

## POSITIVE CESÀRO MEANS OF NUMERICAL SERIES

RICHARD ASKEY<sup>1</sup>

ABSTRACT. If a series has positive Cesàro means of order  $\gamma$ , then its Abel means have positive Cesàro means of order  $\alpha$  for  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$ ,  $-1 < \alpha < \gamma$ .

1. Introduction. Fejér [4] proved the nonnegativity of the partial sums of the series

$$(1.1) \quad 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\theta$$

when  $0 \leq r \leq 1/2$ . In this paper he gave two proofs, one using the convexity of the sequence  $1, r, r^2, \dots, r^n, 0, 0, \dots$ , for  $0 \leq r \leq 1/2$ , and a second by explicitly summing the series

$$1 + 2 \sum_{k=1}^n r^k \cos k\theta = \frac{1 - r^2 + 2r^{n+1}[r \cos n\theta - \cos(n+1)\theta]}{1 - 2r \cos \theta + r^2}.$$

This result brought forth many similar results. Schur and Szegő [9] used the convexity argument to prove that the  $(C, \alpha)$  means of (1.1) are nonnegative for  $0 \leq r \leq (\alpha + 1)/2$ ,  $-1 < \alpha < 1$ . Using a different argument, a summation by parts and the nonnegativity of  $\sum_{k=0}^n P_k(x)$ , Szegő [10] proved that the partial sums of

$$(1.2) \quad \sum_{n=0}^{\infty} r^n (2n + 1) P_n(x), \quad -1 \leq x \leq 1,$$

are nonnegative for  $0 \leq r \leq 1/3$ . Here  $P_n(x)$  is the Legendre polynomial. Finally Fejér [5] observed that these theorems have nothing to do with cosines or Legendre polynomials; they are theorems about numerical series which have a positive Cesàro mean of some order. The  $(C, 1)$  means of (1.1) and the  $(C, 2)$  means of (1.2) are positive for  $0 \leq r \leq 1$  and this is the

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basic fact behind these results. Fejér then proved that the  $(C, 1)$  means of (1.2) are positive for  $0 \leq r \leq 2/3$ . Fejér [6] later proved that

$$\sum_{k=0}^n r^k(k+1) \sin(k+1)\theta \geq 0$$

for  $0 \leq r \leq 1/4$ . Here the  $(C, 3)$  means are positive.

To complete the above results we will prove

**Theorem 1.** *Let  $-1 < \alpha < \gamma$  and assume that the  $(C, \gamma)$  means of  $\sum_{n=0}^{\infty} a_n$  are nonnegative. Then the  $(C, \alpha)$  means of  $\sum_{n=0}^{\infty} a_n r^n$  are nonnegative for  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$ , and this interval is best possible.*

For the convenience of the reader, we recall the definition of the Cesàro means  $(C, \alpha)$ . The  $(C, \alpha)$  means of the formal series

$$(1.3) \quad \sum_{n=0}^{\infty} a_n$$

are defined by

$$(1.4) \quad \sigma_n^\alpha = \frac{n!}{(\alpha + 1)_n} \sum_{k=0}^n \frac{(\alpha + 1)_{n-k}}{(n-k)!} a_k, \quad \alpha > -1,$$

where  $(\alpha)_k$  is defined by

$$(1.5) \quad (\alpha)_k = \Gamma(k + \alpha)/\Gamma(\alpha) = \alpha(\alpha + 1) \cdots (\alpha + k - 1).$$

2. **Proof of Theorem 1.** Since we will only be concerned with positivity properties, and  $\alpha > -1$ , it will be sufficient to consider

$$(2.1) \quad g_n(r) = \sum_{k=0}^n \frac{(\alpha + 1)_{n-k}}{(n-k)!} a_k r^k.$$

Multiply by  $s^n$  and sum. This gives

$$(2.2) \quad \begin{aligned} \sum_{n=0}^{\infty} g_n(r) s^n &= \sum_{k=0}^{\infty} a_k (rs)^k \sum_{n=k}^{\infty} \frac{(\alpha + 1)_{n-k}}{(n-k)!} s^{n-k} \\ &= f(rs)(1-s)^{-\alpha-1} = \frac{(1-rs)^{\gamma+1}}{(1-s)^{\alpha+1}} \cdot \frac{f(rs)}{(1-rs)^{\gamma+1}}. \end{aligned}$$

The assumption that  $\sum_{n=0}^{\infty} a_n$  has nonnegative Cesàro means of order  $\gamma$  means that  $(1-t)^{-\gamma-1} f(t)$  has nonnegative power series coefficients. Thus Theorem 1 will be proved when we show that

$$(1 - rs)^{\gamma+1}(1 - s)^{-\alpha-1} = \sum_{n=0}^{\infty} h_n(r; \alpha, \gamma) s^n$$

with  $h_n(r; \alpha, \gamma) \geq 0$ ,  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$ . A simple calculation gives

$$\begin{aligned} h_n(r; \alpha, \gamma) &= \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (-\gamma - 1)_k}{(-n - \alpha)_k k!} r^k \\ &= \frac{(\alpha + 1)_n}{n!} {}_2F_1(-n, -\gamma - 1; -n - \alpha; r). \end{aligned}$$

Thus Theorem 1 reduces to proving

**Theorem 2.** For  $-1 < \alpha < \gamma$  and  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$ ,

$$(2.3) \quad k_n = {}_2F_1(-n, -\gamma - 1; -n - \alpha; r) \geq 0, \quad n = 0, 1, \dots$$

We will prove Theorem 2 by induction. When  $n = 0$ ,  $k_0 = 1$ , and when  $n = 1$ ,

$$k_1 = 1 - (\gamma + 1)r/(\alpha + 1) \geq 0 \quad \text{if } r \leq (\alpha + 1)/(\gamma + 1).$$

When  $n = 2$ ,

$$k_2 = 1 - \frac{2(\gamma + 1)r}{(\alpha + 2)} + \frac{(\gamma + 1)\gamma r^2}{(\alpha + 2)(\alpha + 1)}$$

and this is nonnegative when

$$l(r) = \gamma(\gamma + 1)r^2 - 2(\gamma + 1)(\alpha + 1)r + (\alpha + 1)(\alpha + 2) \geq 0.$$

Assume that  $\gamma \neq 0$ .

$l(0) > 0$  and  $l(r)$  takes its extreme value when  $r = (\alpha + 1)/\gamma$ , which does not lie in the interval  $[0, (\alpha + 1)/(\gamma + 1)]$ . Thus  $l(r) \geq 0$ ,  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$  if  $l((\alpha + 1)/(\gamma + 1)) \geq 0$ . But  $l((\alpha + 1)/(\gamma + 1)) = (\alpha + 1)(\gamma - \alpha)/(\gamma + 1)$  and this is nonnegative. When  $\gamma = 0$ ,  $l(r)$  is linear so  $l(r) > 0$  follows from the above inequalities.

Now we can start the induction. We would like to use one of the Gauss contiguous relations on the hypergeometric function (2.3). This can be done after we apply the Euler transformation

$${}_2F_1(a, b; c; r) = (1 - r)^{-b} {}_2F_1(c - a, b; c; r/(r - 1)), \quad [2, 2.1.4(22)]$$

to  $k_n$ . This gives

$$k_n = (1 - r)^{\gamma+1} {}_2F_1(-\alpha, -\gamma - 1; -n - \alpha; r/(r - 1))$$

and this is nonnegative if

$$(2.4) \quad l_n(t) = {}_2F_1(-\alpha, -\gamma - 1; -n - \alpha; t) \geq 0, \quad -(\alpha + 1)/(\gamma - \alpha) \leq t \leq 0.$$

This has been proven for  $n = 0, 1, 2$ . The Gauss contiguous relation applied to  $l_{n-1}(t), l_n(t)$  and  $l_{n+1}(t)$  is

$$(2.5) \quad (n + \alpha)(n + \alpha - 1)(1 - t)l_{n+1}(t) = (n + \alpha)[(n + \alpha + 1) - (2n + \alpha - \gamma)t]l_n(t) + n(n + \alpha - \gamma - 1)l_{n-1}(t), \quad [2, 2.8(45)].$$

The coefficient of  $l_{n+1}(t)$  is positive, and the coefficients of  $l_n(t)$  and  $l_{n-1}(t)$  are nonnegative if  $n \geq 2, -(\alpha + 1)/(\gamma - \alpha) \leq t \leq 0$ , and  $\gamma - 1 \leq \alpha \leq \gamma$ . Thus by induction  $l_{n+1}(t)$  is nonnegative if  $l_n(t)$  and  $l_{n-1}(t)$  are nonnegative when we make the extra assumptions that  $\gamma - 1 \leq \alpha < \gamma$  and  $n \geq 2$ . The condition  $n \geq 2$  is no problem, since we have shown  $l_0(t), l_1(t)$  and  $l_2(t)$  are nonnegative. And surprisingly the assumption  $\gamma - 1 \leq \alpha < \gamma$  is also no problem. Let  $\gamma - 2 \leq \alpha < \gamma - 1$ . Then (2.4) holds for  $n = 0, 1, \dots$ , if and only if  $(1 - rs)^{\gamma+1}(1 - s)^{-\alpha-1}$  has nonnegative power series coefficients for  $0 \leq r \leq (\alpha + 1)/(\gamma + 1)$ . Consider

$$(2.6) \quad \frac{(1 - rs)^{\gamma+1}}{(1 - s)^{\alpha+1}} = \frac{(1 - \rho us)^{\gamma+1}}{(1 - us)^{\alpha+2}} \cdot \frac{(1 - us)^{\alpha+2}}{(1 - s)^{\alpha+1}}$$

where  $n = (\alpha + 1)/(\alpha + 2), \rho = (\alpha + 2)r/(\alpha + 1)$ . The first factor on the right in (2.6) has nonnegative power series coefficients because

$$0 \leq \rho \leq \left(\frac{\alpha + 2}{\alpha + 1}\right)\left(\frac{\alpha + 1}{\gamma + 1}\right) = \frac{\alpha + 2}{\gamma + 1}$$

and  $\gamma - 1 \leq \alpha + 1 < \gamma$ . And the second also has nonnegative power series coefficients. Therefore the product does. This argument can be iterated to complete the proof.

The result is best possible, for  $k_1 < 0$  when  $r > (\alpha + 1)/(\gamma + 1)$  and by choosing  $a_{n-1} = 1, a_n = -(\gamma + 1), a_j = 0, j \neq n - 1, n, g_n(r)$  in (2.2) is  $(\alpha + 1)k_1$ .

One application of this result to an interesting class of expansions is given below. For notation the reader is referred to the Bateman project [3]. For ultraspherical expansions these results imply

**Theorem 3.** *Let  $\lambda > 0$  and assume that  $\int_{-1}^1 |f(x)|(1 - x^2)^{\lambda - 1/2} dx < \infty$ . Define*

$$a_n = \int_{-1}^1 f(x) \frac{C_n^\lambda(x)}{C_n^\lambda(1)} (1 - x^2)^{\lambda - 1/2} dx.$$

*Then if  $f(x) \geq 0$ , the  $(C, \alpha)$  means of*

$$(2.7) \quad \sum_{n=0}^{\infty} r^n a_n \frac{(n + \lambda)}{\lambda} C_n^\lambda(x)$$

are nonnegative for  $0 \leq r \leq (\alpha + 1)/(2\lambda + 2)$ ,  $-1 < \alpha \leq 2\lambda + 1$ .

**Proof.** (2.7) is

$$\int_{-1}^1 f(y) \sum_{n=0}^{\infty} \frac{r^n(n + \lambda) C_n^\lambda(x) C_n^\lambda(y)}{\lambda C_n^\lambda(1)} (1 - y^2)^{\lambda - 1/2} dy$$

and the  $(C, 2\lambda + 1)$  means of  $\sum_{n=0}^{\infty} r^n(n + \lambda) C_n^\lambda(x) C_n^\lambda(y)/\lambda C_n^\lambda(1)$  are nonnegative. This was first proven by Kogbetliantz [7], and a simple proof is given in [1].

If we let  $\lambda \rightarrow 0$  this gives the Schur-Szegő result.

In the case of Fourier series, (1.1), and probably for the corresponding sums associated with the spherical harmonic expansion on the  $k$ -dimensional sphere, there is a refinement of Theorem 1. In the case of (1.1), Schur and Szegő called  $r_N$  the largest  $r$  for which  $1 + 2\sum_{n=1}^N r^n \cos n\theta \geq 0$ ,  $0 \leq \theta \leq \pi$ . They proved

$$r_N = 1 - \frac{\log 2N}{N} + \frac{\log \log 2N}{N} + \frac{\epsilon_N}{N}, \quad \epsilon_N \rightarrow 0.$$

Related results for Fourier sine series were given by Robertson [8]. These are deep results and it is unlikely we could obtain results of similar precision in the case of general spherical harmonic expansions.

### 3. Further questions. There are many problems suggested by Theorem

1. The Cesàro means can be replaced by many families of summability methods, or by summability methods applied to integrals. Somewhat surprisingly the most natural analogue for integrals of Theorem 1 fails.

The  $(C, 1)$  (or Riesz) mean of  $\cos yt$  is

$$(3.1) \quad \int_0^x \left(1 - \frac{t}{x}\right) \cos yt dt = \frac{1 - \cos yx}{xy^2} \geq 0, \quad x > 0.$$

The analogue of (1.1) is

$$(3.2) \quad f(x) = \int_0^x e^{-\epsilon t} \cos yt dt = \frac{\epsilon - (\epsilon \cos yx - y \sin yx) e^{-\epsilon x}}{\epsilon^2 + y^2}.$$

If  $f(3\pi/(2y)) \geq 0$  then an integration by parts shows that  $f(x) \geq 0$ ,  $x \geq 0$  when  $\epsilon > 0$ . So it is necessary and sufficient to consider  $f(3\pi/(2y))$ . But  $f(3\pi/(2y)) < 0$  if  $\epsilon - y \exp(-3\pi\epsilon/(2y)) < 0$  or

$$(3.3) \quad \epsilon/y < \exp(-3\pi\epsilon/2y).$$

Let  $\epsilon_1$  be the unique positive root of  $g(x) = x - \exp(-3\pi x/2) = 0$ . It exists since  $g(0) = -1$  and  $g(1) > 0$  and is unique because  $g'(x) > 0$ . To seven decimal places  $\epsilon_1 = 0.2744106$ .

Thus if  $\epsilon > 0$  is given then  $y$  can be chosen so that  $y > \epsilon/\epsilon_1$  and then  $\int_0^{3\pi/(2y)} e^{-\epsilon t} \cos yt \, dt < 0$ . Therefore the most natural analogue of Theorem 1 fails for integrals.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN  
53706