \frac{1}{2} \int (V(x) - V_\infty^+) (\beta \bar{u}_n)^2 &= \frac{\beta^2}{2} \int (V(x + x_n) - V_\infty^+) \bar{u}(x)^2 \\ &\leq -C_2 \int e^{-\gamma|x+x_n|} \bar{u}(x)^2 \\ &\leq -C_2 e^{-\gamma n} \int e^{-\gamma|x|} \bar{u}(x)^2 = -C_3 e^{-\gamma n}. \end{aligned} \tag{4.11}$$

On the other hand, it follows from [1, Lemma 2.4] and (2.1) that

$$\begin{aligned} |J_{\alpha,\beta,n}| &\leq \int f(\alpha u_1) \bar{u}_n + f(\beta \bar{u}_n) u_1 \\ &\leq C_4 \left(\int u_1^q \bar{u}_n + \int u_1^\eta \bar{u}_n + \int u_1 \bar{u}_n^q + \int u_1 \bar{u}_n^\eta \right). \end{aligned} \tag{4.12}$$

Setting $A_n := B_{n/(q+1)}(0)$ we can use Holder’s inequality and (2.4) to write

$$\begin{aligned} \int_{A_n} u_1^q \bar{u}_n &\leq \left(\int u_1^{q+1} \right)^{q/(q+1)} \left(\int_{A_n} \bar{u}_n^{q+1} dx \right)^{1/(q+1)} \\ &\leq C_5 \left(\int_{A_n} e^{-\delta(q+1)|x-x_n|} dx \right)^{1/(q+1)}. \end{aligned}$$

Since $|x_n - x| \geq |x_n| - |x| = n - |x|$, we get

$$\begin{aligned} \int_{A_n} u_1^q \bar{u}_n dx &\leq C_5 e^{-\delta n} \left(\int_{A_n} e^{\delta(q+1)|x|} dx \right)^{1/(q+1)} \\ &= C_6 e^{-\delta n} \left(\int_0^{n/(q+1)} e^{\delta(q+1)r} r^{N-1} dr \right)^{1/(q+1)}. \end{aligned} \tag{4.13}$$

We now recall that, for any fixed $t > 0$ we have

$$\int e^{tr} r^{N-1} dr = e^{tr} P(r),$$

where

$$P(r) := \frac{r^{N-1}}{t} - \frac{(N-1)}{t^2} r^{N-2} + \frac{(N-1)(N-2)}{t^3} r^{N-3} + \dots + (-1)^{N+1} \frac{(N-1)!}{t^N}.$$

Hence, by taking $t := \delta(q+1)$, we get

$$\int_0^{n/(q+1)} e^{\delta(q+1)r} r^{N-1} dr = e^{\delta n} P(n/(q+1)) = C_7 e^{n\delta} + C_8,$$

where C_7 and C_8 depend only on δ . The above estimate and (4.13) provide

$$\int_{A_n} u_1^q \bar{u}_n dx \leq C_9 \left(e^{-\delta n} e^{\delta n/(q+1)} + e^{-\delta n} \right) \leq C_{10} e^{-\delta n(\frac{q}{q+1})}. \tag{4.14}$$

Analogously we have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus A_n} u_1^q \bar{u}_n dx &\leq \left(\int_{\mathbb{R}^N \setminus A_n} u_1^{q+1} dx \right)^{q/(q+1)} \left(\int (\bar{u}(x - x_n))^{q+1} dx \right)^{1/(q+1)} \\ &\leq C_{11} \left(\int_{\mathbb{R}^N \setminus A_n} e^{-\delta(q+1)|x|} dx \right)^{q/(q+1)} \\ &= C_{12} \left(\int_{n/(q+1)}^\infty e^{-\delta(q+1)r} r^{N-1} dr \right)^{q/(q+1)} \\ &\leq C_{13} e^{-\delta n(\frac{q}{q+1})}. \end{aligned}$$

This together with (4.14) implies that

$$\int u_1^q \bar{u}_n \leq C_{14} e^{-\delta n(\frac{q}{q+1})}. \tag{4.15}$$

Since $q < \eta$, we can proceed as above to obtain $C_{15} > 0$ such that

$$\max \left\{ \int u_1^\eta \bar{u}_n, \int u_1 \bar{u}_n^q, \int u_1 \bar{u}_n^\eta \right\} \leq C_{15} e^{-\delta n(\frac{q}{q+1})}.$$

The above estimate, (4.15), (4.12) and (4.11) imply that

$$\frac{1}{2} \int (V(x) - V_\infty^+) (\beta \bar{u}_n)^2 - J_{\alpha,\beta,n} \leq -C_3 e^{-\gamma n} + C_{16} e^{-\delta n(\frac{q}{q+1})}.$$

Since $\gamma < \sqrt{V_\infty^+} q/(q+1)$, we can choose $0 < \delta < \sqrt{V_\infty^+}$ sufficiently close to $\sqrt{V_\infty^+}$ in such way that $\gamma < \delta q/(q+1)$. It follows for this choice and the above expression that (4.10) is satisfied for n large enough. This concludes the proof of the proposition.

In the our next result, we adapt some ideas from [10] (see also [29, 9]).

Proposition 4.2 *There exists a sequence (u_n) in \mathcal{N}_\pm satisfying*

$$I(u_n) \rightarrow c_2 \text{ and } I'(u_n) \rightarrow 0. \tag{4.16}$$

Proof. Since I is bounded from below on \mathcal{N}_\pm , we can apply the Ekeland variational principle to obtain a sequence $(u_n) \subset \mathcal{N}_\pm$ satisfying

$$c_2 \leq I(u_n) \leq c_2 + \frac{1}{n}$$

and

$$I(v) \geq I(u_n) - \frac{1}{n} \|v - u_n\|_X, \quad \text{for all } v \in \overline{\mathcal{N}_\pm}. \tag{4.17}$$

For any fixed $\varphi \in X, n \in \mathbb{N}$, we introduce the C^1 -functions $h_n^\pm : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$\begin{aligned} h_n^\pm(t, s, l) &:= \int |\nabla(u_n + t\varphi + su_n^+ + lu_n^-)^\pm|^2 \\ &\quad + \int V(x) |(u_n + t\varphi + su_n^+ + lu_n^-)^\pm|^2 \\ &\quad - \int f((u_n + t\varphi + su_n^+ + lu_n^-)^\pm)(u_n + t\varphi + su_n^+ + lu_n^-)^\pm. \end{aligned}$$

Note that $h_n^\pm(0, 0, 0) = 0, (\partial h_n^+ / \partial l)(0, 0, 0) = 0$ and $(\partial h_n^- / \partial s)(0, 0, 0) = 0$. Moreover, recalling that $I'(u_n^+)u_n^+ = 0$ and using (\widehat{f}_3) , we get

$$\begin{aligned} \frac{\partial h_n^+}{\partial s}(0, 0, 0) &= 2 \int (|\nabla u_n^+|^2 + V(x)(u_n^+)^2) \\ &\quad - \int f'(u_n^+)(u_n^+)^2 + f(u_n^+)(u_n^+) \\ &= \int f(u_n^+)(u_n^+) - f'(u_n^+)(u_n^+)^2 \\ &\leq -C \int |u_n^\pm|^{\sigma+1}. \end{aligned} \tag{4.18}$$

Claim 1. There exists $C_1 > 0$ such that

$$\int |u_n^\pm|^{\sigma+1} \geq C_1 > 0.$$

In order to prove the claim we first notice that (u_n) is bounded in X . Moreover, for any given $\delta > 0$, we can use $(f_1), I'(u_n^+)u_n^+ = 0$ and (2.2) to obtain $C_\delta > 0$ such that

$$\left(1 - \frac{\|V^-\|_{N/2}}{S}\right) \|u_n^+\|_X^2 \leq \delta \int |u_n^+|^{q+1} + C_\delta \int |u_n^+|^{\eta+1}.$$

Arguing by contradiction we suppose that, up to a subsequence, $\int |u_n^+|^{\sigma+1} \rightarrow 0$. Since (u_n) is bounded in X and $2 \leq q+1 < \eta+1 \leq \sigma+1$, we can use interpolation to conclude that the right hand side of the above expression goes to zero, contradicting (4.1). The same can be done for u_n^- and therefore the claim holds.

It follows from (4.18) and the above claim that

$$\frac{\partial h_n^+}{\partial s}(0, 0, 0) < 0.$$

Similarly, we can check that

$$\frac{\partial h_n^-}{\partial l}(0, 0, 0) < 0.$$

Hence, we can apply the Implicit Function Theorem to obtain $\delta_n > 0$ and two C^1 -functions $s_n, l_n : (-\delta_n, \delta_n) \rightarrow \mathbb{R}$ such that $s_n(0) = l_n(0) = 0$ and

$$h_n^\pm(t, s_n(t), l_n(t)) = 0. \tag{4.19}$$

Claim 2. There exists $C_2 > 0$ such that

$$|s'_n(0)| \leq C_2, \quad |l'_n(0)| \leq C_2.$$

Indeed, by using (4.19) we get

$$\begin{aligned} s'_n(0) &= \frac{(\partial h_n^+ / \partial t)(0, 0, 0)}{(\partial h_n^+ / \partial s)(0, 0, 0)} \\ &= - \frac{2 \int (\nabla u_n^+ \nabla \varphi + V(x) u_n^+ \varphi) - \int (f'(u_n^+) u_n^+ + f(u_n^+)) \varphi}{\int f'(u_n^+) (u_n^+)^2 - f(u_n^+) u_n^+}. \end{aligned}$$

This, (4.19), (f_1) and the boundedness of u_n in X imply that $|s'_n(0)| \leq C_2$ for some $C_2 > 0$. A similar argument can be applied for the sequence $(l'_n(0))$.

We now note that, by (4.19), we have that

$$u_n + t\varphi + s_n(t)u_n^+ + l_n(t)u_n^- \in \mathcal{N}_\pm, \quad \text{for any } t \in (-\delta_n, \delta_n)$$

and therefore we can use (4.17) to get

$$I(u_n + t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-) - I(u_n) \geq -\frac{1}{n} \|t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-\|_X, \tag{4.20}$$

for any $t \in (-\delta_n, \delta_n)$.

Claim 3. We have that

$$I'(u_n)\varphi \geq -\frac{1}{n} \|\varphi\|_X - \frac{C_3}{n}. \tag{4.21}$$

If this is true, it follows that

$$\|I'(u_n)\| \leq \frac{C_4}{n},$$

and therefore the second statement in (4.2) holds and the proof is complete.

It remains to prove Claim 3. Let $w_n := t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-$ and notice that, since $I'(u_n)u_n^\pm = I'(u_n^\pm)u_n^\pm = 0$, we have that

$$I(u_n + w_n) - I(u_n) = I'(u_n)w_n + r(t, n) = tI'(u_n)\varphi + r(t, n), \tag{4.22}$$

where $r(t, n) = o(\|t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-\|_X)$ as $t \rightarrow 0$. This expression, (4.20), the boundedness of (u_n) and Claim 2 imply that

$$\begin{aligned} I'(u_n)\varphi + \frac{r(t, n)}{t} &\geq -\frac{1}{n} \|\varphi\|_X - \frac{1}{n} \left\| \frac{s_n(t)}{t} u_n^+ + \frac{l_n(t)}{t} u_n^- \right\|_X \\ &\geq \frac{1}{n} \|\varphi\|_X - \frac{C_3}{n}, \end{aligned} \tag{4.23}$$

for any $t > 0$. By using Claim 2 again we conclude that

$$\frac{r(t, n)}{t} = \frac{r(t, n)}{\|t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-\|_X} \cdot \frac{\|t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-\|_X}{t} = o(1),$$

as $t \rightarrow 0$. Letting $t \rightarrow 0^+$ in (4.23) we get (4.21) and therefore the proposition is proved.

In view of the two previous propositions we can argue as in [9] to prove Theorem 1.2. For the sake of completeness, we present the proof here.

Proof of Theorem 1.2. Let $(u_n) \subset \mathcal{N}_\pm$ be the sequence given by the above proposition. We can easily check that (u_n) is bounded in X . Hence, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in X with $I'(u_0) = 0$. In view of Lemma 2.3 we have either $u_n \rightarrow u_0$ strongly in X or there exists u^1, \dots, u^k nontrivial solutions of (P_∞) satisfying the conclusions of Lemma 2.3(b). Since $c_1 < c_\infty$ it follows from (2.5) that $k \leq 1$.

Suppose that $u_0 \equiv 0$. In this case, since $c_2 > 0$, we have that $k = 1$ and therefore

$$\|u_n - u^1(\cdot - y_n^1)\|_X \rightarrow 0.$$

Since $|y_n^1| \rightarrow \infty$ and $(u_n) \subset \mathcal{N}_\pm$, the convergence above and (4.1) imply that $(u^1)^\pm \in \mathcal{N}_\infty$. Hence,

$$c_1 + c_\infty > c_2 = I_\infty(u^1) = I_\infty((u^1)^+) + I_\infty((u^1)^-) \geq 2c_\infty$$

contradicting $c_1 < c_\infty$. Thus $u_0 \not\equiv 0$ and we can use $c_2 < c_1 + c_\infty$ again to conclude that $k = 0$, that is, $u_n \rightarrow u_0$ strongly in X . It follows from (4.1) that $u_0 \in \mathcal{N}_\pm$ is a sign changing solution of (P) .

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